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Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series

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Abstract

The purpose of the present paper is to investigate some characterization for Poisson distribution series to be in the new subclasses $\mathcal{G}(\lambda, \alpha)$ and $\mathcal{K}(\lambda, \alpha)$ of analytic functions.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ [19] the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots. \tag{1.2}$$

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Also, for functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \le \alpha < 1)$ may be defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}.$$

The class $\mathcal{S}^*(\alpha)$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \le \alpha < 1)$

$$\begin{aligned} \mathcal{K}(\alpha) &= \left\{ f \in \mathcal{A} : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\} \\ &= \left\{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*(\alpha) \right\} \end{aligned}$$

were introduced by Robertson in [17]. We also write $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Further, $\mathcal{K}(0) = \mathcal{K}$ is the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^{\tau}(A, B)$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1.$$

where $z \in \mathbb{U}, \tau \in \mathbb{C}\backslash\{0\}$, $-1 \le B < A \le 1$. The class $\Re^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [6]. If we put

$$\tau = 1, \ A = \alpha \text{ and } \ B = -\alpha \quad (0 < \alpha \le 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha \quad (z \in \mathbb{U}; 0 < \alpha \le 1)$$

which was studied by (among others) Padmanabhan [12] and Caplinger and Causey [4].

Very recently, Porwal [13] introduce a power series whose coefficients are probabilities of Poisson distribution

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad (z \in \mathbb{U})$$

and we note that, by ratio test the radius of convergence of above series is infinity. In [13], Porwal also defined the series

$$F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad (z \in \mathbb{U}).$$

Now, we considered the linear operator

$$\mathcal{I}(m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(m)f = K(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n. \tag{1.4}$$

Motivated by results on connections between various subclasses of analytic univalent functions by using generalized Bessel functions [1, 2, 3, 8, 11, 13, 15, 22], hypergeometric functions by Srivastava et al. [20] (see [5, 7, 9, 10, 18]) we obtain necessary and sufficient condition for functions $F(m,z)$ in $\mathcal{G}^*(\lambda,\alpha)$ and $\mathcal{K}^*(\lambda,\alpha)$. Further due to the works of Ramesha et al. [16], Padmanabhan [12],we estimate certain inclusion relations between the classes $\mathfrak{R}^\tau(A,B)$, and $\mathcal{G}^*(\lambda,\alpha)$ and $\mathcal{K}^*(\lambda,\alpha)$.

For $0 \le \lambda < 1$ and $0 \le \alpha < 1$, we let $\mathcal{G}(\lambda,\alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{zf'(z)+\lambda z^2 f''(z)}{f(z)}\right) > \alpha, \ (z \in \mathbb{U}). \tag{1.5}$$

and also let $\mathcal{K}(\lambda,\alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{z[zf'(z)+\lambda z^2 f''(z)]'}{zf'(z)}\right) > \alpha, \ (z \in \mathbb{U}). \tag{1.6}$$

Also denote $\mathcal{G}^*(\lambda,\alpha) = \mathcal{G}(\lambda,\alpha) \cap \mathcal{T}$ and $\mathcal{K}^*(\lambda,\alpha) = \mathcal{K}(\lambda,\alpha) \cap \mathcal{T}$.

1.1. Remark. It is of interest to note that for $\lambda = 0$, we have $\mathcal{G}(\lambda,\alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\lambda,\alpha) \equiv \mathcal{K}(\alpha)$

To prove the main results, we need the following Lemmas.

1.2. Lemma. [21] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha.$$

1.3. Lemma. [21] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\lambda,\alpha)$ if*

$$\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha.$$

Further we can easily prove that the conditions are also necessary if $f \in \mathcal{T}$.

1.4. Lemma. *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{G}^*(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha.$$

1.5. Lemma. *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{K}^*(\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\alpha)|a_n| \le 1-\alpha.$$

2. Main Results

2.1. Theorem. *If $m > 0$, then $F(m,z)$ is in $\mathcal{G}^*(\lambda,\alpha)$ if and only if*

$$e^m\left[\lambda m^2 + (1+2\lambda)m\right] \le 1-\alpha. \tag{2.1}$$

Proof. Since $F(m,z) = z - \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n$ and by virtue of Lemma 1.4, it suffices to show that

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{m^{n-1}}{(n-1)!}e^{-m} \le 1-\alpha.$$

Let

$$\mathcal{L}_1(m,\lambda,\alpha)=\sum_{n=2}^{\infty}(n^2\lambda+n(1-\lambda)-\alpha)\frac{m^{n-1}}{(n-1)!}e^{-m}$$

Writing $n^2=(n-1)(n-2)+3(n-1)+1$ and $n=(n-1)+1$, and by simple computation, we get

$$\begin{aligned}\mathcal{L}_1(m,\lambda,\alpha)&=\sum_{n=2}^{\infty}\lambda(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}e^{-m}\\&\quad+(1+2\lambda)\sum_{n=2}^{\infty}(n-1)\frac{m^{n-1}}{(n-1)!}e^{-m}+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\\&=\lambda\sum_{n=3}^{\infty}\frac{m^{n-1}}{(n-3)!}e^{-m}+(1+2\lambda)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}e^{-m}\\&\quad+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\\&=e^{-m}\left[\lambda m^2e^m+(1+2\lambda)me^m+(1-\alpha)(e^m-1)\right]\\&=\lambda m^2+(1+2\lambda)m+(1-\alpha)(1-e^{-m}).\end{aligned}$$

But, this last expression is bounded above by $1-\alpha$ if and only if (2.1) is satisfied. □

2.2. Theorem. *If $m>0$, then $F(m,z)$ is in $\mathcal{K}^*(\lambda,\alpha)$ if and only if*

$$e^m\left[\lambda m^3+(1+5\lambda)m^2+(3+4\lambda-\alpha)m\right]\le 1-\alpha. \tag{2.2}$$

Proof. Since $F(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n$ and by virtue of Lemma 1.5, it suffices to show that

$$\sum_{n=2}^{\infty}(n^3\lambda+n^2(1-\lambda)-n\alpha)\frac{m^{n-1}}{(n-1)!}e^{-m}\le 1-\alpha.$$

Let

$$\mathcal{L}_2(m,\lambda,\alpha)=\sum_{n=2}^{\infty}(n^3\lambda+n^2(1-\lambda)-n\alpha)\frac{m^{n-1}}{(n-1)!}e^{-m}$$

Writing $n^3=(n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1$, $n^2=(n-1)(n-2)+3(n-1)+1$ and $n=(n-1)+1$, we can rewrite the above terms as

$$\begin{aligned}\mathcal{L}_2(m,\lambda,\alpha)&=\lambda\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\frac{m^{n-1}}{(n-1)!}e^{-m}\\&\quad+(1+5\lambda)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}e^{-m}\\&\quad+(3+4\lambda-\alpha)\sum_{n=2}^{\infty}(n-1)\frac{m^{n-1}}{(n-1)!}e^{-m}\\&\quad+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\end{aligned}$$

$$
\begin{aligned}
&= \lambda \sum_{n=4}^{\infty} \frac{m^{n-1}}{(n-4)!} e^{-m} + (1+5\lambda) \sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} e^{-m} \\
&\quad + (3+4\lambda-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} e^{-m} + (1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&= e^{-m} \left[\lambda m^3 e^m + (1+5\lambda) m^2 e^m + (3+4\lambda-\alpha) m e^m \right. \\
&\quad \left. + (1-\alpha)(e^m - 1)\right] \\
&= \lambda m^3 + (1+5\lambda) m^2 + (3+4\lambda-\alpha) m + (1-\alpha)(1-e^{-m}).
\end{aligned}
$$

But, this last expression is bounded above by $1-\alpha$ if and only if (2.2) is satisfied. □

By taking $\lambda = 0$,in Theorem 2.1and 2.2 we state the following corollaries:

2.3. Corollary. *If $m > 0$, then $F(m,z)$ is in $\mathcal{S}^*(\alpha)$ if*

$$me^m \leq 1 - \alpha. \tag{2.3}$$

2.4. Corollary. *If $m > 0$, then $F(m,z)$ is in $\in \mathcal{K}(\alpha)$ if*

$$e^m m(m + 3 - \alpha) \leq 1 - \alpha. \tag{2.4}$$

3. Inclusion Properties

Making use of the following lemma, we will study the action of the Poisson distribution series on the classes $\mathcal{K}(\lambda, \alpha)$.

3.1. Lemma. [6] *A function $f \in \Re^{\tau}(A, B)$ is of form (1.1), then*

$$|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{3.1}$$

The bound given in (3.1) is sharp for

$$f(z) = \int_0^z \left(1 + \frac{(A-B)|\tau| z^{n-1}}{1 + Bz^{n-1}}\right) dz \quad (n \geq 2;\ z \in \mathbb{U})$$

3.2. Theorem. *Let $m > 0$. If $f \in \Re^{\tau}(A, B)$,then $\mathcal{I}(m)f \in \mathcal{K}(\lambda, \alpha)$ if and only if*

$$\frac{(A-B)|\tau| e^m \left[\lambda m^2 + (1+2\lambda) m\right]}{1 - (A-B)|\tau|(1-e^{-m})} \leq 1 - \alpha \tag{3.2}$$

where $\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1$.

Proof. Let f be of the form (1.1) belong to the class $\Re^{\tau}(A, B)$ then by virtue of Lemma 1.5, it suffices to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 1 - \alpha.$$

Let

$$\mathcal{L}_3(m, \lambda, \alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n|.$$

Since $f \in \Re^{\tau}(A, B)$ by Lemma 3.1 we have $|a_n| \leq (A-B)\frac{|\tau|}{n}$, $n \in \mathbb{N}\backslash\{1\}$, hence we get

$$
\begin{aligned}
\mathcal{L}_3(m, \lambda, \alpha) &\leq e^{-m} \sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha) \frac{m^{n-1}}{(n-1)!} (A-B)|\tau| \\
&\leq (A-B)|\tau| e^{-m} \sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha) \frac{m^{n-1}}{(n-1)!}
\end{aligned}
$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and by using the similar arguments as in the proof of Theorem 2.1, we get

$$\mathcal{L}_3(m, \lambda, \alpha) \leq (A - B)|\tau| \left[\lambda m^2 + (1 + 2\lambda)m + (1 - \alpha)(1 - e^{-m})\right].$$

But,the last expression is bounded above by $1 - \alpha$ if and only if (3.2) is satisfied. Hence the proof is completed. □

3.3. Corollary. *Let* $m > 0$ *and* $\lambda = 0$. *If* $f \in \Re^\tau(A, B)$, *then* $\mathcal{I}(m)f \in \mathcal{K}(\alpha)$ *if and only if*

$$(A - B)|\tau| m \left[1 - (A - B)|\tau|(1 - e^{-m}\right]^{-1} \leq 1 - \alpha$$

where $\tau \in \mathbb{C}\backslash\{0\} - 1 \leq B < A \leq 1$.

3.4. Theorem. *Let* $m > 0$, *then*

$$G(m, z) = \int_0^z \frac{F(m, t)}{t})dt$$

is in $\mathcal{K}^*(\lambda, \alpha)$ *if and only if*

$$e^m \left[\lambda m^2 + (1 + 2\lambda)m\right] \leq 1 - \alpha. \tag{3.3}$$

Proof. Since

$$G(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} \frac{z^n}{n} = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{n!} z^n$$

by Lemma 1.5, we need only to show that

$$\sum_{n=2}^{\infty} n(n^2\lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{n!} e^{-m} \leq 1 - \alpha.$$

Now, let

$$\begin{aligned} \mathcal{L}_4(m, \lambda, \alpha) &= \sum_{n=2}^{\infty} n(n^2\lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{n!} e^{-m} \\ &= \sum_{n=2}^{\infty} (n^2\lambda + n(1 - \lambda) - \alpha) \frac{m^{n-1}}{(n-1)!} e^{-m}. \end{aligned}$$

Hence ,writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and by using the similar arguments as in the proof of Theorem 2.1, we have

$$\mathcal{L}_4(m, \lambda, \alpha) \leq \lambda m^2 + (1 + 2\lambda)m + (1 - \alpha)(1 - e^{-m}),$$

which is bounded above by $1 - \alpha$ if and only if (3.3) holds. □

3.5. Theorem. *Let* $m > 0$, *then* $G(m, z) = \int_0^z \frac{F(m,t)}{t})dt$ *is in* $\mathcal{G}^*(\lambda, \alpha)$ *if and only if*

$$m\lambda + \left(1 - \frac{\alpha}{m}\right)\left(1 - e^{-m}\right) + \alpha e^{-m} \leq 1 - \alpha. \tag{3.4}$$

Proof. The proof of theorem is similar to that of Theorem 3.4, hence we omit the details involved. □

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