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Hom-coalgebra cleft extensions and braided tensor Hom-categories of Hom-entwining structures

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Abstract

We investigate how the category of Hom-entwined modules can be made into a monoidal category. The sufficient and necessary conditions making the category of Hom-entwined modules have a braiding are given. Also, we formulate the concept of Hom-cleft extension for a Hom-entwining structure, and prove that if (A, α) is a (C, γ)-cleft extension, then there is an isomorphism of Hom-algebras between (A, α) and a crossed product Hom-algebra of A^{coC} and C.

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1. Introduction

Entwined modules were introduced by Brzeziński and Majid [2, 3], which contained the Long modules, Yetter-Drinfeld modules and Doi-Koppinen modules, etc. So it is very important to study entwined module. As a generalization of entwined modules, Hom-entwined modules were defined by Karacuha [14] as special examples of Hom-corings.

As we know, braided monoidal categories are special categories, whose importance is that the "braiding" structures provide a class of solutions to quantum Yang-Baxter equations. Thus constructing a class of braided monoidal categories is an interesting job. Caenepeel et al. studied how the category of Doi-Hopf modules can be made into a braided monoidal category [5], which have been generalized to entwined modules and Doi-Hom-Hopf modules [13, 17].

The definition of the normal basis for extension associated to a Hopf algebra was introduced by Kreimer and Takeuchi [15]. Using this notion, Doi and Takeuchi [11] characterized H-Galois extensions with normal basis in terms of H-cleft extensions. This result can be extended for Hopf algebras living in symmetric closed categories [12]. A more general formulation in the context of (weak)entwining structures can be found in [1, 3].

The main goal of this paper shall discuss how to make the category of Hom-entwined modules into a monoidal category, and introduce a definition of cleft extension for Hom-entwining structures and with it to obtain a general cleft extension theory. In Section 3, we construct a monoidal category of Hom-etwined modules and give the sufficient and

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necessary conditions making the monoidal category into a braided category. In Section 4, we introduce the notion of (C, γ)-Hom-cleft extension $(A^{coC}, \alpha|_{A^{coC}}) \hookrightarrow (A, \alpha)$, being (A, α) a Hom-algebra, (C, γ) a Hom-coalgebra and A^{coC} a sub-Hom-algebra of A. We prove that if (A, α) is a (C, γ)-Hom-cleft extension, then there is an isomorphism of Hom-algebras between (A, α) and a crossed product Hom-algebra of A^{coC} and C.

2. Preliminaries

Throughout this paper, k will be a field. More knowledge about monoidal Hom-(co)algebra, monoidal Hopf Hom-algebra, Hom-entwined modules, etc. can be found in [4, 6–10, 13, 14, 16, 18–24]. Let $\mathcal{M} = (\mathcal{M}, \otimes, k, a, l, r)$ be the monoidal category of vector spaces over k. We can construct a new monoidal category $\mathcal{H}(\mathcal{M})$ whose objects are ordered pairs (M, μ) with $M \in \mathcal{M}$ and $\mu \in Aut(M)$ and morphisms $f : (M, \mu) \to (N, \nu)$ are morphisms $f : M \to N$ in $\mathcal{M}$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and (k, id_k). All monoidal Hom-structures are objects in the tensor category $\widetilde{\mathcal{H}}(\mathcal{M}) = (\mathcal{H}(\mathcal{M}), \otimes, (k, id_k), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [4] with the associativity and unit constraints given by

$$\widetilde{a}_{M,N,C}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \gamma^{-1}(c)),$$

$$\widetilde{l}(x \otimes m) = \widetilde{r}(m \otimes x) = x\mu(m),$$

for $(M, \mu), (N, \nu)$ and (C, γ). The category $\widetilde{\mathcal{H}}(\mathcal{M})$ is termed Hom-category associated to $\mathcal{M}$.

2.1. Monoidal Hom-algebra

Recall from [4] that a monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $m_A : A \otimes A \to A$, $m_A(a \otimes b) = ab$ and an element $1 \in A$ such that

$$\alpha(ab) = \alpha(a)\alpha(b), \alpha(a)(bc) = (ab)\alpha(c), \tag{2.1}$$

$$\alpha(1) = 1,\ a1 = \alpha(a) = 1a, \tag{2.2}$$

for all $a, b, c \in A$.

A right (A, α)-Hom-module consists of an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\psi : M \otimes A \to M, \psi(m \otimes a) = ma$ satisfying the following conditions:

$$\mu(m)(ab) = (ma)\alpha(b), m1 = \mu(m), \tag{2.3}$$

for all $m \in M$ and $a, b \in A$. For ψ to be a morphism in $\widetilde{\mathcal{H}}(\mathcal{M})$ means

$$\mu(ma) = \mu(m)\alpha(a). \tag{2.4}$$

We call that ψ is a right Hom-action of (A, α) on (M, μ).

Let (M, μ) and (M', μ') be two right (A, α)-Hom-modules. We call a morphism $f : M \to M'$ right (A, α)-linear, if $f \circ \mu = \mu \circ f$ and $f(ma) = f(m)a$. $\mathcal{M}_A$ denotes the category of all right (A, α)-Hom-modules.

2.2. Monoidal Hom-coalgebras

Recall from [4] that a monoidal Hom-coalgebra is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with two linear maps $\Delta_C : C \to C \otimes C$, $\Delta_C(c) = c_1 \otimes c_2$ (summation implicitly understood) and $\varepsilon_C : C \to k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta_C(c_2) = c_{1_1} \otimes (c_{1_2} \otimes \gamma^{-1}(c_2)), \Delta_C(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \tag{2.5}$$

$$\varepsilon_C(\gamma(c)) = \varepsilon_C(c),\ c_1\varepsilon_C(c_2) = \gamma^{-1}(c) = \varepsilon_C(c_1)c_2, \tag{2.6}$$

for all $c \in C$.

A right (C,γ)-Hom-comodule consists of an object $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\rho_M : M \to M \otimes C, \rho_M(m) = m_{[0]} \otimes m_{[1]}$ (summation implicitly understood) satisfying the following conditions:

$$\mu^{-1}(m_{[0]}) \otimes \Delta(m_{[1]}) = m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})), \tag{2.7}$$

$$m_{[0]}\varepsilon_C(m_{[1]}) = \gamma^{-1}(m), \tag{2.8}$$

$$\mu(m)_{[0]} \otimes \mu(m)_{[1]} = \mu(m_{[0]}) \otimes \gamma(m_{[1]}), \tag{2.9}$$

for all $m \in M$. We call that ρ_M is a right Hom-coaction of (A,α) on (M,μ).

Let (M,μ) and (M',μ') be two right (C,γ)-Hom-comodules. We call a morphism $f : M \to M'$ right (A,α)-colinear, if $f \circ \mu = \mu \circ f$ and $f(m)_{[0]} \otimes f(m)_{[1]} = f(m_{[0]}) \otimes m_{[1]}$. $\mathcal{M}^C$ denotes the category of all right (C,γ)-Hom-comodules.

2.3. Monoidal Hom-Hopf algebra

A monoidal Hom-bialgebra $H = (H,\beta,m_H,1,\Delta_H,\varepsilon_H)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M})$. This means that $(H,\beta,m_H,1)$ is a monoidal Hom-algebra and $(H,\beta,\Delta_H,\varepsilon_H)$ is a monoidal Hom-coalgebra such that Δ_H and ε_H are Hom-algebra maps, that is, for any $h,g \in H$,

$$\Delta_H(hg) = \Delta_H(h)\Delta_H(g), \Delta_H(1) = 1 \otimes 1, \tag{2.10}$$

$$\varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g), \varepsilon_H(1) = 1. \tag{2.11}$$

A monoidal Hom-bialgebra (H,β) is called a monoidal Hom-Hopf algebra, if there exists a morphism (called the antipode) $S : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ such that

$$S(h_1)h_2 = \varepsilon_H(h)1 = h_1S(h_2), \tag{2.12}$$

for all $h \in H$.

2.4. Hom-Doi-Koppinen datum

Let (H,β) be a monoidal Hom-bialgebra. Recall from [14] that a right (H,β)-Hom-comodule algebra (A,α) is a monoidal Hom-algebra and a right (H,β)-Hom-comodule with a Hom-coaction ρ_A such that ρ_A is a Hom-algebra morphism, i.e., for any $a,a' \in A$,

$$(aa')_{[0]} \otimes (aa')_{[1]} = a_{[0]}a'_{[0]} \otimes a_{[1]}a'_{[1]}, \tag{2.13}$$

$$\rho_A(1) = 1 \otimes 1, \rho_A \circ \alpha = (\alpha \otimes \beta) \circ \rho_A. \tag{2.14}$$

A right (H,β)-Hom-module coalgebra (C,γ) is a monoidal Hom-coalgebra and a right (H,β)-Hom-module such that, for any $c \in C$ and $h \in H$,

$$(ch)_1 \otimes (ch)_2 = c_1h_1 \otimes c_2h_2, \tag{2.15}$$

$$\varepsilon_C(ch) = \varepsilon_C(c)\varepsilon_H(h), \gamma(ch) = \gamma(c)\beta(h). \tag{2.16}$$

A Hom-Doi-Koppinen datum is a triple $[(H,\beta),(A,\alpha),(C,\gamma)]$, where (H,β) is a monoidal Hom-Hopf algebra, (A,α) a right (H,β)-Hom-comodule algebra and (C,γ) a left (H,β)-Hom-module coalgebra. A Doi-Koppinen Hom-Hopf module (M,μ) is a left (A,α)-Hom-module which is also a right (C,γ)-Hom-comodule with the coaction structure ρ_M such that

$$\rho_M(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]},$$

for all $m \in M$ and $a \in A$.

2.5. Hom-entwining structure

A (right-right) Hom-entwining structure is a $[(A,\alpha),(C,\ \gamma)]_\psi$ consisting of a monoidal Hom-algebra (A,α), a monoidal Hom-coalgebra (C,γ) and a linear map $\psi : C\otimes A \to A\otimes C$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ satisfying the following conditions, for all $a, a' \in A$, $c \in C$,

$$(aa')_\psi \otimes \gamma(c)^\psi = a_\psi a'_\Psi \otimes \gamma(c^{\psi\Psi}), \tag{2.17}$$

$$\alpha^{-1}(a_\psi) \otimes c^\psi{}_1 \otimes c^\psi{}_2 = \alpha^{-1}(a)_{\psi\Psi} \otimes c_1{}^\Psi \otimes c_2{}^\psi, \tag{2.18}$$

$$1_{A\psi} \otimes c^\psi = 1_A \otimes c, \tag{2.19}$$

$$a_\psi \varepsilon_C(c^\psi) = a\varepsilon_C(c). \tag{2.20}$$

Here we use the following notation $\psi(c\otimes a) = a_\psi \otimes c^\psi$ for the so-called entwining map ψ. $\psi \in \widetilde{\mathcal{H}}(\mathcal{M})$ means that the relation

$$\alpha(a)_\psi \otimes \gamma(c)^\psi = \alpha(a_\psi) \otimes \gamma(c^\psi). \tag{2.21}$$

If the map ψ occurs more than once in the same expression, then we use different sub- and superscripts: $\psi, \Psi, \psi_1, \psi_2, \cdots$.

Given a Hom-entwining structure $[(A,\alpha),(C,\gamma)]_\psi$. A right-right $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-module is an object (M,μ) in $\widetilde{\mathcal{H}}(\mathcal{M})$ is a right (A,α)-Hom-module, and a right (C,γ)-Hom-comodule with coaction $\rho_M : M \to M\otimes C$, $m \mapsto m_{[0]}\otimes m_{[1]}$ satisfying the condition, for any $m \in M, a \in A$,

$$\rho_M(ma) = m_{[0]}\alpha^{-1}(a)_\psi \otimes \gamma(m_{[1]}{}^\psi).$$

We use $\widetilde{\mathcal{M}}_A^C(\psi)$ to denote the category of $[(A,\alpha),(C,\gamma)]_\psi$-entwined Hom-modules together with the morphisms in which are both right (A,α)-linear and right (C,γ)-colinear.

3. Braiding on the Hom-category of Hom-entwined modules

Definition 3.1. We call $[(A,\alpha),(C,\gamma)]_\psi$ a momoidal Hom-entwining datum, if $[(A,\alpha),(C,\gamma)]_\psi$ is a Hom-entwining structure and A and C are monoidal Hom-bialgebras with the additional compatibility relations, for all $a\in A$ and $c, c' \in C$,

$$a_{1\psi} \otimes a_{2\Psi} \otimes c^\psi c'^\Psi = \Delta_A(a_\psi) \otimes (cc')^\psi, \tag{3.1}$$

$$\varepsilon_A(a)1_C = \varepsilon_A(a_\psi)1_C^\psi. \tag{3.2}$$

Proposition 3.2. *Let $[(A,\alpha),(C,\gamma)]_\psi$ be a momoidal Hom-entwining structure. Then the tensor product of two Hom-entwined modules (M,μ) and (N,ν) is again a Hom-entwined module $(M\otimes N, \mu\otimes\nu)$ with the structure maps given by*

$$\rho_{M\otimes N}(m\otimes n) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]}n_{[1]}, \tag{3.3}$$

$$(m\otimes n)a = ma_1 \otimes na_2, \tag{3.4}$$

for all $m\in M, n\in N$ and $a\in A$. Thus the category $\widetilde{\mathcal{M}}_A^C(\psi)$ is a Hom-category.

Proof. We show that $(M\otimes N,\mu\otimes\nu)$ is a Hom-entwined module. For all $m\in M, n\in N$ and $a\in A$, we have

$$\begin{aligned}
\rho_{M\otimes N}((m\otimes n)a) &= (ma_1)_{[0]} \otimes (na_2)_{[0]} \otimes (ma_1)_{[1]}(na_2)_{[1]}\\
&= m_{[0]}\alpha^{-1}(a_1)_\psi \otimes n_{[0]}\alpha^{-1}(a_2)_\Psi \otimes \gamma(m_{[1]}{}^\psi)\gamma(n_{[1]}{}^\Psi)\\
&= m_{[0]}\alpha^{-1}(a)_{1\psi} \otimes n_{[0]}\alpha^{-1}(a)_{2\Psi} \otimes \gamma(m_{[1]}{}^\psi n_{[1]}{}^\Psi)\\
&= m_{[0]}\alpha^{-1}(a)_{\psi 1} \otimes n_{[0]}\alpha^{-1}(a)_{\psi 2} \otimes \gamma((m_{[1]}n_{[1]})^\psi)(by(3.1))\\
&= (m_{[0]}\otimes n_{[0]})\alpha^{-1}(a)_\psi \otimes \gamma((m_{[1]}n_{[1]})^\psi).
\end{aligned}$$

Thus $(M\otimes N,\mu\otimes\nu)$ is an object of $\widetilde{\mathcal{M}}_A^C(\psi)$. Let (M,μ), (N,ν) and (W,ς) be Hom-entwined modules. The isomorphisms

$$\begin{aligned}\widetilde{a}_{M,N,W}:(M\otimes N)\otimes W&\to M\otimes(N\otimes W)\\ (m\otimes n)\otimes w&\mapsto\mu(m)\otimes(\nu(n)\otimes\varsigma^{-1}(w)),\end{aligned}$$

$$\widetilde{r}_M:M\otimes k\to M, m\otimes x\mapsto x\mu(m),$$

$$\widetilde{l}_M:k\otimes M\to M, x\otimes m\mapsto x\mu(m),$$

obviously satisfy the pentagon axiom and the triangle axiom. We observe that (k,id) is an object of $\widetilde{\mathcal{M}}_A^C(\psi)$ via the trivial (A,α)-Hom-action and (C,γ)-Hom-coaction given by $xa=\varepsilon_A(a)x$ and $\rho_k=x\otimes 1_C$. It is clear that (k,id) is a unit object of $\widetilde{\mathcal{M}}_A^C(\psi)$. Hence $\widetilde{\mathcal{M}}_A^C(\psi)$ is a Hom-category. □

Let $[(A,\alpha),(C,\gamma)]_\psi$ be a momoidal Hom-entwining datum. We know that a braiding on $\widetilde{\mathcal{M}}_A^C(\psi)$ is a natural family of isomorphisms

$$t_{M,N}:M\otimes N\to N\otimes M$$

in $\widetilde{\mathcal{M}}_A^C(\psi)$ such that, for all (M,μ), (N,ν) and (W,ς),

$$(id_N\otimes t_{M,W})\circ\widetilde{a}_{N,M,W}\circ(t_{M,N}\otimes id_W)\circ\widetilde{a}^{-1}_{M,N,W}=\widetilde{a}_{N,W,M}\circ t_{M,N\otimes W},\tag{3.5}$$

$$\widetilde{a}^{-1}_{P,M,N}\circ t_{M,P}\otimes id_N\circ\widetilde{a}^{-1}_{M,P,N}\circ id_M\otimes t_{N,P}\circ\widetilde{a}_{M,N,P}=t_{M\otimes N,P}.\tag{3.6}$$

Consider a map $Q:C\otimes C\to A\otimes A$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ with twisted convolution inverse R. We use the following notations $Q(c\otimes d)=Q^1(c\otimes d)\otimes Q^2(c\otimes d)$ and $R(c\otimes d)=R^1(c\otimes d)\otimes R^2(c\otimes d)$, for all $c,d\in C$. Thus we have

$$Q^1(c_2\otimes d_2)R^1(c_1\otimes d_1)\otimes Q^2(c_2\otimes d_2)R^2(c_1\otimes d_1)=\varepsilon_C(c)1_A\otimes\varepsilon_C(d)1_A,\tag{3.7}$$

$$R^1(c_2\otimes d_2)Q^1(c_1\otimes d_1)\otimes R^2(c_2\otimes d_2)Q^2(c_1\otimes d_1)=\varepsilon_C(c)1_A\otimes\varepsilon_C(d)1_A.\tag{3.8}$$

Consider two Hom-entwined modules (M,μ) and (N,ν), we define

$$t_{M,N}:M\otimes N\to N\otimes M, m\otimes n\mapsto(n_{[0]}\otimes m_{[0]})Q(n_{[1]}\otimes m_{[1]}),$$

for all $m\in M, n\in N$. It follows from (3.7) and (3.8) that $t_{M,N}$ is bijective.

Example 3.3. Let $[(A,\alpha),(C,\gamma)]_\psi$ a Hom-entwining structure. The $(A\otimes C,\alpha\otimes\gamma)$ can become a Hom-entwined module with the right (A,α)-Hom-action and right (C,γ)-Hom-coaction given by

$$(a\otimes c)b=a\alpha^{-1}(b)\otimes\gamma(c),\tag{3.9}$$

$$\rho_{A\otimes C}(a\otimes c)=(\alpha^{-1}(a)_\psi\otimes c_1)\otimes\gamma(c_2{}^\psi),\tag{3.10}$$

for all $a\in A$ and $c\in C$.

Proof. It is straightforward to check that $(A\otimes C,\alpha\otimes\gamma)$ is a right (A,α)-Hom-module. Here we shall check that $(A\otimes C,\alpha\otimes\gamma)$ is also a right (C,γ)-Hom-comodule. In fact, for $a\in A$ and $c\in C$,

$$\begin{aligned}&(\alpha^{-1}\otimes\gamma^{-1})((a\otimes c)_{[0]})\otimes\Delta_C((a\otimes c)_{[1]})\\ &=\alpha^{-1}(\alpha^{-1}(a)_\psi)\otimes\gamma^{-1}(c_1)\otimes(\gamma(c_2{}^\psi{}_1)\otimes\gamma(c_2{}^\psi{}_2))\\ &=\alpha^{-2}(a)_{\psi\Psi}\otimes\gamma^{-1}(c_1)\otimes(\gamma(c_{2_1}{}^\Psi)\otimes\gamma(c_{2_2}{}^\psi))\\ &=\alpha^{-1}(\alpha^{-1}(a)_{\psi\Psi})\otimes\gamma^{-1}(c_1)\otimes(\gamma(c_{2_1})^\Psi\otimes\gamma(c_{2_2})^\psi)\\ &=\alpha^{-1}(\alpha^{-1}(a)_{\psi\Psi})\otimes c_{1_1}\otimes(\gamma(c_{1_2})^\Psi\otimes c_2{}^\psi)\\ &=\alpha^{-1}(\alpha^{-1}(a)_\psi)_\Psi\otimes c_{1_1}\otimes(\gamma(c_{1_2}{}^\Psi)\otimes c_2{}^\psi)\\ &=(a\otimes c)_{[0][0]}\otimes((a\otimes c)_{[0][1]}\otimes\gamma^{-1}((a\otimes c)_{[1]})),\end{aligned}$$

which proves that (2.7) holds. The other conditions can be checked straightforwardly. The compatibility can be proved as follows: for $a, b \in A, c \in C$,

$$\begin{aligned}
\rho_{A\otimes C}((b\otimes c)a) &= \rho_{A\otimes C}(b\alpha^{-1}(a)\otimes\gamma(c))\\
&= (\alpha^{-1}(b\alpha^{-1}(a))_{\psi}\otimes\gamma(c_1))\otimes\gamma(\gamma(c_2)^{\psi})\\
&= ((\alpha^{-1}(b)\alpha^{-2}(a))_{\psi}\otimes\gamma(c_1))\otimes\gamma(\gamma(c_2)^{\psi})\\
&= (\alpha^{-1}(b)_{\psi}\alpha^{-2}(a)_{\Psi}\otimes\gamma(c_1))\otimes\gamma(\gamma(c_2{}^{\psi\Psi}))\\
&= (\alpha^{-1}(b)_{\psi}\alpha^{-1}(\alpha^{-1}(a)_{\Psi})\otimes\gamma(c_1))\otimes\gamma(\gamma(c_2{}^{\psi})^{\Psi})\\
&= (\alpha^{-1}(b)_{\psi}\otimes c_1)\alpha^{-1}(a)_{\Psi}\otimes\gamma(\gamma(c_2{}^{\psi})^{\Psi})
\end{aligned}$$

as desired. □

Lemma 3.4. *With notations as above, the map $t_{M,N}$ is right (A,α)-linear for all Hom-entwined modules (M,μ) and (N,ν) if and only if*

$$(b_{2\psi}\otimes b_{1\Psi})Q(c'^{\psi}\otimes c^{\Psi}) = Q(c'\otimes c)\Delta_A(b), \tag{3.11}$$

for all $b \in A$ and $c, c' \in C$.

Proof. Suppose that $t_{A\otimes C,A\otimes C}$ is (A,α)-linear. Then, for $a, a', b \in A$ and $c, c' \in C$, we have

$$t_{A\otimes C,A\otimes C}(((a\otimes c)\otimes(a'\otimes c'))b) = t_{A\otimes C,A\otimes C}((a\otimes c)\otimes(a'\otimes c'))b. \tag{3.12}$$

Since

$$\begin{aligned}
\text{LHS} &= t_{A\otimes C,A\otimes C}((a\otimes c)b_1\otimes(a'\otimes c')b_2)\\
&= t_{A\otimes C,A\otimes C}((a\alpha^{-1}(b_1)\otimes\gamma(c))\otimes(a'\alpha^{-1}(b_2)\otimes\gamma(c')))\\
&= (\alpha^{-1}(a'\alpha^{-1}(b_2))_{\psi}\otimes\gamma(c')_1)Q^1(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi}))\\
&\quad\otimes(\alpha^{-1}(a\alpha^{-1}(b_1))_{\Psi}\otimes\gamma(c)_1)Q^2(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi}))\\
&= (\alpha^{-1}(a'\alpha^{-1}(b_2))_{\psi}\alpha^{-1}(Q^1(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi})))\otimes\gamma(\gamma(c')_1))\\
&\quad\otimes(\alpha^{-1}(a\alpha^{-1}(b_1))_{\Psi}\alpha^{-1}(Q^2(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi})))\otimes\gamma(\gamma(c)_1))
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= (((\alpha^{-1}(a')_{\psi}\otimes c'_1)\otimes(\alpha^{-1}(a)_{\Psi}\otimes c_1))Q(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi})))b\\
&= ((\alpha^{-1}(a')_{\psi}\otimes c'_1)Q^1(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi})))b_1\\
&\quad\otimes((\alpha^{-1}(a)_{\Psi}\otimes c_1)Q^2(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi})))b_2\\
&= ((\alpha^{-1}(a')_{\psi}\alpha^{-1}(Q^1(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi}))))\alpha^{-1}(b_1)\otimes\gamma^2(c'_1))\\
&\quad\otimes((\alpha^{-1}(a)_{\Psi}\alpha^{-1}(Q^2(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi}))))\alpha^{-1}(b_2)\otimes\gamma^2(c_1)),
\end{aligned}$$

we have

$$\begin{aligned}
&(\alpha^{-1}(a'\alpha^{-1}(b_2))_{\psi}\alpha^{-1}(Q^1(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi})))\otimes\gamma(\gamma(c')_1))\\
&\quad\otimes(\alpha^{-1}(a\alpha^{-1}(b_1))_{\Psi}\alpha^{-1}(Q^2(\gamma(\gamma(c')_2{}^{\psi})\otimes\gamma(\gamma(c)_2{}^{\Psi})))\otimes\gamma(\gamma(c)_1))\\
&= ((\alpha^{-1}(a')_{\psi}\alpha^{-1}(Q^1(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi}))))\alpha^{-1}(b_1)\otimes\gamma^2(c'_1))\\
&\quad\otimes((\alpha^{-1}(a)_{\Psi}\alpha^{-1}(Q^2(\gamma(c_2'^{\psi})\otimes\gamma(c_2{}^{\Psi}))))\alpha^{-1}(b_2)\otimes\gamma^2(c_1)).
\end{aligned}$$

By taking $a = a' = 1_A$ in the above equality and then applying $id_A\otimes\varepsilon_C\otimes id_A\otimes\varepsilon_C$ to both sides, we can get

$$(b_{2\psi}\otimes b_{1\Psi})Q(c'^{\psi}\otimes c^{\Psi}) = Q(c'\otimes c)\Delta_A(b). \tag{3.13}$$

Conversely, suppose that (3.11) holds, and consider two Hom-entwined modules (M,μ) and (N,ν). For all $m\in M, n\in N$ and $a\in A$, we have

$$\begin{aligned}
t_{M,N}((m\otimes n)a) &= t_{M,N}(ma_1\otimes na_2)\\
&= ((na_2)_{[0]}\otimes (ma_1)_{[0]})Q((na_2)_{[1]}\otimes (ma_1)_{[1]})\\
&= (n_{[0]}\alpha^{-1}(a_2)_\psi\otimes m_{[0]}\alpha^{-1}(a_1)_\Psi)Q(\gamma({n_{[1]}}^\psi)\otimes\gamma({m_{[1]}}^\Psi))\\
&= (n_{[0]}\alpha^{-1}(a)_{2\psi}\otimes m_{[0]}\alpha^{-1}(a)_{1\Psi})Q(\gamma({n_{[1]}}^\psi)\otimes\gamma({m_{[1]}}^\Psi))\\
&= \nu(n_{[0]})\alpha^{-1}(a_{2\psi}Q^1(\gamma(n_{[1]})^\psi\otimes\gamma(m_{[1]})^\Psi))\\
&\quad\otimes\mu(m_{[0]})\alpha^{-1}(a_{1\Psi}Q^2(\gamma(n_{[1]})^\psi\otimes\gamma(m_{[1]})^\Psi))\\
&= \nu(n_{[0]})\alpha^{-1}(Q^1(\gamma(n_{[1]})\otimes\gamma(m_{[1]}))a_1)\\
&\quad\otimes\mu(m_{[0]})\alpha^{-1}(Q^2(\gamma(n_{[1]})\otimes\gamma(m_{[1]}))a_2)\\
&= (n_{[0]}\alpha^{-1}(Q^1(\gamma(n_{[1]})\otimes\gamma(m_{[1]}))))a_1\\
&\quad\otimes(m_{[0]}\alpha^{-1}(Q^2(\gamma(n_{[1]})\otimes\gamma(m_{[1]}))))a_2\\
&= (n_{[0]}Q^1(n_{[1]}\otimes m_{[1]}))a_1\otimes(m_{[0]}Q^2(n_{[1]}\otimes m_{[1]}))a_2\\
&= t_{M,N}(m\otimes n)a,
\end{aligned}$$

which follows that $t_{M,N}$ is (A,α)-linear. □

Lemma 3.5. *With notations as above, the map $t_{M,N}$ is right (C,γ)-colinear for all Hom-entwined modules (M,μ) and (N,ν) if and only if*

$$Q^1(c'_2\otimes c_2)_\psi\otimes Q^2(c'_2\otimes c_2)_\Psi\otimes {c'_1}^\psi {c_1}^\Psi = Q^1(c'_1\otimes c_1)\otimes Q^2(c'_1\otimes c_1)\otimes c_2c'_2, \tag{3.14}$$

for all $c,c'\in C$.

Proof. Suppose that $t_{A\otimes C,A\otimes C}$ is (C,γ)-colinear. Then, for $c,c'\in C$, we have

$$\begin{aligned}
&(\alpha^{-1}(Q^1(\gamma(c'_2)\otimes\gamma(c_2))_\psi)\otimes\gamma(c'_{11}))\\
&\quad\otimes(\alpha^{-1}(Q^2(\gamma(c'_2)\otimes\gamma(c_2))_\Psi)\otimes\gamma(c_{11}))\otimes\gamma^2(c'_{12})^\psi\gamma^2(c_{12})^\Psi\\
&= (Q^1(\gamma(c'_{12})\otimes\gamma(c_{12}))\otimes\gamma(c'_{11}))\otimes(Q^2(\gamma(c'_{12})\otimes\gamma(c_{12}))\otimes\gamma(c_{11}))\otimes\gamma(c_2)\gamma(c'_2).
\end{aligned}$$

Applying $id_A\otimes\varepsilon_C\otimes id_A\otimes\varepsilon_C\otimes id_C$ to both sides, we can have (3.14).

Conversely, assume that (3.14) holds. Take two Hom-entwined modules (M,μ) and (N,ν). Then, for $m\in M$, and $n\in N$, we have

$$\begin{aligned}
&\rho_{M\otimes N}(t_{M,N}(m\otimes n))\\
&= \rho_{M\otimes N}((n_{[0]}\otimes m_{[0]})Q(n_{[1]}\otimes m_{[1]}))\\
&= (n_{[0][0]}\alpha^{-1}(Q^1(n_{[1]}\otimes m_{[1]}))_\psi\\
&\quad\otimes m_{[0][0]}\alpha^{-1}(Q^2(n_{[1]}\otimes m_{[1]}))_\Psi)\otimes\gamma({n_{[0][1]}}^\psi)\gamma({m_{[0][1]}}^\Psi)\\
&= (\nu^{-1}(n_{[0]})\alpha^{-1}(Q^1(\gamma(n_{[1]2})\otimes\gamma(m_{[1]2})))_\psi\\
&\quad\otimes\mu^{-1}(m_{[0]})\alpha^{-1}(Q^2(\gamma(n_{[1]2})\otimes\gamma(m_{[1]2})))_\Psi)\otimes\gamma({n_{[1]1}}^\psi)\gamma({m_{[1]1}}^\Psi)\\
&= ((\nu^{-1}(n_{[0]})\alpha^{-1}(Q^1(\gamma(n_{[1]})_2\otimes\gamma(m_{[1]})_2)_\psi))\\
&\quad\otimes(\mu^{-1}(m_{[0]})\alpha^{-1}(Q^2(\gamma(n_{[1]})_2\otimes\gamma(m_{[1]})_2))_\Psi))\otimes\gamma(n_{[1]})_1{}^\psi\gamma(m_{[1]})_1{}^\Psi\\
&= ((\nu^{-1}(n_{[0]})\alpha^{-1}(Q^1(\gamma(n_{[1]1})\otimes\gamma(m_{[1]1}))))\\
&\quad\otimes(\mu^{-1}(m_{[0]})\alpha^{-1}(Q^2(\gamma(n_{[1]1})\otimes\gamma(m_{[1]1}))))))\otimes\gamma(m_{[1]2})\gamma(n_{[1]2})\\
&= (n_{[0][0]}\alpha^{-1}(Q^1(\gamma(n_{[0][1]})\otimes\gamma(m_{[0][1]})))\\
&\quad\otimes(m_{[0][0]}\alpha^{-1}(Q^2(\gamma(n_{[0][1]})\otimes\gamma(m_{[0][1]})))))\otimes m_{[1]}n_{[1]}
\end{aligned}$$

$$= ((n_{[0][0]}Q^1(n_{[0][1]} \otimes m_{[0][1]})) \otimes (m_{[0][0]}Q^2(n_{[0][1]} \otimes m_{[0][1]}))) \otimes m_{[1]}n_{[1]}$$
$$= (t_{M,N} \otimes id_C)(\rho_{M\otimes N}((m \otimes n))),$$

which follows that $(t_{M,N})$ is (C, γ)-colinear. □

Lemma 3.6. *With notations as above, (3.5) holds for all Hom entwined modules (M, μ), (N, ν) and (W, ς) if and only if*

$$(\Delta_A \otimes id_A)Q(c'c'' \otimes c) = Q^1(c' \otimes c_2) \otimes Q^1(c'' \otimes {c_1}^{\psi}) \otimes Q^2(c' \otimes c_2)_{\psi}Q^2(c'' \otimes {c_1}^{\psi}), \quad (3.15)$$

for all $c, c', c'' \in C$.

Proof. Suppose that (3.5) holds. We take $M = N = W = (A\otimes C, \alpha\otimes\gamma)$. For $c, c', c'' \in C$, on the one hand, we have

$$(id_N \otimes t_{M,W}) \circ \widetilde{a}_{N,M,W} \circ (t_{M,N} \otimes id_W) \circ \widetilde{a}^{-1}_{M,N,W}((1 \otimes c) \otimes ((1 \otimes c') \otimes (1 \otimes c'')))$$
$$= (\alpha(Q^1(\gamma(c'_2) \otimes c_2)) \otimes \gamma^2(c'_1)) \otimes ((Q^1(\gamma(c''_2) \otimes \gamma({c_{1_2}}^{\psi})) \otimes \gamma(c''_1))$$
$$\otimes \alpha^{-1}(Q^2(\gamma(c'_2) \otimes c_2)_{\psi}Q^2(\gamma(c''_2) \otimes \gamma({c_{1_2}}^{\psi}))) \otimes \gamma(c_{1_1}))$$
$$= (\alpha(Q^1(\gamma(c'_2) \otimes \gamma(c_{2_2}))) \otimes \gamma^2(c'_1)) \otimes ((Q^1(\gamma(c''_2) \otimes \gamma({c_{2_1}}^{\psi})) \otimes \gamma(c''_1))$$
$$\otimes \alpha^{-1}(Q^2(\gamma(c'_2) \otimes \gamma(c_{2_2}))_{\psi}Q^2(\gamma(c''_2) \otimes \gamma({c_{2_1}}^{\psi}))) \otimes c_1).$$

On the other hand,

$$\widetilde{a}_{N,W,M} \circ t_{M,N\otimes W}((1 \otimes c) \otimes ((1 \otimes c') \otimes (1 \otimes c'')))$$
$$= \alpha(Q^1(\gamma(c'_2c''_2) \otimes \gamma(c_2))_1) \otimes \gamma^2(c'_1)$$
$$\otimes ((Q^1(\gamma(c'_2c''_2) \otimes \gamma(c_2))_2 \otimes \gamma(c''_1)) \otimes (\alpha^{-1}(Q^2(\gamma(c'_2c''_2) \otimes \gamma(c_2))) \otimes c_1))$$
$$= (\alpha(Q^1(\gamma(c'_2) \otimes \gamma(c_{2_2}))) \otimes \gamma^2(c'_1)) \otimes ((Q^1(\gamma(c''_2) \otimes \gamma({c_{2_1}}^{\psi})) \otimes \gamma(c''_1))$$
$$\otimes \alpha^{-1}(Q^2(\gamma(c'_2) \otimes \gamma(c_{2_2}))_{\psi}Q^2(\gamma(c''_2) \otimes \gamma({c_{2_1}}^{\psi}))) \otimes c_1).$$

Applying $id_A \otimes \varepsilon_C \otimes id_A \otimes \varepsilon_C \otimes \otimes id_A \otimes \varepsilon_C$ to both sides, we get (3.15).

Conversely, if (3.15) holds. Let (M, μ), (N, ν) and (W, ς) be Hom-entwined modules. We easily compute that

$$(id_N \otimes t_{M,W}) \circ \widetilde{a}_{N,M,W} \circ (t_{M,N} \otimes id_W) \circ \widetilde{a}^{-1}_{M,N,W}(m \otimes (n \otimes w))$$
$$= \nu(n_{[0]})\alpha(Q^1(n_{[1]} \otimes \gamma^{-1}(m_{[1]}))) \otimes (w_{[0]}Q^1(w_{[1]} \otimes \gamma((\mu^{-1}(m_{[0]})_{[1]})^{\psi}))$$
$$\otimes (\mu^{-1}(m_{[0]})_{[0]}\alpha^{-1}(Q^2(n_{[1]} \otimes \gamma^{-1}(m_{[1]})))_{\psi})Q^2(w_{[1]} \otimes \gamma((\mu^{-1}(m_{[0]})_{[1]})^{\psi})))$$
$$= \nu(n_{[0]})\alpha(Q^1(n_{[1]} \otimes \gamma^{-1}(m_{[1]}))) \otimes (w_{[0]}Q^1(w_{[1]} \otimes \gamma((\gamma^{-1}(m_{[0][1]})^{\psi})))$$
$$\otimes (\mu^{-1}(m_{[0][0]})\alpha^{-1}(Q^2(n_{[1]} \otimes \gamma^{-1}(m_{[1]})))_{\psi})Q^2(w_{[1]} \otimes \gamma((\gamma^{-1}(m_{[0][1]})^{\psi}))))$$
$$= \nu(n_{[0]})\alpha(Q^1(n_{[1]} \otimes \gamma^{-1}(m_{[1]}))) \otimes (w_{[0]}Q^1(w_{[1]} \otimes (m_{[0][1]})^{\psi})$$
$$\otimes (\mu^{-1}(m_{[0][0]})\alpha^{-1}(Q^2(n_{[1]} \otimes \gamma^{-1}(m_{[1]}))_{\psi}))Q^2(w_{[1]} \otimes (m_{[0][1]})^{\psi}))$$
$$= \nu(n_{[0]})\alpha(Q^1(n_{[1]} \otimes m_{[1]_2})) \otimes (w_{[0]}Q^1(w_{[1]} \otimes (m_{[1]_1})^{\psi})$$
$$\otimes (\mu^{-2}(m_{[0]})\alpha^{-1}(Q^2(n_{[1]} \otimes m_{[1]_2})_{\psi}))Q^2(w_{[1]} \otimes (m_{[1]_1})^{\psi}))$$
$$= \nu(n_{[0]})\alpha(Q^1(n_{[1]} \otimes m_{[1]_2})) \otimes (w_{[0]}Q^1(w_{[1]} \otimes (m_{[1]_1})^{\psi})$$
$$\otimes \mu^{-1}(m_{[0]})(\alpha^{-1}(Q^2(n_{[1]} \otimes m_{[1]_2})_{\psi})\alpha^{-1}(Q^2(w_{[1]} \otimes (m_{[1]_1})^{\psi}))))$$
$$= \widetilde{a}_{N,W,M} \circ t_{M,N\otimes W}(m \otimes (n \otimes w)),$$

which proves that (3.5) holds. □

The proof of The next lemma is similar to the proof of Lemma 3.6, so we omit it.

Lemma 3.7. *With notations as above, (3.6) holds for all Hom entwined modules (M,μ), (N,ν) and (W,ς) if and only if*

$$(id_A\otimes\Delta_A)Q(c\otimes c'c'')=Q^1(c_2\otimes c'')_\psi Q^1(c_1{}^\psi\otimes c')\otimes Q^2(c_1{}^\psi\otimes c')\otimes Q^2(c_2\otimes c''),\quad (3.16)$$

for all $c,c',c''\in C$.

We summarize our results as follows:

Theorem 3.8. *Let $[(A,\alpha),(C,\gamma)]_\psi$ a monoidal Hom-entwining datum, and $Q:C\otimes C\to A\otimes A$ a twisted convolution invertible map in $\widetilde{\mathcal{H}}(\mathcal{M})$. Then the family of maps*

$$t_{M,N}:M\otimes N\to N\otimes M, m\otimes n\mapsto (n_{[0]}\otimes m_{[0]})Q(n_{[1]}\otimes m_{[1]})$$

defines a braiding on the category of Hom-entwined modules $\widetilde{\mathcal{M}}_A^C(\psi)$ if and only if Q satisfies Equations (3.11) and (3.14)-(3.16).

Now, we shall apply Theorem 3.8 to Doi-Koppinen Hom-Hopf modules. Given a Hom-Doi-Koppinen datum $[(H,\beta),(A,\alpha),(C,\gamma)]$, we have a Hom-entwining datum $[(A,\alpha),(C,\gamma)]_\psi$ with ψ given by

$$\psi:C\otimes A\to A\otimes C,\ c\otimes a\mapsto \alpha(a_{[0]})\otimes\gamma^{-1}(c)a_{[1]}=a_\psi\otimes c^\psi. \quad (3.17)$$

The Hom-category $\widetilde{\mathcal{M}}_A^C(\psi)$ of Hom-entwined modules associated to the induced Hom-entwining datum $[(A,\alpha),(C,\gamma)]_\psi$ is denoted by $\widetilde{\mathcal{M}}(H)_A^C$.

A Hom-Doi-Koppinen datum $[(H,\beta),(A,\alpha),(C,\gamma)]$ is called a monoidal Hom-Doi-Koppinen datum, if it satisfies the following condition,

$$a_{1[0]}\otimes a_{2[0]}\otimes(ca_{1[1]})(c'a_{2[1]})=a_{[0]1}\otimes a_{[0]2}\otimes(cc')a_{[1]}, \quad (3.18)$$

for all $a\in A$ and $c\in C$.

From Theorem 3.8, we have the following result.

Corollary 3.9. *Let $[(H,\beta),(A,\alpha),(C,\gamma)]$ be a monoidal Hom-Doi-Koppinen datum, and $Q:C\otimes C\to A\otimes A$ a twisted convolution invertible map in $\widetilde{\mathcal{H}}(\mathcal{M})$. Then the family of maps*

$$t_{M,N}:M\otimes N\to N\otimes M, m\otimes n\mapsto (n_{[0]}\otimes m_{[0]})Q(n_{[1]}\otimes m_{[1]})$$

defines a braiding on the category of Doi-Koppinen Hom-Hopf modules $\widetilde{\mathcal{M}}(H)_A^C$ if and only if Q satisfies the following equations, for any $b\in A$ and $c,c',c''\in C$,

(1) $(\alpha(b_{2[0]})\otimes\alpha(b_{1[0]}))Q(c'b_{2[1]}\otimes cb_{1[1]})=Q(\gamma(c')\otimes\gamma(c))\Delta_A(b)$,

(2)

$$\begin{aligned}&\alpha(Q^1(c_2'\otimes c_2)_{[0]})\otimes\alpha(Q^2(c_2'\otimes c_2)_{[0]})\\ &\quad\otimes(\gamma^{-1}(c_1')Q^1(c_2'\otimes c_2)_{[1]})(\gamma^{-1}(c_1)Q^2(c_2'\otimes c_2)_{[1]})\\ &=Q^1(c_1'\otimes c_1)\otimes Q^2(c_1'\otimes c_1)\otimes c_2c_2',\end{aligned}$$

(3)

$$\begin{aligned}(\Delta_A\otimes id_A)Q(c'c''\otimes c)&=Q^1(c'\otimes c_2)\otimes Q^1(c''\otimes\gamma^{-1}(c_1)Q^2(c'\otimes c_2)_{[1]})\\ &\quad\otimes\alpha(Q^2(c'\otimes c_2)_{[0]})Q^2(c''\otimes\gamma^{-1}(c_1)Q^2(c'\otimes c_2)_{[1]}),\end{aligned}$$

(4)

$$\begin{aligned}(id_A\otimes\Delta_A)Q(c\otimes c'c'')&=\alpha(Q^1(c_2\otimes c'')_{[0]})Q^1(\gamma^{-1}(c_1)Q^1(c_2\otimes c'')_{[1]}\otimes c')\\ &\quad\otimes Q^2(\gamma^{-1}(c_1)Q^1(c_2\otimes c'')_{[1]}\otimes c')\otimes Q^2(c_2\otimes c'').\end{aligned}$$

4. Hom-coalgebra cleft extensions for Hom-entwining structures

Let (A,α) be a object of $\widetilde{\mathcal{M}}_A^C(\psi)$ with the Hom-coaction ρ_A. For $(M,\mu)\in\widetilde{\mathcal{M}}_A^C(\psi)$, The Hom-invariants of C on M are the set

$$M^{coC}=\{m\in M|\rho_M(m)=\mu^{-2}(m)1_{[0]}\otimes\gamma(1_{[1]})\}.$$

Specially, we have $A^{coC}=\{a\in A|\rho_A(a)=\alpha^{-2}(a)1_{[0]}\otimes\gamma(1_{[1]})\}$. For $m\in M^{coC}$, it follows that $\mu(a)\in M^{coC}$. We use $\mu|_{M^{coC}}$ for denoting the restriction map of μ on M^{coC}.

Lemma 4.1. *For $(A,\alpha),(M,\mu)$ in $\widetilde{\mathcal{M}}_A^C(\psi)$, we have*

(1) *$(A^{coC},\alpha|_{A^{coC}},1)$ is a Hom-algebra.*
(2) *$(M^{coC},\mu|_{M^{coC}})$ is a right $(A^{coC},\alpha|_{A^{coC}})$-Hom-module.*

Proof. Straightforward. □

Let us put $\mathrm{Hom}^C(C,A)$ consisting of right (C,γ)-colinear morphisms $f:C\to A$, that is, $f(c)_{[0]}\otimes f(c)_{[1]}=f(c_{[0]})\otimes c_{[1]}$, for $c\in C$ and $f\circ\gamma=\alpha\circ f$.

Lemma 4.2. $\mathrm{Hom}^C(C,A)$ *is an associative algebra with the unit $\varepsilon_C 1_A$ and multiplication*

$$(f*g)(c)=f(c_1)g(c_2),$$

for $f,g\in\mathrm{Hom}^C(C,A)$ and $c\in C$.

Proof. Straightforward. □

By $\mathrm{Reg}(C,A)$ we denote the set of morphisms $\omega\in\mathrm{Hom}^C(C,A)$ which are invertible under the convolution $*$ in Lemma 4.2.

Definition 4.3. We say that $(A^{coC},\alpha|_{A^{coC}})\hookrightarrow(A,\alpha)$ is a (C,γ)-Hom-cleft extension, if there exists a morphism $\omega\in\mathrm{Reg}(C,A)$.

Proposition 4.4. *If $(A^{coC},\alpha|_{A^{coC}})\hookrightarrow(A,\alpha)$ is a (C,γ)-Hom-cleft extension, we have*

$$\omega^{-1}(c_2)_\psi\otimes {c_1}^\psi=\alpha^{-2}(\omega^{-1}(c))1_{[0]}\otimes\gamma(1_{[1]}),\tag{4.1}$$

for all $c\in C$.

Proof. Since $(A,\alpha)\in\widetilde{\mathcal{M}}_A^C(\psi)$, the Hom-coaction can be written as $\rho_A(a)=1_{[0]}\alpha^{-2}(a)_\psi\otimes\gamma({1_{[1]}}^\psi)$. Then we have, for any $c\in C$,

$$\begin{aligned}
&\varepsilon_C(c)\alpha(1_{[0]})\otimes 1_{[1]}\\
&\quad=1_{[0]}\psi(1_{[1]}\otimes\omega(c_1)\omega^{-1}(c_2))\\
&\quad=1_{[0]}(\omega(c_1)\omega^{-1}(c_2))_\psi\otimes {1_{[1]}}^\psi\\
&\quad=\alpha(1_{[0]})(\omega(c_1)\omega^{-1}(c_2))_\psi\otimes\gamma(1_{[1]})^\psi\\
&\quad=\alpha(1_{[0]})(\omega(c_1)_\psi\omega^{-1}(c_2)_\Psi)\otimes\gamma({1_{[1]}}^{\psi\Psi})\\
&\quad=(1_{[0]}\omega(c_1)_\psi)\alpha(\omega^{-1}(c_2)_\Psi)\otimes\gamma({1_{[1]}}^{\psi\Psi})\\
&\quad=\alpha^2(\omega(c_1)_{[0]})\alpha(\omega^{-1}(c_2)_\Psi)\otimes\gamma(\gamma(\omega(c_1)_{[1]})^\Psi)\\
&\quad=\alpha^2(\omega(c_{1_1}))\alpha(\omega^{-1}(c_2)_\Psi)\otimes\gamma(\gamma(c_{1_2})^\Psi)\\
&\quad=\alpha(\omega(c_1))\alpha(\omega^{-1}(\gamma(c_2)_2)_\Psi)\otimes\gamma(\gamma(c_2)_1{}^\Psi),
\end{aligned}$$

which implies that Eq (4.1) holds. □

Lemma 4.5. *Assume that $(A^{coC},\alpha|_{A^{coC}})\hookrightarrow(A,\alpha)$ is a (C,γ)-Hom-cleft extension via ω and $(M,\mu)\in\widetilde{\mathcal{M}}_A^C(\psi)$. Then, for any $m\in M$, $m_{[0]}\omega^{-1}(m_{[1]})\in M^{coC}$. As a consequence, if $M=A$, we have $a_{[0]}\omega^{-1}(a_{[1]})\in A^{coC}$*

Proof. We compute

$$
\begin{aligned}
&\rho_M(m_{[0]}\omega^{-1}(m_{[1]}))\\
&\quad= m_{[0][0]}\alpha^{-1}(\omega^{-1}(m_{[1]}))_\psi \otimes \gamma(m_{[0][1]}{}^{\psi})\\
&\quad= \mu^{-1}(m_{[0]})\alpha^{-1}(\omega^{-1}(\gamma(m_{[1]2})))_\psi \otimes \gamma(m_{[1]1}{}^{\psi})\\
&\quad= \mu^{-1}(m_{[0]})\alpha^{-1}(\omega^{-1}(\gamma(m_{[1]})_2{}_\psi) \otimes (\gamma(m_{[1]})_1)^{\psi}\\
&\quad= \mu^{-1}(m_{[0]})(\alpha^{-2}(\omega^{-1}(m_{[1]}))\alpha^{-1}(1_{[0]})) \otimes \gamma(1_{[1]})\\
&\quad= (\mu^{-2}(m_{[0]})\alpha^{-2}(\omega^{-1}(m_{[1]})))1_{[0]} \otimes \gamma(1_{[1]})\\
&\quad= \mu^{-2}(m_{[0]}\omega^{-1}(m_{[1]}))1_{[0]} \otimes \gamma(1_{[1]}).
\end{aligned}
$$

Hence $m_{[0]}\omega^{-1}(m_{[1]}) \in M^{coC}$. □

Theorem 4.6. *Suppose that $(A^{coC}, \alpha|_{A^{coC}}) \hookrightarrow (A, \alpha)$ is a (C, γ)-Hom-cleft extension via ω. For $(M, \mu) \in \widetilde{\mathcal{M}}_A^C(\psi)$, then $(M, \mu) \cong (M^{coC} \otimes C, \mu|_{M^{coC}} \otimes \gamma)$ as right (C, γ)-Hom-comodules, where the (C, γ)-Hom-coaction on $(M^{coC} \otimes C, \mu|_{M^{coC}} \otimes \gamma)$ is*

$$\rho_{M^{coC}\otimes C}(m \otimes c) = (\mu^{-1}(m) \otimes c_1) \otimes \gamma(c_2).$$

In particular, if we consider $M = A$, we have $(A, \alpha) \cong (A^{coC} \otimes C, \alpha|_{A^{coC}} \otimes \gamma)$ as both right (C, γ)-Hom-comodules and left $(A^{coC}, \alpha|_{A^{coC}})$-Hom-modules, where the $(A^{coC}, \alpha|_{A^{coC}})$-Hom-action on $A^{coC} \otimes C$ defined by $b \cdot (a \otimes c) = \alpha^{-1}(b)a \otimes \gamma(c)$, for all $a, b \in A^{coC}$ and $c \in C$.

Proof. We define $\Theta_M : (M^{coC} \otimes C, \mu|_{M^{coC}} \otimes \gamma) \to (M, \mu)$ by $\Theta_M(m \otimes c) = m\omega(c)$ and $\Omega_M : (M, \mu) \to (M^{coC} \otimes C, \mu|_{M^{coC}} \otimes \gamma)$ by $\Omega_M(m) = m_{[0][0]}\omega^{-1}(m_{[0][1]}) \otimes m_{[1]}$. For $m \in M$, we have

$$
\begin{aligned}
\Theta_M \circ \Omega_M(m) &= (m_{[0][0]}\omega^{-1}(m_{[0][1]}))\omega(m_{[1]})\\
&= (\mu^{-1}(m_{[0]})\omega^{-1}(m_{[1]_1}))\omega(\gamma(m_{[1]_2}))\\
&= (\mu^{-1}(m_{[0]})\omega^{-1}(m_{[1]_1}))\alpha(\omega(m_{[1]_2}))\\
&= m_{[0]}(\omega^{-1}(m_{[1]_1})\omega(m_{[1]_2}))\\
&= m_{[0]}\varepsilon_C(m_{[1]})1_A = m,
\end{aligned}
$$

which follows that $\Theta_M \circ \Omega_M = id$. Next, we check that $\Omega_M \circ \Theta_M = id$ holds. In fact, for any $m \in M^{coC}$ and $c \in C$, we compute

$$
\begin{aligned}
&\Omega_M \circ \Theta_M(m \otimes c)\\
&\quad= (m\omega(c))_{[0][0]}\omega^{-1}((m\omega(c))_{[0][1]}) \otimes (m\omega(c))_{[1]}\\
&\quad= (m_{[0][0]}\alpha^{-1}(\alpha^{-1}(\omega(c))_\psi)_\Psi)\omega^{-1}(\gamma(m_{[0][1]}{}^{\Psi})) \otimes \gamma(m_{[1]}{}^{\psi})\\
&\quad= (m_{[0][0]}\alpha^{-1}(\alpha^{-1}(\omega(c))_{\psi\Psi}))\omega^{-1}(\gamma(m_{[0][1]})^{\Psi}) \otimes \gamma(m_{[1]}{}^{\psi})\\
&\quad= (\mu^{-1}(m_{[0]})\alpha^{-1}(\alpha^{-1}(\omega(c))_{\psi\Psi}))\omega^{-1}(\gamma(m_{[1]})_1{}^{\Psi}) \otimes \gamma(\gamma(m_{[1]})_2{}^{\psi})\\
&\quad= (\mu^{-1}(\mu^{-2}(m)1_{[0]})\alpha^{-1}(\alpha^{-1}(\omega(c))_{\psi\Psi}))\omega^{-1}(\gamma^2(1_{[1]})_1{}^{\Psi}) \otimes \gamma(\gamma^2(1_{[1]})_2{}^{\psi})\\
&\quad= (\mu^{-2}(m)(\alpha^{-1}(1_{[0]})\alpha^{-2}(\alpha^{-1}(\omega(c)_{\psi\Psi}))))\omega^{-1}(\gamma^2(1_{[1]})_1{}^{\Psi}) \otimes \gamma(\gamma^2(1_{[1]})_2{}^{\psi})\\
&\quad= (\mu^{-2}(m)(\alpha^{-1}(1_{[0]})\alpha^{-3}(\omega(c)_\psi)))\omega^{-1}(\gamma^2(1_{[1]})^{\psi}{}_1) \otimes \gamma(\gamma^2(1_{[1]})^{\psi}{}_2)\\
&\quad= (\mu^{-2}(m)\alpha^{-3}(1_{[0]}\omega(c)_\psi))\omega^{-1}(1_{[1]}{}^{\psi}{}_1) \otimes \gamma(1_{[1]}{}^{\psi}{}_2)\\
&\quad= (\mu^{-2}(m)\alpha^{-1}(\omega(c)_{[0]}))\omega^{-1}(\gamma(\omega(c)_{[1]})_1) \otimes \gamma(\gamma(\omega(c)_{[1]})_2)\\
&\quad= (\mu^{-2}(m)\alpha^{-1}(\omega(c_1)))\omega^{-1}(\gamma(c_2)_1) \otimes \gamma(\gamma(c_2)_2)\\
&\quad= (\mu^{-2}(m)\alpha^{-1}(\omega(c_1)))\omega^{-1}(\gamma(c_{2_1})) \otimes \gamma(\gamma(c_{2_2}))
\end{aligned}
$$

$$= (\mu^{-2}(m)\omega(c_{1_1}))\omega^{-1}(\gamma(c_{1_2})) \otimes \gamma(c_2)$$
$$= \mu^{-1}(m)(\omega(c_{1_1})\omega^{-1}(c_{1_2})) \otimes \gamma(c_2)$$
$$= \mu^{-1}(m)1_A \otimes c$$
$$= m \otimes c,$$

as desired. □

Let $(A^{coC}, \alpha|_{A^{coC}}) \hookrightarrow (A, \alpha)$ be a (C, γ)-Hom-cleft extension via ω. From Theorem 4.6, we have that Ω_A is an isomorphism of monoidal Hom-algebras, where the monoidal Hom-algebra structure on $A^{coC} \otimes C$ can be induced by Ω_A:

$$1_{A^{coC}\times C} = \Omega_A(1_A), \widetilde{m}_{A^{coC}\times C} = \Omega_A \circ m_A \circ (\Omega_A^{-1} \otimes \Omega_A^{-1}).$$

The induced monoidal Hom-algebras on $A^{coC} \otimes C$ is called a crossed product Hom-algebra of A^{coC} and C, and denoted by $A^{coC} \ltimes C$.

Next, we can obtain $\widetilde{m}_{A^{coC}\times C}$ in other way. First, we need some preliminary results.

Lemma 4.7. *Suppose that $(A^{coC}, \alpha|_{A^{coC}}) \hookrightarrow (A, \alpha)$ is a (C, γ)-Hom-cleft extension via ω. We define a morphism $\varpi : C \otimes A \to A$ by*

$$\varpi(c, a) = (\omega(c_1)\alpha^{-1}(a)_\psi)\omega^{-1}(\gamma(c_2{}^\psi)).$$

Then $\rho_A(\varpi(c, a)) \in A^{coC}$, for all $c \in C$ and $a \in A$.

Proof. For if $c \in C, a \in A$, then

$$\begin{aligned}
\rho_A(\varpi(c,a)) &= \alpha(\omega(c_1))_{[0]}\alpha^{-1}(\alpha^{-1}(a)_\psi\omega^{-1}(c_2{}^\psi))_\Psi \otimes \gamma(\alpha(\omega(c_1))_{[1]}{}^\Psi)\\
&= \alpha(\omega(c_1)_{[0]})\alpha^{-1}(\alpha^{-1}(a)_\psi\omega^{-1}(c_2{}^\psi))_\Psi \otimes \gamma(\gamma(\omega(c_1)_{[1]})^\Psi)\\
&= \alpha(\omega(c_1)_{[0]})\alpha^{-1}(\alpha^{-1}(a)_\psi\omega^{-1}(c_2{}^\psi))_\Psi \otimes \gamma(\gamma(\omega(c_1)_{[1]})^\Psi)\\
&= \alpha(\omega(c_1)_{[0]})\alpha^{-1}((\alpha^{-1}(a)_\psi\omega^{-1}(c_2{}^\psi))_\Psi) \otimes \gamma(\gamma(\omega(c_1)_{[1]}))^\Psi\\
&= \alpha(\omega(c_1)_{[0]})\alpha^{-1}(\alpha^{-1}(a)_{\psi\Psi}\omega^{-1}(c_2{}^\psi)_{\Psi'}) \otimes \gamma(\gamma(\omega(c_1)_{[1]})^{\Psi\Psi'})\\
&= \alpha(\omega(c_{1_1}))\alpha^{-1}(\alpha^{-1}(a)_{\psi\Psi}\omega^{-1}(c_2{}^\psi)_{\Psi'}) \otimes \gamma(\gamma(c_{1_2})^{\Psi\Psi'})\\
&= \omega(c_1)\alpha^{-1}(\alpha^{-1}(a)_{\psi\Psi}\omega^{-1}(\gamma(c_2)_2{}^\psi)_{\Psi'}) \otimes \gamma(\gamma(c_2)_1{}^{\Psi\Psi'})\\
&= \omega(c_1)\alpha^{-1}(\alpha^{-1}(a_\psi)\omega^{-1}(\gamma(c_2)^\psi{}_2)_{\Psi'}) \otimes \gamma(\gamma(c_2)^\psi{}_1{}^{\Psi'})\\
&= \omega(c_1)\alpha^{-1}(\alpha^{-1}(a_\psi)(\alpha^{-2}(\omega^{-1}(\gamma(c_2)^\psi)1_{[0]}))) \otimes \gamma(\gamma(1_{[1]}))\\
&= (\alpha^{-1}(\omega(c_1))\alpha^{-3}(a_\psi\omega^{-1}(\gamma(c_2)^\psi)))1_{[0]} \otimes \gamma(1_{[1]})\\
&= ((\alpha^{-2}(\omega(c_1))\alpha^{-3}(a_\psi))\alpha^{-2}(\omega^{-1}(\gamma(c_2)^\psi)))1_{[0]} \otimes \gamma(1_{[1]})\\
&= \alpha^{-2}((\omega(c_1)\alpha^{-1}(a_\psi))\omega^{-1}(\gamma(c_2)^\psi))1_{[0]} \otimes \gamma(1_{[1]})\\
&= \alpha^{-2}((\omega(c_1)\alpha^{-1}(a)_\psi)\omega^{-1}(\gamma(c_2^\psi)))1_{[0]} \otimes \gamma(1_{[1]}).
\end{aligned}$$

Thus $\varpi(c, a) \in A^{coC}$. □

Now, we construct a morphism Λ as follows:

$$\Lambda : \ C \otimes A \to A \otimes C, \ \Lambda(c \otimes d) = \varpi(c_1 \otimes \alpha^{-1}(\omega(d))_\psi) \otimes \gamma(c_2{}^\psi).$$

By Lemma 4.7, we have $\Lambda(c\otimes d) \in A^{coC} \otimes C$. Using Λ, we define a multiplication $m_{A^{coC}\otimes C}$ on $A^{coC} \otimes C$ by

$$\begin{aligned}
m_{A^{coC}\otimes C} =&(m_A \otimes id_C) \circ (m_A \otimes \Lambda \circ (id_C \otimes \omega)) \circ \widetilde{a}_{A,A,C\otimes C}\\
&\circ (id_C \otimes \widetilde{a}_{A,C,C}) \circ (id_C \otimes \Lambda \otimes id_C) \circ (id_A \otimes \widetilde{a}^{-1}_{C,A,C}) \circ \widetilde{a}_{A,C,A\otimes C}.
\end{aligned}$$

Concretely,

$$\begin{aligned}(a\otimes c)(b\otimes d) =&(\alpha^{-1}(a)((\alpha^{-2}(\omega(c_1))\alpha^{-2}(\alpha^{-1}(b)_{\psi}))\omega^{-1}(c_2{}^{\psi}{}_1)))\\ &\times((\omega(\gamma(c_2{}^{\psi}{}_{2_1}))\alpha^{-2}(\omega(d)_{\Psi}))\omega^{-1}(\gamma^3(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_1))\otimes\gamma^5(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_2.\end{aligned}$$

Proposition 4.8. *Suppose that $(A^{coC},\alpha|_{A^{coC}})\hookrightarrow(A,\alpha)$ is a (C,γ)-Hom-cleft extension via ω. Then $m_{A^{coC}\otimes C}=\widetilde{m}_{A^{coC}\otimes C}$.*

Proof. It suffice to prove that $m_{A^{coC}\times C}=\Omega_A\circ m_A\circ(\Omega_A^{-1}\otimes\Omega_A^{-1})$ holds. Indeed, for any $a,b\in A$ and $c,d\in C$, we have

$$\begin{aligned}&\Omega_A^{-1}\circ m_{A^{coC}\otimes C}((a\otimes c)\otimes(b\otimes d))\\
=&((\alpha^{-1}(a)((\alpha^{-2}(\omega(c_1)\alpha^{-1}(b)_{\psi})\omega^{-1}(c_2{}^{\psi}{}_1))))\\
&\times((\omega(\gamma(c_2{}^{\psi}{}_{2_1}))\alpha^{-2}(\omega(d)_{\Psi}))\omega^{-1}(\gamma^3(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_1)))\omega(\gamma^5(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_2)\\
=&(a((\alpha^{-1}(\omega(c_1)\alpha^{-1}(b)_{\psi})\alpha(\omega^{-1}(c_2{}^{\psi}{}_1)))))\\
&\times(((\omega(\gamma(c_2{}^{\psi}{}_{2_1}))\alpha^{-2}(\omega(d)_{\Psi}))\omega^{-1}(\gamma^3(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_1))\omega(\gamma^4(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_2))\\
=&(a((\alpha^{-1}(\omega(c_1)\alpha^{-1}(b)_{\psi})\alpha(\omega^{-1}(c_2{}^{\psi}{}_1)))))\\
&\times((\alpha(\omega(\gamma(c_2{}^{\psi}{}_{2_1})))\alpha^{-1}(\omega(d)_{\Psi}))(\omega^{-1}(\gamma^3(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_1)\omega(\gamma^3(c_2{}^{\psi}{}_{2_2})^{\Psi}{}_2)))\\
=&(a((\alpha^{-1}(\omega(c_1)\alpha^{-1}(b)_{\psi})\alpha(\omega^{-1}(c_2{}^{\psi}{}_1)))))(\alpha^2(\omega(c_2{}^{\psi}{}_2)\omega(d)))\\
=&\alpha(a)((\alpha^{-1}(\omega(c_1)\alpha^{-1}(b)_{\psi})\alpha(\omega^{-1}(c_2{}^{\psi}{}_1)))(\alpha(\omega(c_2{}^{\psi}{}_2))\alpha^{-1}(\omega(d))))\\
=&\alpha(a)((\omega(c_1)\alpha^{-1}(b)_{\psi})(\alpha(\omega^{-1}(c_2{}^{\psi}{}_1))(\omega(c_2{}^{\psi}{}_2)\alpha^{-2}(\omega(d)))))\\
=&\alpha(a)((\omega(c_1)\alpha^{-1}(b)_{\psi})((\omega^{-1}(c_2{}^{\psi}{}_1)\omega(c_2{}^{\psi}{}_2))\alpha^{-1}(\omega(d))))\\
=&\alpha(a)((\omega(\gamma^{-1}(c))\alpha^{-1}(b))\omega(d))\\
=&\alpha(a)(\omega(c)(\alpha^{-1}(b)\alpha^{-1}(\omega(d))))\\
=&(a\omega(c))(b\omega(d))\\
=&m_A\circ(\Omega_A^{-1}\otimes\Omega_A^{-1})((a\otimes c)\otimes(b\otimes d)).\end{aligned}$$

Thus we gain the desired result. □

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