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A data driven runs test to identify first order positive Markovian dependence

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Abstract

We propose a data driven test to identify first order positive Markovian dependence in a Bernoulli sequence, based on a combination of two runs tests: a well known runs test for the same purpose conditional on the numbers of ones in the sequence, and a modified runs test independent of the number of ones. We give analytic expressions for the exact distribution of the modified runs test statistic and for its power; also we built an algorithm to calculate it explicitly. To compare the power of the tests, we calculated these for some values of the proportion of ones and the success probability. We show that there are some intervals for the success probability in which the new runs test surpasses the power of the conditional test, and that the data driven test improves the power of the two runs tests, when they are considered separately.

Keywords: Markov-dependent Bernoulli trials, Data driven runs test, Runs distributions, Hypothesis of randomness, Power of a test.

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1. Introduction

Since the pioneering work performed by [1], in which he calculated the power of a test for randomness based on the total number of runs conditioned on the number of successes in a binary sequence versus the first order Markovian dependence alternative, many other works have appeared. For instance, one year later [2] calculated the conditional distribution of the longest success run for a second order Markovian dependence alternative. Later [3] studied the power of the conditional David's test with a parametrization of the transition probabilities. [4] used the total number of runs conditioned on the number of symbols of each type for pattern sequences and calculated critical values for the distribution of the number of runs conditioned on the number of symbols of each type for pattern sequences for randomness tests. [5] proposed a randomness test for the Markovian first order alternative based on the length of the longest run and developed methods of computing the probability of the occurrence of a given success-failure run as a function of the composition of the run, the number n of trials and the probabilities of the possible outcomes at each trial. [6] used the total number of success runs of length greater than k and the total number of success runs of length k as test statistics to test the randomness hypotheses versus three alternatives: First order Markov-dependence, non-systematic unimodal and bimodal clustering and cyclical clustering. By a Monte Carlo study they compared the powers of their tests with the power of two known tests; some based on the total number of success (or failure) runs and others based on the length of the longest success run, and they found that the test based on the number of overlapping success runs of length k is slightly less sensitive than its competitors for success probabilities near to 1. [7] studied a randomness test based on the conditional distribution of the sum of the exact lengths of runs of length greater than k successes. They found that when the type I error must be kept low ($\alpha = 0.01$), their test is more powerful than a test based on the number of runs of exact length k, for the first order Markov-dependence alternative.

Many other researchers have focused their research to calculate explicit expressions for the distributions of runs statistics in many contexts; for example, [5], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18].

We propose a data driven test to identify first order positive Markovian dependence in a Bernoulli sequence, based on a combination of two runs tests: the well known Barton and David's runs test, conditional on the number of ones in the sequence, for the Markovian alternative, and an extension of this test to an unconditional test (on the number of ones). We give analytic expressions for the exact distribution of the original and of the extended runs test statistic and for its power; we built an algorithm and we developed the R code to calculated it explicitly. To compare the power of the tests, we calculate the exact powers of both tests for some values of the proportion of ones and the success probability. We found intervals for the success probability for which the unconditional runs test surpasses the power of the conditional test, and we show by calculating the powers, that the data driven test optimizes the power of the two runs tests, when they are considered separately. Finally, as a bonus product, we developed an algorithm and implemented it in R code to solve a polynomial with matrix coefficients, to find explicitly the distribution of the Markov-Binomial Distribution.

2. Two Runs Tests for Markovian Dependence

Let $\eta_1, \ldots, \eta_N$ be a two state Markov chain and let p be the success probability such that:

$$P(\eta_t = 1) = p, \quad P(\eta_t = 0) = 1 - p, \quad 0 < p < 1 \quad \text{for } t = 1, 2, \ldots, N,$$

and stationary transition probabilities ([3]):

$$\begin{aligned} P_{11} &= P(\eta_t = 1 \,|\, \eta_{t-1} = 1) = (1-\theta)p + \theta, \\ P_{10} &= P(\eta_t = 0 \,|\, \eta_{t-1} = 1) = (1-\theta)(1-p), \\ P_{01} &= P(\eta_t = 1 \,|\, \eta_{t-1} = 0) = (1-\theta)p, \\ P_{00} &= P(\eta_t = 0 \,|\, \eta_{t-1} = 0) = 1 - (1-\theta)p, \end{aligned} \tag{2.1}$$

where θ is the coefficient of correlation between η_{t-1} and η_t for $t = 2, 3, \ldots, N$.

Although Barton and David gave the bounds ± 1 for θ, they can be improved as follows:

From (2.1), the following is true:

$$\begin{aligned} 0 \le (1-\theta)p + \theta \le 1 \quad \text{or} \quad 0 \le (1-\theta)(1-p) \le 1 \quad \text{implies} \quad -\frac{p}{1-p} \le \theta \le 1, \\ 0 \le (1-\theta)p \le 1 \quad \text{or} \quad 0 \le 1 - (1-\theta)p \le 1 \quad \text{implies} \quad -\frac{1-p}{p} \le \theta \le 1. \end{aligned} \tag{2.2}$$

Now from (2.2) we conclude:

$$\begin{aligned} \text{for} \quad p = 1/2 \quad \text{it follows that} \quad & -1 \le \theta \le 1, \\ \text{for} \quad 0 < p < 1/2 \quad \text{it follows that} \quad & -\frac{p}{1-p} \le \theta \le 1, \\ \text{and for} \quad 1/2 < p < 1 \quad \text{it follows that} \quad & -\frac{1-p}{p} \le \theta \le 1. \end{aligned} \tag{2.3}$$

The conditions (2.3) on θ are represented graphically in Figure 1.

2.1. The Barton-David Test. From now on, we will consider the following test problem:

$$H_0 : \theta = 0 \quad \text{against} \quad H_1 : \theta > 0 \quad \text{(positive Markovian dependence)}$$

Let m be the fixed number of ones (successes), $n = N - m$ the number of zeros (failures), and let R_m be the total number of runs in $\eta_1, \ldots, \eta_N$. [3] gave a conditional (on m) runs test based on R_m, which rejects H_0 in favor of the positive Markovian alternative for few runs. The critical region was justified as follows: under H_0, $\theta = 0$ holds true and hence $\eta_1, \ldots, \eta_N$ is a Bernoulli sequence of independent and identically distributed (i.i.d.) random variables; under H_1, either $\theta > 0$ (positive dependence), which implies $P_{11} > P_{01}$ or $P_{00} > P_{10}$, and then we expect few runs.

Let $\zeta = \sum_{t=1}^{N} \eta_t$ be a random variable denoting the number of ones in the sequence $\eta_1 \ldots, \eta_N$. [3] gave the following expressions for:

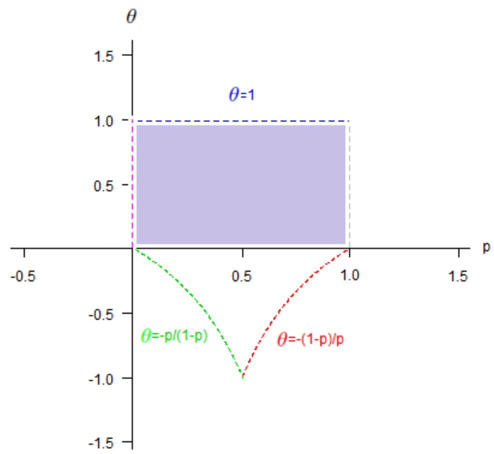

Figure 1. Relation between p and θ

a) the null distribution of R_m

$$(2.4) \qquad P_0(R_m = r \,|\, \zeta = m) = \begin{cases} \dfrac{2\binom{m-1}{\frac{r}{2}-1}\binom{n-1}{\frac{r}{2}-1}}{\binom{N}{m}} & \text{if } r \text{ is even,} \\[2ex] \dfrac{\binom{m-1}{\frac{r-1}{2}}\binom{n-1}{\frac{r-1}{2}-1}+\binom{m-1}{\frac{r-1}{2}-1}\binom{n-1}{\frac{r-1}{2}}}{\binom{N}{m}} & \text{if } r \text{ is odd.} \end{cases}$$

b) the power of the conditional R_m test

$$(2.5) \quad P_\theta(R_m = r \,|\, \zeta = m) = \begin{cases} \frac{2}{S}\frac{1}{1-\theta}\binom{m-1}{\frac{r}{2}-1}\binom{n-1}{\frac{r}{2}-1}\left(\frac{p(1-p)(1-\theta)^2}{(p(1-\theta)+\theta)(1-p(1-\theta))}\right)^{\frac{r}{2}} & \text{if } r \text{ is even,} \\[2ex] \frac{1}{S}\binom{m-1}{\frac{r-1}{2}-1}\binom{n-1}{\frac{r-1}{2}-1}\frac{1}{(p(1-\theta)+\theta)(1-p(1-\theta))}\times \\ \left(-2p(1-p)-\theta(p^2+(1-p)^2)+\frac{Np(1-p(1-\theta))+n\theta(1-2p)}{\frac{r-1}{2}}\right)\times \\ \left(\frac{p(1-p)(1-\theta)^2}{(p(1-\theta)+\theta)(1-p(1-\theta))}\right)^{\frac{r-1}{2}} & \text{if } r \text{ is odd,} \end{cases}$$

where

$$S = \sum_{k=1}^{n}\left[\frac{p(1-p)(1-\theta)^2}{(p(1-\theta)+\theta)(1-p(1-\theta))}\right]^k \binom{m-1}{k-1}\binom{n-1}{k-1}$$
$$\left[\frac{\theta(1-\theta)k+(1-\theta)\left[Np(1-p)+\theta(Np^2+n(1-2p))\right]}{k(1-\theta)(p(1-\theta)+\theta)(1-p(1-\theta))}\right].$$

2.2. A Modified Runs Test. The conditional R_m test in (2.4) can be modified as follows: as we noted above, under H_0, $\eta_1, \ldots, \eta_N$ is an i.i.d. Bernoulli sequence and hence ζ is Binomial distributed with parameters N and $p = P(\eta_t = 1)$, for $t = 1, \ldots, N$. Now let R be the total number of runs, without taking into account the number of ones in the sequence $\eta_1 \ldots, \eta_N$. The modified R test rejects H_0 in favor of the positive Markovian dependence alternative for few runs, with the same arguments as for the R_m test, but now the reject region must be calculated from the unconditional distribution of R, by means of the theorem of total probabilities, as follows:

$$P_0(R = r) = \sum_{m=0}^{N} P_0(R_m = r \mid \zeta = m) \binom{N}{m} p^m (1-p)^{N-m},$$

for $r = 1, \ldots, N$, where the conditional distribution of R_m is calculated as in (2.4).

2.2.1. *The Power of the Modified R Test.* We obtain the distribution of the R, under the Markovian alternative H_1 as follows:

$$P_\theta(R = r) = \sum_{m=0}^{N} P_\theta(R_m = r \mid \zeta = m) P_\theta(\zeta = m), \tag{2.6}$$

for $r = 1, \ldots, N$, where $P_\theta(R_m = r \mid \zeta = m)$ is given in (2.5), and $P_\theta(\zeta = m)$ under H_1 is calculated by means of the probability generating function (pgf) of the Markov-Binomial Distribution ([19]) as a function of the dummy variable s:

$$\begin{aligned} G_N(s) &= \begin{pmatrix} ps & 1-p \end{pmatrix} \begin{pmatrix} ((1-\theta)p+\theta)s & (1-\theta)(1-p) \\ (1-\theta)ps & 1-(1-\theta)p \end{pmatrix}^{N-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} ps & 1-p \end{pmatrix} A^{N-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{2.7}$$

$N = 1, 2, \ldots,$ for $0 \le s \le 1$, where $A = \begin{pmatrix} ((1-\theta)p+\theta)s & (1-\theta)(1-p) \\ (1-\theta)ps & 1-(1-\theta)p \end{pmatrix}$.

2.2.2. *Algorithm to Calculate the Markov-Binomial Distribution Explicitly.* To calculate the power of the R test in (2.6), we need to extract the coefficients of s in the pgf (2.7) of the Markov-Binomial Distribution, which contains the probability distribution of ζ. For this, the following algorithm is useful:

$$\begin{aligned} A^{N-1} &= \begin{pmatrix} ((1-\theta)p+\theta)s & (1-\theta)(1-p) \\ (1-\theta)ps & 1-(1-\theta)p \end{pmatrix}^{N-1} \\ &= \begin{pmatrix} c_1^{11}s + c_2^{11}s^2 + \cdots + c_{N-1}^{11}s^{N-1} & c_0^{12} + c_1^{12}s + \cdots + c_{N-2}^{12}s^{N-2} \\ c_1^{21}s + c_2^{21}s^2 + \cdots + c_{N-1}^{21}s^{N-1} & c_0^{22} + c_1^{22}s + \cdots + c_{N-2}^{22}s^{N-2} \end{pmatrix} \\ &= \begin{pmatrix} c_1^{11} & 0 \\ c_1^{21} & 0 \end{pmatrix} s + \begin{pmatrix} c_2^{11} & 0 \\ c_2^{21} & 0 \end{pmatrix} s^2 + \cdots + \begin{pmatrix} c_{N-1}^{11} & 0 \\ c_{N-1}^{21} & 0 \end{pmatrix} s^{N-1} + \\ &\qquad \begin{pmatrix} 0 & c_0^{12} \\ 0 & c_0^{22} \end{pmatrix} + \begin{pmatrix} 0 & c_1^{12} \\ 0 & c_1^{22} \end{pmatrix} s + \cdots + \begin{pmatrix} 0 & c_{N-2}^{12} \\ 0 & c_{N-2}^{22} \end{pmatrix} s^{N-2} \\ &= \begin{pmatrix} 0 & c_0^{12} \\ 0 & c_0^{22} \end{pmatrix} + \begin{pmatrix} c_1^{11} & c_1^{12} \\ c_1^{21} & c_1^{22} \end{pmatrix} s + \\ &\qquad \cdots + \begin{pmatrix} c_{N-2}^{11} & c_{N-2}^{12} \\ c_{N-2}^{21} & c_{N-2}^{22} \end{pmatrix} s^{N-2} + \begin{pmatrix} c_{N-1}^{11} & 0 \\ c_{N-1}^{21} & 0 \end{pmatrix} s^{N-1} \\ &= C^{(0)} + C^{(1)}s + \cdots + C^{(N-2)}s^{N-2} + C^{(N-1)}s^{N-1} \\ &= \sum_{m=0}^{N-1} C^{(m)} s^m, \end{aligned} \tag{2.8}$$

where $C^{(m)}$ are matrices with the role of coefficients of the polynomial in s.

For example, for $N = 3$ the coefficients of the polynomial can be calculated as follows: let A be the transition matrix whose first column is multiplied by the auxiliary variable s. Then the power of the matrix in (2.7) can be written as:

$$\begin{aligned}
A^2 &= \begin{pmatrix} ((1-\theta)p+\theta)s & (1-\theta)(1-p) \\ (1-\theta)ps & 1-(1-\theta)p \end{pmatrix}^2 \\
&= \begin{pmatrix} c_1^{11}s + c_2^{11}s^2 & c_0^{12} + c_1^{12}s \\ c_1^{21}s + c_2^{21}s^2 & c_0^{22} + c_1^{22}s \end{pmatrix} \\
&= \begin{pmatrix} 0 & c_0^{12} \\ 0 & c_0^{22} \end{pmatrix} + \begin{pmatrix} c_1^{11} & c_1^{12} \\ c_1^{21} & c_1^{22} \end{pmatrix} s + \begin{pmatrix} c_2^{11} & 0 \\ c_2^{21} & 0 \end{pmatrix} s^2 \\
&= C^{(0)} + C^{(1)}s + C^{(2)}s^2,
\end{aligned}$$

where

$$\begin{aligned}
c_0^{12} &= (1-(1-\theta)p)(1-\theta)(1-p) & c_2^{11} &= ((1-\theta)p+\theta)^2 \\
c_0^{22} &= (1-(1-\theta)p)^2 & c_2^{21} &= ((1-\theta)p+\theta)(1-\theta)p \\
c_1^{11} &= (1-\theta)^2(1-p)p & c_1^{12} &= ((1-\theta)p+\theta)(1-\theta)(1-p) \\
c_1^{21} &= (1-(1-\theta)p)(1-\theta)p & c_1^{22} &= (1-\theta)^2(1-p)p
\end{aligned}$$

Using the polynomial expression for the power of the matrix A introduced in (2.8), the pgf of ζ can be expressed as:

$$\begin{aligned}
G_N(s) &= \begin{pmatrix} ps & 1-p \end{pmatrix} \left[\sum_{m=0}^{N-1} C^{(m)} s^m\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \{\begin{pmatrix} p & 0 \end{pmatrix} s + \begin{pmatrix} 0 & 1-p \end{pmatrix}\} \left[\sum_{m=0}^{N-1} C^{(m)} s^m\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} p & 0 \end{pmatrix} \sum_{m=1}^{N} C^{(m-1)} s^m \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1-p \end{pmatrix} \left[\sum_{m=0}^{N-1} C^{(m)} s^m\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} p & 0 \end{pmatrix} \left[\sum_{m=1}^{N-1} C^{(m-1)} s^m + C^{(N-1)} s^N\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \\
&\qquad \begin{pmatrix} 0 & 1-p \end{pmatrix} \left[C^{(0)} + \sum_{m=1}^{N-1} C^{(m)} s^m\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} p & 0 \end{pmatrix} C^{(N-1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} s^N + \\
&\qquad \sum_{m=1}^{N-1} \left[\begin{pmatrix} p & 0 \end{pmatrix} C^{(m-1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1-p \end{pmatrix} C^{(m)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] s^m + \\
&\qquad \begin{pmatrix} 0 & 1-p \end{pmatrix} C^{(0)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned} \tag{2.9}$$

2.2.3. *Algorithm to calculate the coefficients $C^{(m)}$.* To obtain the coefficients $C^{(m)}, m = 0, \ldots, N$ in (2.9) we have to decompose the matrix A as follows:

$$\begin{aligned}
A &= \begin{pmatrix} ((1-\theta)p+\theta)s & (1-\theta)(1-p) \\ (1-\theta)ps & 1-(1-\theta)p \end{pmatrix} = \\
&\qquad \begin{pmatrix} 0 & (1-\theta)(1-p) \\ 0 & 1-(1-\theta)p \end{pmatrix} + \begin{pmatrix} ((1-\theta)p+\theta) & 0 \\ (1-\theta)p & 0 \end{pmatrix} s = U + Vs.
\end{aligned}$$

All summands of the $(N-1)$-th power of A can be generated by iterating the following binomial expression:

$$
\begin{aligned}
A^{N-1} &= (U+Vs)^{N-1} \\
&= \underbrace{(UU\dots U)}_{\binom{N-1}{0}\text{ summands}} + \underbrace{(UU\dots UV + UU\dots VU + \dots + VU\dots UU)}_{\binom{N-1}{1}\text{ summands}} s + \\
&\quad \underbrace{(UUU\dots UVV + UU\dots VUV + \dots + VVU\dots UUU)}_{\binom{N-1}{2}\text{ summands}} s^2 + \dots + \\
&\quad \underbrace{(UV\dots VV + VU\dots VV + \dots + VV\dots VU)}_{\binom{N-1}{N-2}\text{ summands}} s^{N-2} + \underbrace{(VV\dots V)}_{\binom{N-1}{N-1}\text{ summands}} s^{N-1}.
\end{aligned}
\tag{2.10}
$$

Comparing the coefficients in (2.8) and (2.10) we obtain:

$$
\begin{aligned}
C^{(0)} &= \underbrace{UU\dots U}_{\binom{N-1}{0}\text{ summands}} \\
C^{(1)} &= \underbrace{UU\dots UV + UU\dots VU + \dots + VU\dots UU}_{\binom{N-1}{1}\text{ summands}} \\
C^{(2)} &= \underbrace{UUU\dots UVV + UU\dots VUV + \dots + VVU\dots UUU}_{\binom{N-1}{2}\text{ summands}} \\
&\vdots \\
C^{(N-2)} &= \underbrace{VV\dots VU + VV\dots UV + \dots + UV\dots VV}_{\binom{N-1}{N-2}\text{ summands}} \\
C^{(N-1)} &= \underbrace{VV\dots V}_{\binom{N-1}{N-1}\text{ summands}}
\end{aligned}
$$

Note that m in $C^{(m)}$ corresponds to the number of times that the matrix V is in the products and it is also the number of ones in the sample. In order to generate all summands, we can iterate over all binary numbers with N bits:

$$
\begin{pmatrix}
U & U & U & \dots & U & U & U \\
 & & & & & & \\
U & U & U & \dots & U & U & V \\
U & U & U & \dots & U & V & U \\
 & & & \vdots & & & \\
V & U & U & \dots & U & U & U \\
 & & & & & & \\
U & U & U & \dots & U & V & V \\
U & U & U & \dots & V & U & V \\
 & & & \vdots & & & \\
V & V & U & \dots & U & U & U \\
 & & & & & & \\
\vdots & & & \vdots & & & \vdots \\
 & & & & & & \\
V & V & V & \dots & V & V & U \\
V & V & V & \dots & V & U & V \\
 & & & \vdots & & & \\
U & V & V & \dots & V & V & V \\
 & & & & & & \\
V & V & V & \dots & V & V & V
\end{pmatrix}_{2^N \times (N-1)}
\longmapsto
\begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
 & & & & & & \\
0 & 0 & 0 & \dots & 0 & 0 & 1 \\
0 & 0 & 0 & \dots & 0 & 1 & 0 \\
 & & & \vdots & & & \\
1 & 0 & 0 & \dots & 0 & 0 & 0 \\
 & & & & & & \\
0 & 0 & 0 & \dots & 0 & 1 & 1 \\
0 & 0 & 0 & \dots & 1 & 0 & 1 \\
 & & & \vdots & & & \\
1 & 1 & 0 & \dots & 0 & 0 & 0 \\
 & & & & & & \\
\vdots & & & \vdots & & & \vdots \\
 & & & & & & \\
1 & 1 & 1 & \dots & 1 & 1 & 0 \\
1 & 1 & 1 & \dots & 1 & 0 & 1 \\
 & & & \vdots & & & \\
0 & 1 & 1 & \dots & 1 & 1 & 1 \\
 & & & & & & \\
1 & 1 & 1 & \dots & 1 & 1 & 1
\end{pmatrix}_{2^N \times (N-1)}
$$

The first row in the second matrix indicates that in $C^{(0)}$ the matrix U must be multiplied $(N-1)$ times. The following $\binom{N-1}{1}$ rows indicate that for $C^{(1)}$ there are $\binom{N-1}{1}$ summands, each one of them containing the product of $(N-2)$ Us and one V, and so on. These iterations are helpful to identify the summands to calculate $C^{(m)}$.

2.2.4. *Algorithm to Calculate the Power of the R_m Test and of the Modified R Test.* To compare the power of the modified R test with the power of the R_m test, we will calculate it explicitly for some values of θ, p, m and N with the following algorithm:

(1) Calculate the conditional probability distribution of R_m under the alternative as in (2.5), the probability distribution of R under the alternative as in (2.6), and the probability distribution of ζ using (2.9):
   (a) $P(\zeta = N) = \begin{pmatrix} p & 0 \end{pmatrix} C^{(N-1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   (b) $P(\zeta = m) = \left[\begin{pmatrix} p & 0 \end{pmatrix} C^{(m-1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1-p \end{pmatrix} C^{(m)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$, for $m = 1, \dots, N-1$
   (c) $P(\zeta = 0) = \begin{pmatrix} 0 & 1-p \end{pmatrix} C^{(0)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   where the matrix $C^{(m)}$ for $m = 0, \dots, (N-1)$, is calculated with the algorithm 2.2.3.
(2) Calculate the conditional cumulative distributions of R_m and of R under the alternative.
(3) Fix significance level $\alpha = 0.05$ and find critical values c_m and c such that $P_0(R_m \le c_m \mid \zeta = m) \le \alpha$, and $P_0(R \le c) \le \alpha$ respectively.
(4) Randomize the R_m and R tests such that for $0 < \gamma < 1$ and $0 < \gamma' < 1$

$$\alpha = P_0(R_m \le c_m \mid \zeta = m) + \gamma\, P_0(c_m < R_m \le c_m + 1 \mid \zeta = m)$$

and

$\alpha = P_0(R \leq c) + \gamma' P_0(c < R \leq c+1)$

(5) Calculate the power of the randomized R_m and R tests as follows:

$\pi_{R_m}(\theta) = P_\theta(R_m \leq c_m \,|\, \zeta = m) + \gamma\, P_\theta(c_m < R_m \leq c_m + 1 \,|\, \zeta = m).$

and

$\pi_R(\theta) = P_\theta(R \leq c) + \gamma' P_\theta(c < R \leq c+1).$

respectively.

3. A Comparative Power Study and Main Results

We calculated the exact power of the R and R_m tests explicitly, for sample sizes[§] $N = 7(1)22, 30, 40, 50$, for $p = 0.1(0.05)0.9$ and for $\theta = 0(0.05)0.9$. The R_m test was compared with the R test for each value of p, for each N and each m. We have not included the extreme cases $m = 0$ and $m = N$ because for these the power of the R_m test is zero.

We show the main results for $N = 10(10)50$, for $\theta = 0(0.1)0.9$ and for some number of ones in the observed sequence obtained as percents ([N(10%)] and [N(20%)]) of the sample size, to find typical patterns of the powers of the compared tests[¶]. They are in Tables 1 to 15, ordered as follows: Tables 1, 4, 7, 10 and 13 contain the powers of the R test. The other tables are for the R_m test distinguished by the number of ones.

To facilitate the reading of the tables, we built three dimensional graphic illustrations, each containing five graphics denoted by $\pi(p, \theta)$ for each combination of p and θ and the five sample sizes considered. Intersections of the red lines are powers of the R test and the blue ones are powers of the R_m test for fixed values of m.

All figures and all graphics show that the powers of the R test increase with θ as expected, that the powers increase faster for values of p around 0.5 and that the speed of increase is lower when p tends to zero or to one. The power of the R_m test shows small decreases for values of p around 0.5.

In Figure 4, for example, with 10% ones in the observed sequence, for success probabilities p between 0.3 and 0.7, and for sample sizes $N = 10(10)50$, it can be noted that the R test is more powerful than the R_m test. We specially note that for $N = 10$, the same result occurs in a bit larger interval for $0.2 \leq p \leq 0.8$, as can be verified in Tables 1 and 2, with some exceptions for values of θ less than or equal to 0.5, where the power of the R_m test is greater than the power of the R test. The same situation occurs for the powers showed in figure 5 for 20% ones, but now it holds for a smaller interval of values of p: $0.4 \leq p \leq 0.6$.

In general, the interval of values of p for which the power of the R test overtakes the power of the R_m test is smaller when m increases up to 50% ones, the case in which the power of the R_m test overtakes the power of the R test for all values of p. From 50% to

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[§] $N = a(step)b, c, d, \ldots$ means that N goes from a to b, jumping a "step" each time, and after value b, taking values c,d,...

[¶] All these calculations of this section are available on the web page *http://www.docentes.unal.edu.co/jacorzos/docs/*, document: WebPotencia_R_R_m.pdf.

100% ones, the interval of values of p for which the R test overtakes the R_m test increases.

For a fixed percent of ones, it can be observed that the length of the interval of values of p in which the power of the R test overtakes the power of the R_m test tends to be constant, for all compared sample sizes. It can be noted that the R test seems to be more powerful than the R_m test because the larger the correlation between observations, the fewer and larger are the runs.

In Tables 1, 4, 7, 10 and 13 we can see that the power of the R test increases with N around $p = 1/2$. On the other hand, the power of the R_m test increases with N and with m around $N/2$ when N is even, around $(N-1)/2$ and $(N+1)/2$ when N is odd [‖].

Although these types of Markov chains could seem rare, we highlight the conditions under which they can occur. In Figure 2, side (a), we see that P_{11} increases with θ and p, whilst P_{01} decreases with θ and increases with p. Moreover, in part (b) we see that P_{00} increases with θ and decreases with p, whilst P_{10} decreases with both θ and p. We can also see in part (a), that when θ increases it holds true that $P_{11} > P_{01}$, and P_{11} tends to be much larger than P_{01} when θ tends to one, and that implies few runs and large runs of ones. In part (b), the situation is analogous, but with the zeros instead of the ones.

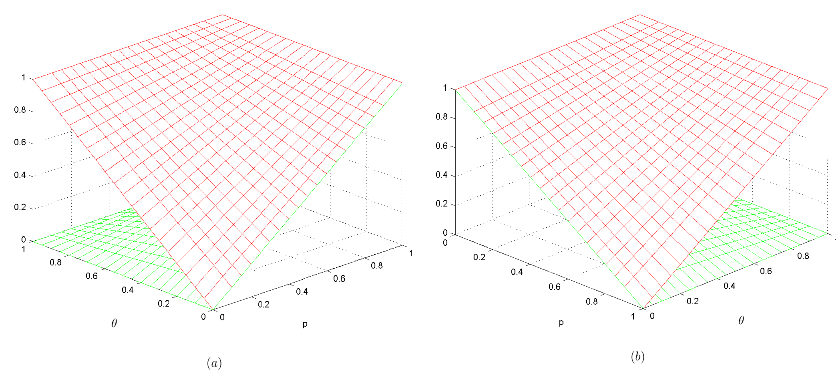

Figure 2. Positive association: (a) Transition Probabilities P_{11} (red) and P_{01}(green), (b) Transition Probabilities P_{00} (red) and P_{10}(green)

A Data Driven Test. As we said, the power of the R test is greater than the power of the R_m test for some intervals of values of the success probability p and for some values of m. This suggests the following selection process of the appropriate test for a fixed level $\alpha = 0.05$:

(1) Calculate the number of ones m in the observed sequence.
(2) Estimate the success probability p, by means of the [20] estimator[**] $\hat{p} = \frac{1}{N}\sum_{i=1}^{N} x_i$.
(3) Choose the data driven test as follows (see Figure 3):

[‖]See list of tables justifying this comments in our web page, in the cited document.

[**]Although $\hat{p}$ is the old well known plug-in estimator of p given by [21], we reference [20] because he shows, among other things, that the maximum likelihood estimator for $(\hat{p}, \hat{\theta})$ obtained under Markovian dependence, is strongly consistent for (p, θ).

Select the R test if:

- $\hat{p} \in [0.25, 0.75]$ and $m \in (0\%N, 13\%N) \cup (87\%N, 100\%N)$
- $\hat{p} \in [0.30, 0.70]$ and $m \in [13\%N, 17\%N) \cup (83\%N, 87\%N]$
- $\hat{p} \in [0.35, 0.65]$ and $m \in [17\%N, 23\%N) \cup (77\%N, 83\%N]$
- $\hat{p} \in [0.40, 0.60]$ and $m \in [23\%N, 33\%N) \cup (67\%N, 77\%N]$
- $\hat{p} \in [0.45, 0.55]$ and $m \in [33\%N, 43\%N) \cup (57\%N, 67\%N]$
- $\hat{p} = 0.50$ and $m \in [43\%N, 57\%N]$

Select the R_m test otherwise.

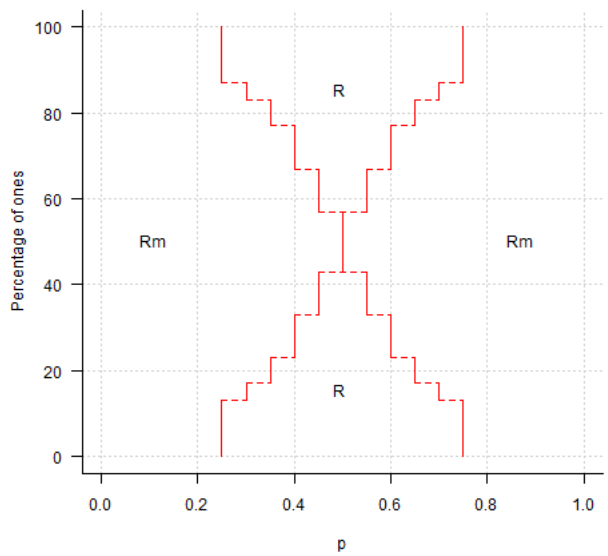

Figure 3. Regions to choose between R and R_m Tests

4. Example

A first order homogeneous Markov Chain (fohMC) can be used to describe the behavior of a buyer as follows[††]: a buyer at a supermarket A switches to buying in a supermarket B on her/his next shopping trip with probability $\lambda > 0$, while she/he switches to supermarket A with probability $\beta > 0$ when her/his last shopping was in supermarket B.

To verify the Markovian assumption of the behavior of the buyer, we simulated first order homogeneous Markov Chains using the Metropolis Hasting Algorithm, for some values of the success probability p and the correlation between successive observations θ. These combinations of values p and θ give values of the transition probabilities λ, β which indicate how Markovian the behavior of the buyer is.

[††]Adapted from [22]

For each combination of (p,θ), $p, \theta = 0.1(0.1)0.9$, we simulate 1000 fohMC and we select those with not all zeros and not all ones, because such stable buyers are not interesting. For the remaining fohMC, we calculate the proportion of rejections of the null hypothesis, and the ratio $(1-\lambda)/\beta$ to find the conditions for which the fohMC is a good assumption for the behavior of the consumer.

The results are in Table 16 for $N = 20$. It can be seen that the empirical powers of the test increase up to 87% for $p = 0.4$ and $\theta = 0.9$. It can be noted also that the ratio $p_{00}/p_{10} = (1-\lambda)/\beta$ grows also with θ, as expected.

The fact that the larger the values of the ratio $(1-\lambda)/\beta$, the greater the empirical power of the runs test, indicates that the behavior of the consumer tends to be most Markovian when the probability of continuing shopping in supermarket A is larger than the probability of switching to supermarket B.

5. Conclusions and Discussion

We have discovered regions of values of m and p where the R test is more powerful than the R_m test, and we have included additional information about p and m to the test statistic, to produce a data driven test which covers the complete $p \times m$ region, and improves the power of the test.

The power of the R test increases with N and especially around $p = 1/2$, while the power of the R_m test increases with N and with m around $N/2$, for N even or $(N-1)/2$ and $(N+1)/2$ for N odd.

Although the proposed data driven test includes information about the length of the runs without being explicit (few runs implies long runs in most cases), a way to improve the power of the test could be to include the length of the runs explicitly, and to use the results of [15] or of [23] about the distribution of the longest runs test.

Acknowledgements: We are thankful to the statistician Gustavo Romero for developing the R code for the algorithm to obtain the coefficients of the pgf.

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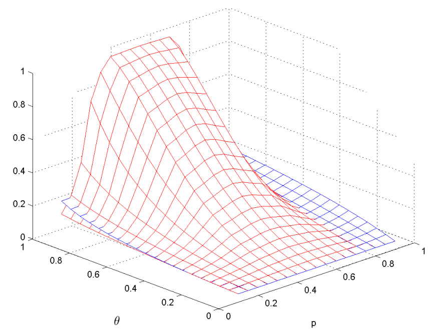
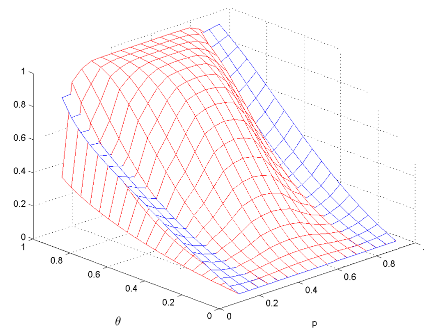
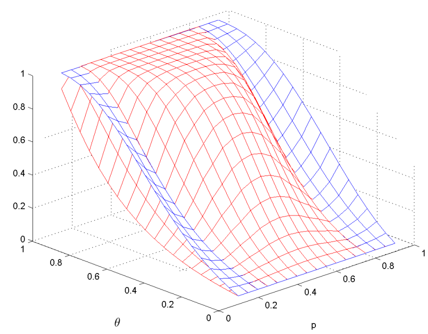
Appendix A. Figures of the Power of the Proposed Test



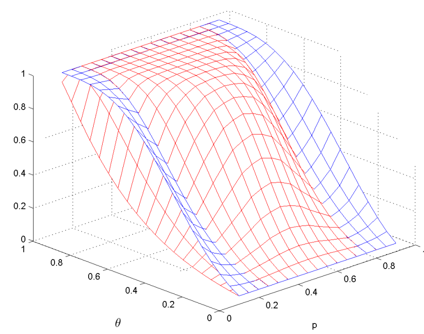
N=10 and m=1



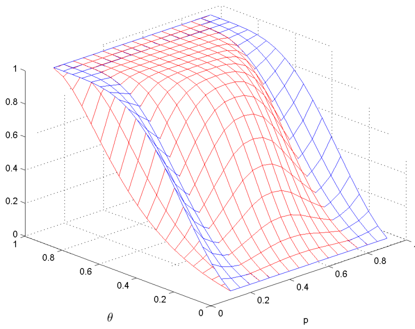
N=20 and m=2



N=30 and m=3

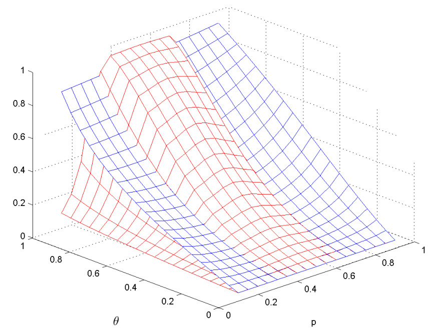


N=40 and m=4

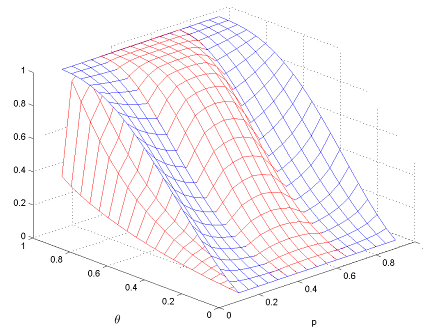


N=50 and m=5

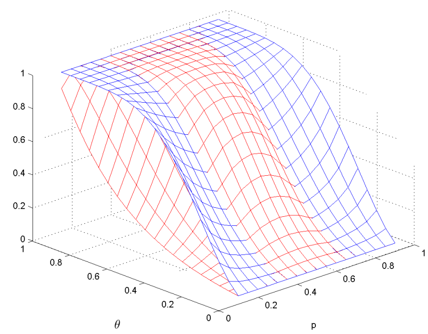
Figure 4. $\pi_R(p,\theta)$ (red) vs. $\pi_{R_m}(p,\theta)$ (blue) for $N = 10, 20, 30, 40, 50$ with 10% ones



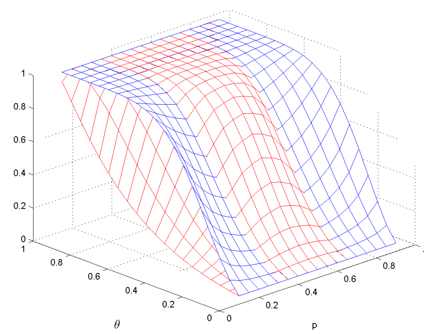
N=10 and m=2



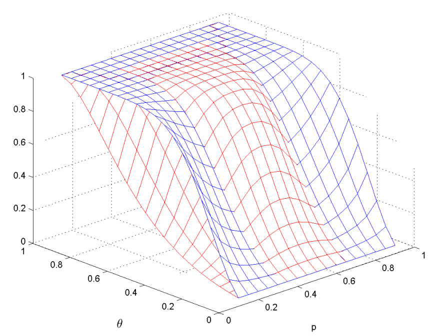
N=20 and m=4



N=30 and m=6



N=40 and m=8



N=50 and m=10

Figure 5. $\pi_R(p,\theta)$ (red) vs. $\pi_{R_m}(p,\theta)$ (blue) for $N = 10, 20, 30, 40, 50$ with 20% ones

Appendix B. Tables of the Power of the Proposed Test

Table 1. Power of the R Test, N=10

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0552	0.0624	0.0729	0.0815	0.0866	0.0815	0.0729	0.0624	0.0552
0.20	0.0609	0.0776	0.1047	0.1274	0.1404	0.1274	0.1047	0.0776	0.0609
0.30	0.0672	0.0959	0.1480	0.1918	0.2148	0.1918	0.1480	0.0959	0.0672
0.40	0.0740	0.1182	0.2059	0.2780	0.3116	0.2780	0.2059	0.1182	0.0740
0.50	0.0814	0.1453	0.2821	0.3879	0.4305	0.3879	0.2821	0.1453	0.0814
0.60	0.0896	0.1788	0.3797	0.5205	0.5671	0.5205	0.3797	0.1788	0.0896
0.70	0.0990	0.2213	0.5016	0.6692	0.7120	0.6692	0.5016	0.2213	0.0990
0.80	0.1100	0.2774	0.6485	0.8196	0.8495	0.8196	0.6485	0.2774	0.1100
0.90	0.1240	0.3546	0.8176	0.9447	0.9560	0.9447	0.8176	0.3546	0.1240

Table 2. Power of the R_m Test, N=10 and m=1

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0548	0.0554	0.0562	0.0571	0.0585	0.0605	0.0638	0.0700	0.0864
0.20	0.0605	0.0618	0.0633	0.0654	0.0682	0.0722	0.0786	0.0900	0.1167
0.30	0.0674	0.0694	0.0718	0.0750	0.0793	0.0853	0.0944	0.1100	0.1423
0.40	0.0758	0.0786	0.0820	0.0864	0.0921	0.1000	0.1115	0.1300	0.1643
0.50	0.0864	0.0900	0.0944	0.1000	0.1071	0.1167	0.1300	0.1500	0.1833
0.60	0.1000	0.1045	0.1100	0.1167	0.1250	0.1357	0.1500	0.1700	0.2000
0.70	0.1183	0.1237	0.1300	0.1375	0.1466	0.1577	0.1717	0.1900	0.2147
0.80	0.1441	0.1500	0.1567	0.1643	0.1731	0.1833	0.1955	0.2100	0.2278
0.90	0.1833	0.1885	0.1940	0.2000	0.2065	0.2136	0.2214	0.2300	0.2395

Table 3. Power of the R_m Test, N=10 and m=2

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0983	0.0816	0.0763	0.0745	0.0749	0.0773	0.0827	0.0947	0.1307
0.20	0.1496	0.1216	0.1114	0.1083	0.1096	0.1153	0.1276	0.1540	0.2255
0.30	0.2042	0.1707	0.1573	0.1535	0.1564	0.1664	0.1869	0.2281	0.3271
0.40	0.2628	0.2298	0.2158	0.2127	0.2183	0.2334	0.2623	0.3165	0.4316
0.50	0.3272	0.3001	0.2887	0.2882	0.2976	0.3182	0.3547	0.4178	0.5361
0.60	0.4008	0.3834	0.3779	0.3822	0.3964	0.4222	0.4637	0.5296	0.6386
0.70	0.4891	0.4836	0.4862	0.4967	0.5158	0.5450	0.5875	0.6484	0.7374
0.80	0.6020	0.6078	0.6184	0.6343	0.6561	0.6848	0.7219	0.7696	0.8312
0.90	0.7586	0.7701	0.7837	0.7994	0.8174	0.8380	0.8616	0.8884	0.9190

Table 4. Power of the R Test, N=20

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0617	0.0736	0.0871	0.1091	0.1091	0.1049	0.0871	0.0736	0.0617
0.20	0.0759	0.1068	0.1451	0.2086	0.2086	0.1971	0.1451	0.1068	0.0759
0.30	0.0932	0.1528	0.2305	0.3525	0.3525	0.3323	0.2305	0.1528	0.0932
0.40	0.1142	0.2155	0.3482	0.5299	0.5299	0.5034	0.3482	0.2155	0.1142
0.50	0.1397	0.2994	0.4971	0.7126	0.7126	0.6861	0.4971	0.2994	0.1397
0.60	0.1704	0.4097	0.6650	0.8635	0.8635	0.8440	0.6650	0.4097	0.1704
0.70	0.2076	0.5504	0.8255	0.9565	0.9565	0.9470	0.8255	0.5504	0.2076
0.80	0.2531	0.7205	0.9428	0.9933	0.9933	0.9909	0.9428	0.7205	0.2531
0.90	0.3126	0.8988	0.9940	0.9998	0.9998	0.9997	0.9940	0.8988	0.3126

Table 5. Power of the R_m Test, N=20 and m=2

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1025	0.0815	0.0747	0.0715	0.0717	0.0725	0.0759	0.0840	0.1092
0.20	0.1631	0.1223	0.1079	0.1010	0.1017	0.1035	0.1112	0.1292	0.1818
0.30	0.2302	0.1739	0.1520	0.1412	0.1426	0.1457	0.1587	0.1880	0.2674
0.40	0.3017	0.2372	0.2094	0.1954	0.1979	0.2027	0.2217	0.2627	0.3649
0.50	0.3750	0.3121	0.2819	0.2668	0.2712	0.2780	0.3033	0.3552	0.4723
0.60	0.4489	0.3973	0.3703	0.3582	0.3659	0.3748	0.4062	0.4658	0.5862
0.70	0.5252	0.4922	0.4747	0.4713	0.4838	0.4951	0.5309	0.5925	0.7018
0.80	0.6119	0.6004	0.5971	0.6073	0.6254	0.6385	0.6750	0.7299	0.8133
0.90	0.7351	0.7414	0.7511	0.7728	0.7929	0.8049	0.8334	0.8694	0.9146

Table 6. Power of the R_m Test, N=20 and m=4

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1496	0.1073	0.0940	0.0879	0.0883	0.0898	0.0961	0.1116	0.1614
0.20	0.2784	0.1912	0.1608	0.1463	0.1475	0.1511	0.1667	0.2034	0.3089
0.30	0.4211	0.3014	0.2544	0.2309	0.2333	0.2396	0.2659	0.3244	0.4702
0.40	0.5656	0.4338	0.3754	0.3449	0.3489	0.3579	0.3938	0.4679	0.6265
0.50	0.7011	0.5794	0.5191	0.4866	0.4922	0.5030	0.5439	0.6213	0.7631
0.60	0.8178	0.7244	0.6735	0.6456	0.6522	0.6629	0.7010	0.7668	0.8700
0.70	0.9078	0.8516	0.8185	0.8007	0.8068	0.8151	0.8425	0.8852	0.9426
0.80	0.9665	0.9437	0.9297	0.9229	0.9267	0.9310	0.9439	0.9619	0.9827
0.90	0.9945	0.9908	0.9885	0.9878	0.9888	0.9897	0.9920	0.9949	0.9979

Table 7. Power of the R Test, N=30

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0689	0.0826	0.1025	0.1294	0.1294	0.1229	0.1025	0.0826	0.0689
0.20	0.0946	0.1322	0.1922	0.2724	0.2724	0.2537	0.1922	0.1322	0.0946
0.30	0.1294	0.2048	0.3280	0.4739	0.4739	0.4425	0.3280	0.2048	0.1294
0.40	0.1765	0.3058	0.5055	0.6918	0.6918	0.6564	0.5055	0.3058	0.1765
0.50	0.2398	0.4380	0.6985	0.8661	0.8661	0.8395	0.6985	0.4380	0.2398
0.60	0.3249	0.5972	0.8625	0.9623	0.9623	0.9500	0.8625	0.5972	0.3249
0.70	0.4388	0.7666	0.9610	0.9946	0.9946	0.9917	0.9610	0.7666	0.4388
0.80	0.5907	0.9114	0.9954	0.9998	0.9998	0.9996	0.9954	0.9114	0.5907
0.90	0.7892	0.9889	0.9999	1.0000	1.0000	1.0000	0.9999	0.9889	0.7892

Table 8. Power of the R_m Test, N=30 and m=3

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1127	0.0878	0.0802	0.0772	0.0782	0.0796	0.0851	0.0979	0.1389
0.20	0.1959	0.1420	0.1245	0.1177	0.1204	0.1239	0.1375	0.1685	0.2597
0.30	0.3011	0.2174	0.1884	0.1771	0.1822	0.1886	0.2127	0.2649	0.4016
0.40	0.4271	0.3185	0.2779	0.2619	0.2701	0.2799	0.3155	0.3877	0.5518
0.50	0.5677	0.4474	0.3980	0.3784	0.3897	0.4027	0.4477	0.5322	0.6964
0.60	0.7110	0.5994	0.5485	0.5283	0.5415	0.5561	0.6040	0.6861	0.8215
0.70	0.8395	0.7587	0.7176	0.7013	0.7138	0.7267	0.7670	0.8292	0.9156
0.80	0.9356	0.8963	0.8746	0.8665	0.8747	0.8825	0.9052	0.9365	0.9728
0.90	0.9881	0.9802	0.9757	0.9746	0.9770	0.9790	0.9842	0.9904	0.9964

Table 9. Power of the R_m Test, N=30 and m=6

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.2050	0.1341	0.1128	0.1028	0.1029	0.1047	0.1132	0.1348	0.2057
0.20	0.4221	0.2709	0.2181	0.1918	0.1919	0.1965	0.2182	0.2705	0.4183
0.30	0.6386	0.4498	0.3688	0.3247	0.3245	0.3321	0.3673	0.4461	0.6295
0.40	0.8099	0.6417	0.5515	0.4966	0.4958	0.5053	0.5476	0.6345	0.7992
0.50	0.9193	0.8091	0.7349	0.6837	0.6824	0.6912	0.7294	0.8008	0.9112
0.60	0.9744	0.9235	0.8806	0.8468	0.8455	0.8512	0.8758	0.9175	0.9703
0.70	0.9947	0.9801	0.9650	0.9511	0.9504	0.9527	0.9625	0.9777	0.9935
0.80	0.9995	0.9976	0.9952	0.9927	0.9925	0.9930	0.9947	0.9972	0.9993
0.90	1.0000	1.0000	0.9999	0.9998	0.9998	0.9998	0.9999	0.9999	1.0000

Table 10. Power of the R Test, N=40

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0715	0.0910	0.1138	0.1403	0.1531	0.1403	0.1138	0.0910	0.0715
0.20	0.1013	0.1576	0.2285	0.3099	0.3464	0.3099	0.2285	0.1576	0.1013
0.30	0.1421	0.2579	0.4014	0.5442	0.5986	0.5442	0.4014	0.2579	0.1421
0.40	0.1975	0.3965	0.6126	0.7743	0.8221	0.7743	0.6126	0.3965	0.1975
0.50	0.2716	0.5674	0.8100	0.9259	0.9499	0.9259	0.8100	0.5674	0.2716
0.60	0.3690	0.7476	0.9394	0.9866	0.9926	0.9866	0.9394	0.7476	0.3690
0.70	0.4947	0.8971	0.9904	0.9990	0.9996	0.9990	0.9904	0.8971	0.4947
0.80	0.6532	0.9793	0.9996	1.0000	1.0000	1.0000	0.9996	0.9793	0.6532
0.90	0.8446	0.9993	1.0000	1.0000	1.0000	1.0000	1.0000	0.9993	0.8446

Table 11. Power of the R_m Test, N=40 and m=4

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1372	0.1003	0.0893	0.0854	0.0851	0.0878	0.0947	0.1111	0.1637
0.20	0.2647	0.1787	0.1514	0.1414	0.1404	0.1471	0.1644	0.2040	0.3185
0.30	0.4234	0.2912	0.2441	0.2261	0.2241	0.2357	0.2654	0.3300	0.4907
0.40	0.5949	0.4375	0.3727	0.3464	0.3429	0.3592	0.4003	0.4831	0.6574
0.50	0.7548	0.6064	0.5344	0.5026	0.4978	0.5165	0.5625	0.6479	0.7988
0.60	0.8798	0.7728	0.7108	0.6807	0.6753	0.6920	0.7323	0.8007	0.9024
0.70	0.9572	0.9039	0.8667	0.8466	0.8422	0.8526	0.8774	0.9160	0.9646
0.80	0.9912	0.9767	0.9646	0.9572	0.9552	0.9587	0.9671	0.9792	0.9924
0.90	0.9995	0.9984	0.9974	0.9967	0.9965	0.9968	0.9975	0.9985	0.9995

Table 12. Power of the R_m Test, N=40 and m=8

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.2286	0.1478	0.1239	0.1149	0.1131	0.1172	0.1293	0.1592	0.2579
0.20	0.4804	0.3124	0.2530	0.2296	0.2250	0.2359	0.2675	0.3407	0.5350
0.30	0.7150	0.5226	0.4364	0.3993	0.3918	0.4098	0.4594	0.5615	0.7663
0.40	0.8769	0.7300	0.6450	0.6041	0.5957	0.6163	0.6696	0.7651	0.9079
0.50	0.9606	0.8847	0.8278	0.7969	0.7903	0.8066	0.8457	0.9057	0.9731
0.60	0.9916	0.9670	0.9433	0.9287	0.9254	0.9335	0.9513	0.9748	0.9947
0.70	0.9990	0.9949	0.9899	0.9863	0.9855	0.9876	0.9917	0.9964	0.9994
0.80	1.0000	0.9997	0.9994	0.9991	0.9991	0.9992	0.9995	0.9998	1.0000
0.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 13. Power of the R Test, N=50

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.0766	0.0992	0.1249	0.1678	0.1678	0.1565	0.1249	0.0992	0.0766
0.20	0.1156	0.1831	0.2645	0.3936	0.3936	0.3618	0.2645	0.1831	0.1156
0.30	0.1716	0.3118	0.4708	0.6719	0.6719	0.6292	0.4708	0.3118	0.1716
0.40	0.2499	0.4846	0.7018	0.8841	0.8841	0.8539	0.7018	0.4846	0.2499
0.50	0.3559	0.6800	0.8839	0.9773	0.9773	0.9668	0.8839	0.6800	0.3559
0.60	0.4926	0.8535	0.9743	0.9981	0.9981	0.9966	0.9743	0.8535	0.4926
0.70	0.6568	0.9603	0.9977	1.0000	1.0000	0.9999	0.9977	0.9603	0.6568
0.80	0.8304	0.9961	1.0000	1.0000	1.0000	1.0000	1.0000	0.9961	0.8304
0.90	0.9659	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9659

Table 14. Power of the R_m Test, N=50 and m=5

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.1703	0.1159	0.0995	0.0918	0.0919	0.0933	0.1000	0.1167	0.1715
0.20	0.3459	0.2221	0.1808	0.1606	0.1607	0.1643	0.1813	0.2225	0.3438
0.30	0.5451	0.3706	0.3020	0.2660	0.2659	0.2722	0.3013	0.3682	0.5366
0.40	0.7298	0.5494	0.4629	0.4131	0.4125	0.4210	0.4598	0.5426	0.7166
0.50	0.8697	0.7303	0.6472	0.5934	0.5921	0.6011	0.6413	0.7202	0.8569
0.60	0.9530	0.8766	0.8192	0.7767	0.7750	0.7820	0.8125	0.8672	0.9449
0.70	0.9891	0.9632	0.9386	0.9173	0.9161	0.9195	0.9342	0.9581	0.9861
0.80	0.9988	0.9949	0.9902	0.9854	0.9850	0.9858	0.9890	0.9938	0.9983
0.90	1.0000	0.9999	0.9997	0.9996	0.9996	0.9996	0.9997	0.9998	1.0000

Table 15. Power of the R_m Test, N=50 and m=10

θ	p=0.10	p =0.20	p=0.30	p=0.40	p=0.50	p=0.60	p=0.70	p=0.80	p=0.90
0.00	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
0.10	0.2674	0.1660	0.1357	0.1241	0.1210	0.1246	0.1371	0.1690	0.2761
0.20	0.5604	0.3630	0.2890	0.2584	0.2503	0.2602	0.2933	0.3717	0.5787
0.30	0.7963	0.5992	0.4998	0.4536	0.4410	0.4569	0.5071	0.6121	0.8142
0.40	0.9293	0.8048	0.7195	0.6740	0.6610	0.6781	0.7279	0.8169	0.9397
0.50	0.9828	0.9326	0.8862	0.8575	0.8490	0.8608	0.8923	0.9396	0.9865
0.60	0.9974	0.9857	0.9713	0.9609	0.9577	0.9624	0.9738	0.9880	0.9982
0.70	0.9998	0.9986	0.9966	0.9949	0.9944	0.9952	0.9970	0.9989	0.9999
0.80	1.0000	1.0000	0.9999	0.9998	0.9998	0.9998	0.9999	1.0000	1.0000
0.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Appendix C. Results of the Example

Table 16. Estimated power π (first entry for each value of p) and Ratio $(1-\lambda)/\beta$ second entry for the same value of p), for the simulate Markov Chains

		θ								
	$N=20$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
p	0.1	0.04	0.08	0.19	0.25	0.38	0.49	0.61	0.73	0.78
		1.12	1.28	1.48	1.74	2.11	2.67	3.59	5.44	11
	0.2	0.05	0.11	0.19	0.30	0.42	0.55	0.63	0.77	0.84
		1.14	1.31	1.54	1.83	2.25	2.88	3.92	6	12.25
	0.3	0.06	0.12	0.21	0.35	0.47	0.61	0.74	0.78	0.85
		1.16	1.36	1.61	1.95	2.43	3.14	4.33	6.71	13.86
	0.4	0.06	0.14	0.20	0.39	0.51	0.67	0.75	0.82	0.87
		1.19	1.42	1.71	2.11	2.67	3.5	4.89	7.67	16
	0.5	0.06	0.12	0.24	0.37	0.50	0.67	0.75	0.83	0.84
		1.22	1.5	1.86	2.33	3	4	5.67	9	19
	0.6	0.08	0.13	0.21	0.35	0.52	0.68	0.76	0.82	0.85
		1.28	1.63	2.07	2.67	3.5	4.75	6.83	11	23.5
	0.7	0.06	0.13	0.23	0.32	0.51	0.58	0.73	0.78	0.83
		1.37	1.83	2.43	3.22	4.33	6	8.78	14.33	31
	0.8	0.04	0.11	0.21	0.30	0.43	0.58	0.66	0.76	0.81
		1.56	2.25	3.14	4.33	6	8.5	12.67	21	46
	0.9	0.05	0.08	0.16	0.23	0.35	0.51	0.58	0.69	0.81
		2.11	3.5	5.29	7.67	11	16	24.33	41	91