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Extended generalized extreme value distribution with applications in environmental data

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Abstract

In probability theory and statistics, the generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory, which has wide applicability in several areas including hydrology, engineering, science, ecology and finance. In this paper, we propose three extensions of the GEV distribution that incorporate an additional parameter. These extensions are more flexible than the GEV distribution, i.e., the additional parameter introduces skewness and to vary tail weight. In these three cases, the GEV distribution is a particular case. The parameter estimation of these new distributions is done under the Bayesian paradigm, considering vague priors for the parameters. Simulation studies show the efficiency of the proposed models. Applications to river quotas and rainfall show that the generalizations can produce more efficient results than is the standard case with GEV distribution.

Keywords. **Keywords:** Extreme value theory; Generalized extreme value distribution, Generalized classes of distributions, Environmental and Economic data.

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1. Introduction

In many areas of knowledge, the study of the behavior of a variable is related to the tail of the distribution. These observations, although they can occur with lower frequency than the central part of the distribution, can be of the greatest interest to the researcher; in some situations, an occurrence in the tail can cause a great impact on society, such as an index of very high rainfall or a high level of river flow, among other variables such as temperature. These changes cause more impact than in the mean of the index, according to [16] and [18]. One of the major challenges of analyzing extremes is proposing a model and estimating its parameters with little information due to the scant data available. Often the tail of a statistical distribution is commonly seen as normal, or exponential tails may not be the most suitable for this type of data. Another challenge in analyzing this type of data is estimating with a high level of precision the probability of the occurrence of events that not have been observed. [2] shows in detail the difficulties in estimating extreme events. To answer these questions, the extreme value theory has been developed to analyze these types of occurrences, proposing specific distributions for this type of observation. One approach is to analyze this type of data and group data in maxima every n observations, and then to model the block maxima. A pioneering work in this area was done by [1], who presented an asymptotic result for distribution of maximum block n. There was not much progress in the area until the 1950s, when works such as those by [21] and [8] showed that the only non-trivial limiting distribution of affinely normalised maximum is the generalized extreme value (GEV) distribution.

A random variable X follows the GEV distribution if its cumulative distribution function (cdf) is given by

$$(1.1)\qquad G(x;\mu,\sigma,\xi) = \begin{cases} \exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0, \end{cases}$$

and is defined in the set $\{x : 1+\xi(x-\mu)/\sigma > 0\}$, where $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter and $\xi \in \mathbb{R}$ is a shape parameter. Thus, for $\xi > 0$, the expression just given for the cumulative distribution function is valid for $x > \mu - \sigma/\xi$, while for $\xi < 0$ it is valid for $x < \mu + \sigma/(-\xi)$. In the first case, at the lower end-point it equals 0; in the second case, at the upper end-point, it equals 1. For $\xi = 0$ the expression in (1.1) is interpreted by taking the limit as $\xi \to 0$. The probability density function (pdf) corresponding to (1.1) is given by

$$g(x;\mu,\sigma,\xi) = \begin{cases} \sigma^{-1}[1+\xi(x-\mu)/\sigma]^{-(1/\xi)-1}\exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \sigma^{-1}\exp[-(x-\mu)/\sigma]\exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0. \end{cases}$$

Estimates of extreme quantiles z_u of the annual maximum distribution are then obtained by inverting Equation (1.1)

$$z_u = \begin{cases} \mu + \frac{\sigma}{\xi}\left\{[-\log(u)]^{-\xi} - 1\right\}, & \xi \neq 0, \\ \mu - \sigma\log\left[-\log(u)\right], & \xi \to 0, \end{cases}$$

where $u \in [0,1]$.

In GEV distribution, if x^* is the upper limit of distribution G, according to [5] and [4] the shape parameter ξ satisfies

$$(1.2)\qquad \lim_{x\to x^*} \frac{1 - G(x;\mu,\sigma,\xi)}{x g(x;\mu,\sigma,\xi)} = \xi,$$

if $\xi > 0$ and $x^* = \infty$, and

$$\lim_{x \to x^*} \frac{1 - G(x; \mu, \sigma, \xi)}{(x - x^*)\, g(x; \mu, \sigma, \xi)} = \xi, \tag{1.3}$$

if $\xi < 0$ and $x^* < \infty$.

There has been an increased interest in defining new classes of univariate continuous distributions introducing additional shape parameters to the baseline model. In many applied areas such as lifetime analysis [7], environmental [17], medical [14], economy [11], there is a clear need for extended forms of the classical distributions, that is, new distributions which are more flexible to model real data in these areas since the data can present a high degree of skewness and kurtosis. In the context of extreme values, Papastathopoulos and Tawn (2013) studied three extensions of the generalised Pareto distribution. The extended distributions have attracted several statisticians to develop new models because the computational and analytical facilities available in standard softwares can easily tackle the problems involved in computing special functions in these extended distributions.

In recent years, several common distributions have been generalized via exponentiation. Let $G(x)$ be the cdf of any continuous baseline distribution. The cdf of the exponentiated-G distribution is defined by elevating $G(x)$ to the power α, say $F(x) = G(x)^{\alpha}$, where $\alpha > 0$ denotes an extra shape parameter. The baseline distribution is obtained as a special case when $\alpha = 1$. The pdf corresponding can be written as

$$f(x) = \alpha\, g(x)\, G(x)^{\alpha - 1}, \ x \in \mathbb{R}. \tag{1.4}$$

where $g(x)$ is the pdf of baseline distribution. Following this idea, [6] introduced the exponentiated exponential distribution as a generalization of the exponential distribution. In the same way, [13] proposed four more exponentiated distributions which generalize the gamma, Weibull, Gumbel and Fréchet distributions and provided some mathematical properties for each distribution. Several other authors have considered exponentiated distributions, for example, [12], [7], [20], [9] and [10]. Recently, [17] studied a broad family of univariate distributions through a particular case of Stacy's generalized gamma distribution. This new family stems from the general class: if $G(x)$ denotes the baseline cdf of a random variable, then a generalized class of distributions can be defined by

$$F(x) = 1 - \gamma\{\delta, -\log[G(x)]\}, \quad x \in \mathbb{R}, \ \delta > 0, \tag{1.5}$$

where

$$\gamma(\delta, z) = \frac{1}{\Gamma(\delta)} \int_0^z t^{\delta - 1} \mathrm{e}^{-t} \mathrm{d}t,$$

denotes the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. This family of distributions has pdf given by

$$f(x) = \frac{1}{\Gamma(\delta)} \{-\log[G(x)]\}^{\delta - 1}\, g(x).$$

[19] proposed a class of generalized distributions based on the transmutation map approach. Let F_1 and F_2 be the cdf's of two distributions with a common sample space. The general rank transmutation as given in [19] is defined as $G_{R_{12}}(u) = F_2(F_1^{-1}(u))$ and $G_{R_{21}}(u) = F_1(F_2^{-1}(u))$. Notice that the inverse cdf also known as quantile function is defined as $F^{-1}(y) = \inf_{x \in \mathbb{R}}\{F(x) \geq y\}$, for $y \in [0, 1]$. The functions $G_{R_{12}}(u)$ and $G_{R_{21}}(u)$ are both mapped in the unit interval $I = [0, 1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{R_{ij}}(0) = 0$ and $G_{R_{ij}}(1) = 1$, for $i = 1, 2$. A quadratic rank transmutation map is defined as $G_{R_{12}}(u) = u + \lambda\, u(1 - u), \ |\lambda| \leq 1$ from which follows that the cdf satisfies the relationship

$$F_2(x) = (1 + \lambda) F_1(x) - \lambda [F_1(x)]^2, \tag{1.6}$$

which on differentiation yields $f_2(x) = f_1(x)[1 + \lambda - 2\, \lambda\, F_1(x)]$, where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively.

The aim of this paper is to propose new modifications to GEV models that incorporate an additional parameter, with the hope that it will yield "better" results in certain practical situations. We create three new modifications for the GEV distribution: dual gamma GEV distribution, exponentiated GEV distribution and transmuted GEV distribution. The major benefit of these models is their ability to fit the skewed data better than GEV distribution.

The article is organized as follows. In Section 2, we define the dual gamma generalized extreme value (GGEV), exponentiated generalized extreme value (EGEV) and transmuted generalized extreme value (TGEV) distributions, derive the quantile functions of models and provide plots of such functions for selected parameter values. In Section 3, inference procedure is carried out under the Bayesian paradigm, with prior information playing an important role in the estimation procedures. Section 4 illustrates the method with a few simulated examples. Section 5 presents two applications to extreme data analysis. Concluding remarks are addressed in Section 6.

2. Construction of extreme value models

In this section, we present three new probability density functions that are generalizations of the GEV density. We illustrate the flexibility of these distributions and provide plots of the density function for selected parameter values.

2.1. The dual gamma generalized extreme value distribution (GGEV). Taking the GEV distribution as the baseline model in Equation (1.5), we have

$$F(x;\mu,\sigma,\xi,\delta) = \begin{cases} 1-\gamma(\delta,[1+\xi(x-\mu)/\sigma]^{-1/\xi}), & \xi\neq 0,\\ 1-\gamma(\delta,\exp[-(x-\mu)/\sigma]), & \xi\to 0, \end{cases} \tag{2.1}$$

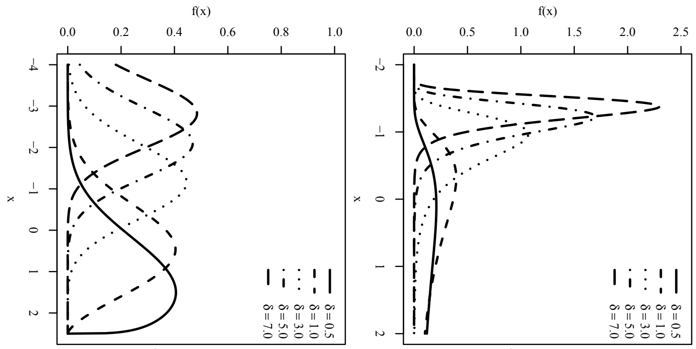
where $\delta>0$. The corresponding pdf has a very simple form

$$f(x;\mu,\sigma,\xi,\delta) = \begin{cases} \frac{\sigma^{-1}}{\Gamma(\delta)}[1+\xi(x-\mu)/\sigma]^{-(\delta/\xi)-1}\exp\left\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi\neq 0,\\ \frac{\sigma^{-1}}{\Gamma(\delta)}\exp\left\{-\delta[(x-\mu)/\sigma]\right\}\exp\left\{-\exp[-(x-\mu)/\sigma]\right\}, & \xi\to 0. \end{cases}$$

The quantile function of GGEV distribution is given by

$$z_u = \begin{cases} \mu+\frac{\sigma}{\xi}\left\{\left[Q^{-1}(\delta,(1-u))\right]^{-\xi}-1\right\}, & \xi\neq 0,\\ \mu-\sigma\log\left[Q^{-1}(\delta,(1-u))\right], & \xi\to 0, \end{cases}$$

where $u\in[0,1]$ and $Q^{-1}(\delta,u)$ is the inverse function of $Q(\delta,x)=\gamma(\delta,x)$. Some plots of the GGEV density functions are displayed in Figure 1. The case where $\delta=1$ is the particular case of standard GEV distribution. As this distribution has not the form of a GEV distribution, it not has some properties as max-stability. However, applications results shown that the flexibility of this class of distribution allow some predictive advantages compared with standard GEV.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 1. Plot for the GGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.1, we provide some useful properties of the GGEV distribution.

2.1. Proposition. *Let $X \sim (\mu, \sigma, \xi, \delta)$. Then, first moment, variance, skewness and kurtosis are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi\Gamma(\delta) + \sigma[\Gamma(\delta-\xi) - \Gamma(\delta)]}{\xi\Gamma(\delta)}$, *$\xi \neq 0$ and $\xi < \delta$. When $\xi = 0$, we have $\mathbb{E}(X) = \mu - \sigma\psi(\delta)$, where $\psi(\cdot)$ is derivative of the logarithm of the gamma function.*

(b) *$Var(X) = \dfrac{\sigma^2[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]}{\xi^2\Gamma^2(\delta)}$, $\xi \neq 0$ and $\xi < \delta/2$. When $\xi = 0$, $Var(X) = \sigma^2\psi(1,\delta)$.*

(c) $$\gamma_1 = \begin{cases} \frac{\Gamma^2(\delta)\Gamma(\delta-3\xi) - 3\Gamma(\delta)\Gamma(\delta-\xi)\Gamma(\delta-2\xi) + 2\Gamma^3(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^{3/2}}, & \textit{if } \xi > 0 \textit{ and } \xi < \delta/3, \\ -\frac{\Gamma^2(\delta)\Gamma(\delta-3\xi) - 3\Gamma(\delta)\Gamma(\delta-\xi)\Gamma(\delta-2\xi) + 2\Gamma^3(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^{3/2}}, & \textit{if } \xi < 0, \\ -\frac{\psi(2,\delta)}{[\psi(1,\delta)]^{3/2}}, & \textit{if } \xi = 0. \end{cases}$$

(d) $$\gamma_2 = \begin{cases} \frac{\Gamma^3(\delta)\Gamma(\delta-4\xi) - 4\Gamma^2(\delta)\Gamma(\delta-3\xi)\Gamma(\delta-\xi) + 6\Gamma(\delta)\Gamma(\delta-2\xi)\Gamma^2(\delta-\xi) - 3\Gamma^4(\delta-\xi)}{[\Gamma(\delta)\Gamma(\delta-2\xi) - \Gamma^2(\delta-\xi)]^2} - 3, & \textit{if } \xi \neq 0 \textit{ and } \xi < \frac{\delta}{4}, \\ \frac{3[\psi(1,\delta)]^2 + \psi(3,\delta)}{[\psi(1,\delta)]^2} - 3, & \textit{if } \xi = 0, \\ \infty, & \textit{if } \xi \geq \delta/4. \end{cases}$$

These results (a), (b), (c) and (d) are directly obtained from the definition of each measure.

2.1. Remark. The density function of X (GGEV distribution) can be expressed as

$$f(x;\mu,\sigma,\xi,\delta) = \frac{[1 + \xi(x-\mu)/\sigma]^{\frac{-(\delta-1)}{\xi}}}{\Gamma(\delta)} \cdot g(x;\mu,\sigma,\xi),$$

where $g(x;\mu,\sigma,\xi)$ is the pdf of the GEV distribution. The multiplying quantity $\frac{[1+\xi(x-\mu)/\sigma]^{\frac{-(\delta-1)}{\xi}}}{\Gamma(\delta)}$ works as a corrected factor for the pdf of the GEV distribution.

2.2. The exponentiated generalized extreme value distribution (EGEV). Now inserting (1.1) into (1.4) we obtain the pdf of exponentiated generalized extreme value (EGEV) distribution

$$f(x;\mu,\sigma,\xi,\alpha) = \begin{cases} \alpha\,\sigma^{-1}[1 + \xi(x-\mu)/\sigma]^{-(1/\xi)-1}\exp\left\{-\alpha[1 + \xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \alpha\,\sigma^{-1}\exp[-(x-\mu)/\sigma]\exp\left\{-\alpha\exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0. \end{cases} \tag{2.2}$$

The EGEV cdf can be expressed as

$$
F(x;\mu,\sigma,\xi,\alpha) \quad = \quad \begin{cases} \exp\left\{-\alpha[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\}, & \xi \neq 0, \\ \exp\left\{-\alpha \exp[-(x-\mu)/\sigma]\right\}, & \xi \to 0, \end{cases} \tag{2.3}
$$

where $\alpha > 0$. The quantile function corresponding to Equation (2.3) is

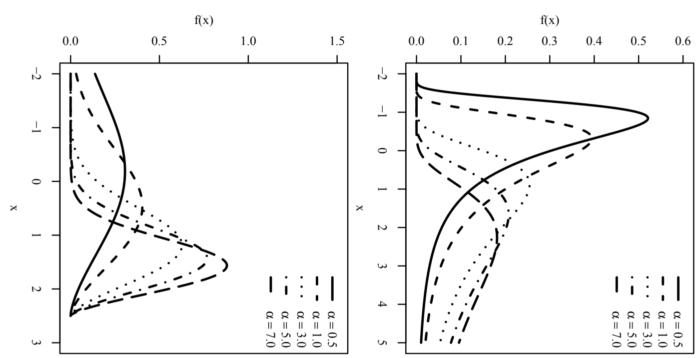
$$
z_u \quad = \quad \begin{cases} \mu + \frac{\sigma}{\xi}\left\{\left[-\frac{1}{\alpha}\log(u)\right]^{-\xi} - 1\right\}, & \xi \neq 0, \\ \mu - \sigma \log\left[-\frac{1}{\alpha}\log(u)\right], & \xi \to 0, \end{cases}
$$

where $u \in [0, 1]$.

2.2. Proposition. *The EGEV distribution is a particular case of GEV distribution. The Proof is shown in appendix.*

The result of this proposition hold important properties for EGEV distribution as for example the max-stability, and the shape parameter form obtained from Equations (1.2) and (2.5) is ξ.

Figure 2 displays some plots of the density function (2.2) for some parameter values. The case where $\alpha = 1$ is the particular case of standard GEV distribution.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 2. Plot for the EGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.3, we provide some useful properties of the EGEV distribution.

2.3. Proposition. *Let $X \sim (\mu, \sigma, \xi, \alpha)$. Then, first moment and variance are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi + \sigma[\alpha^{\xi}\Gamma(1-\xi) - 1]}{\xi}$, $\xi \neq 0$ *and* $\xi < 1$. *When* $\xi = 0$, *we have* $\mathbb{E}(X) = \mu + \sigma[\zeta + \ln \alpha]$, *where* $\zeta = 0.577215$ *is the Euler's constant.*

(b) *Var*$(X) = \dfrac{\sigma^2\alpha^{2\xi}[\Gamma(1-2\xi) - \Gamma^2(1-\xi)]}{\xi^2}$, $\xi \neq 0$ *and* $\xi < 1/2$. *When* $\xi = 0$, *the variance is Var*$(X) = \dfrac{\pi^2\sigma^2}{6}$.

These results (a) and (b) are directly obtained from the definition of each measure.

2.3. The transmuted generalized extreme value distribution (TGEV). If $G(x)$ is the GEV cumulative distribution in (1.1), then, applying it in the function (1.6), the transmuted generalized extreme value (TGEV) cumulative distribution is given by

$$F(x;\mu,\sigma,\xi,\lambda) \quad = \quad \begin{cases} (1+\lambda)\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\} - \lambda\exp\left\{-2[1+z\,\xi]^{-1/\xi}\right\}, \xi \neq 0, \\ (1+\lambda)\exp\left\{-\exp[-z]\right\} - \lambda\exp\left\{-2\exp[-z]\right\}, \ \xi \to 0. \end{cases} \tag{2.4}$$

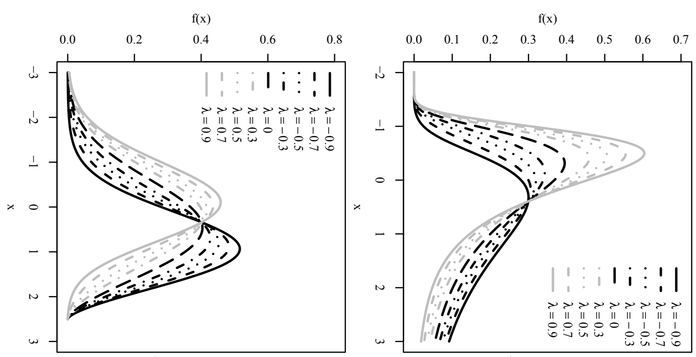
The corresponding pdf is

$$f(x;\mu,\sigma,\xi,\lambda) = \begin{cases} \dfrac{\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\}[1+\lambda-2\lambda\exp\left\{-[1+z\,\xi]^{-1/\xi}\right\}]}{\sigma[1+z\,\xi]^{1+(1/\xi)}}, \xi \neq 0, \\ \dfrac{\exp[-z]\exp\left\{-\exp[-z]\right\}[1+\lambda-2\lambda\exp\left\{-\exp[-z]\right\}]}{\sigma}, \xi \to 0, \end{cases}$$

where $\lambda \in [-1,1]$, $z = (x-\mu)/\sigma$. The quantile function of TGEV distribution, say z_u, is given by

$$z_u \quad = \quad \begin{cases} \mu + \frac{\sigma}{\xi}\left\{\left[-\log\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\,\lambda\,u}}{2\,\lambda}\right]\right]^{-\xi} - 1\right\}, \quad \xi \neq 0, \\ \mu - \sigma\log\left\{-\log\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\,\lambda\,u}}{2\,\lambda}\right]\right\}, \qquad \xi \to 0, \end{cases}$$

for $\lambda \neq 0$ and $u \in [0,1]$. In Figure 3, we plot the density of the TGEV distribution for selected parameter values. The case where $\lambda = 0$ is the particular case of standard GEV distribution.



(a) $\xi = -0.4$ (b) $\xi = 0.4$

Figure 3. Plot for the TGEV density for some parameter values; $\mu = 0$ and $\sigma = 1$.

In the Proposition 2.4, we provide some useful properties of the TGEV distribution.

2.4. Proposition. *Let $X \sim (\mu,\sigma,\xi,\alpha)$. Then, first moment and variance are given by*

(a) $\mathbb{E}(X) = \dfrac{\mu\xi + \sigma[(1+\lambda-2^{\xi}\lambda)\Gamma(1-\xi)-1]}{\xi}$, $\xi \neq 0$ *and* $\xi < 1$. *When* $\xi = 0$ *we have* $\mathbb{E}(X) = (\mu+\sigma\zeta) - \lambda\sigma\ln 2$.

(b) $\mathrm{Var}(X) = \dfrac{\sigma^2\{-[\lambda(2^{\xi}-1)-1]^2\Gamma^2(1-\xi) - [\lambda(4^{\xi}-1)-1]\Gamma(1-2\xi)\}}{\xi^2}$, $\xi \neq 0$ and $\xi < 1/2$. When $\xi = 0$ we have $\mathrm{Var}(X) = \sigma^2\left\{\dfrac{\pi^2}{6} - \lambda(1+\lambda)[\ln 2]^2\right\}$.

These results (a) and (b) are directly obtained from the definition of each measure.

2.5. Proposition. *The density function of X (TGEV) can be expressed as an finite linear combination of densities of GEV(μ,σ,ξ) and EGEV$(\mu,\sigma,\xi,2)$ density functions, i.e.,*

$$f(x;\mu,\sigma,\xi,\lambda) = \beta\cdot g(x;\mu,\sigma,\xi) + (1-\beta)\cdot z(x;\mu,\sigma,\xi,2),$$

where $\beta = 1+\lambda$ and $z(x)$ is pdf of EGEV distribution.

2.1. Corollary. If $\lambda = -1$, then $X \sim$ EGEV$(\mu,\sigma,\xi,2)$.

2.6. Proposition. *If f is the density function of TGEV distribution, with cumulative function F, then*

$$\lim_{x\to x^*}\frac{1-F(x;\mu,\sigma,\xi,\lambda)}{xf(x;\mu,\sigma,\xi,\lambda)} = \xi$$

if $\xi > 0$ and $x^* = \infty$, and

$$\lim_{x\to x^*}\frac{1-F(x;\mu,\sigma,\xi,\lambda)}{(x-x^*)\,f(x;\mu,\sigma,\xi,\lambda)} = \xi, \tag{2.5}$$

if $\xi < 0$ and $x^* < \infty$.

The proof is shown in Appendix.

2.4. Return levels. In extreme values studies, is important to know with which probability a rare event can occur in the next periods of time, or every how many years is expected an event higher than r. For this, we can calculate the return level for every t periods of time. Specifically, the return level r_t is related to the quantile $1-1/t$ of the distribution of extreme values. Thus, for each of the three generalizations the return levels are given by $r_t = z_{1-1/t}$. In Bayesian estimation, as sampled points of the parameters for the respective posteriors, they sampled points with the return levels, obtaining a posterior distribution for r_t. We can verify some relationship between the standard GEV distribution and its generalizations.

2.7. Proposition. *Let $r_{EGEV,t}$ the return level for EGEV distribution with parameters (μ,σ,ξ,α) and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ,σ,ξ). Then*

(1) *If $\alpha > 1$, then $r_{GEV,t} < r_{EGEV,t}$.*
(2) *If $\alpha < 1$, then $r_{GEV,t} > r_{EGEV,t}$.*

The proof is shown in the appendix

2.8. Proposition. *Let $r_{GGEV,t}$ the return level for GGEV distribution with parameters (μ,σ,ξ,δ) and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ,σ,ξ). Then*

(1) *If $\delta > 1$, then $r_{GEV,t} > r_{GGEV,t}$.*
(2) *If $\delta < 1$, then $r_{GEV,t} < r_{GGEV,t}$.*

The proof is shown in the appendix

2.9. Proposition. *Let $r_{TGEV,t}$ the return level for TGEV distribution with parameters (μ,σ,ξ,λ) and let $r_{GEV,t}$ the return level for the GEV distribution with parameters (μ,σ,ξ). Then*

(1) *If $\lambda > 0$, then $r_{GEV,t} > r_{TGEV,t}$.*
(2) *If $\lambda < 0$, then $r_{GEV,t} < r_{TGEV,t}$.*

The proof is shown in the appendix

3. Estimation and inference

3.1. Maximum likelihood estimation. In this section, we discuss maximum likelihood estimation for the new models. We present the log-likelihood function for all models considering the case $\xi \neq 0$. Thus, the log-likelihood are given by

$$\ell^{\text{GGEV}}(\boldsymbol{\theta}_1) = -n\,\log(\sigma\,\Gamma(\delta)) - (\delta/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi},$$

$$\ell^{\text{EGEV}}(\boldsymbol{\theta}_2) = n\,\log(\alpha/\sigma) - (1/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \alpha\sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi},$$

$$\begin{aligned}\ell^{\text{TGEV}}(\boldsymbol{\theta}_3) &= -n\,\log(\sigma) - (1/\xi+1)\sum_{i=1}^{n}\log[1+\xi(x_i-\mu)/\sigma] - \sum_{i=1}^{n}[1+\xi(x_i-\mu)/\sigma]^{-1/\xi} \\ &+ \sum_{i=1}^{n}\log\left(1+\lambda-2\,\lambda\,\exp\left\{-[1+\xi(x_i-\mu)/\sigma]^{-1/\xi}\right\}\right),\end{aligned}$$

where $\boldsymbol{\theta}_1 = (\mu, \sigma, \xi, \delta)$, $\boldsymbol{\theta}_2 = (\mu, \sigma, \xi, \alpha)$ and $\boldsymbol{\theta}_3 = (\mu, \sigma, \xi, \lambda)$, provided that

$$1 + \xi(x_i - \mu)/\sigma > 0, \text{ for } i = 1, \ldots, n.$$

At parameter combinations for which the above result is violated, corresponding to a configuration for which at least one of the observed data falls beyond an end-point of the distribution, the likelihood is zero and the log-likelihood equals $-\infty$.

3.2. Bayesian analysis. In this work, we use the Bayesian paradigm to estimate the posterior parameters of these new class of distributions. We proposed vague prior distributions for the parameters, and perform the estimation combining the information of prior and the likelihood function to provide the posterior points. We have the posterior points by Markov chain Monte Carlo (MCMC) [3]. Base on the parametric space of the parameters, we proposed the following priors:

- $\mu \sim N(\mu_0, \sigma_0^2)$, μ_0 and σ_0^2 known;
- $\sigma \sim \Gamma(a_1, b_1)$, a_1 and b_1 known;
- $\xi \sim N(\mu_\xi, \sigma_\xi^2)$, μ_ξ and σ_ξ^2 known;
- $\delta \sim \Gamma(a_2, b_2)$, a_2 and b_2 known;
- $\alpha \sim \Gamma(a_3, b_3)$, a_3 and b_3 known;
- $\lambda \sim U(-1, 1)$.

Considering a case with a non-informative prior to the parameters, we consider $\mu_0 = \mu_\xi = 0$, $\sigma_0^2 = 1000$, σ_ξ^2, $a_i = 0.001$, $b_i = 0.001$, $i = 1, 2, 3$. Posterior points can be performed using MCMC algorithms. As we not have a closed form for the full conditional distributions for all the three cases, we use the Metropolis-Hastings algorithm technique of sampling.

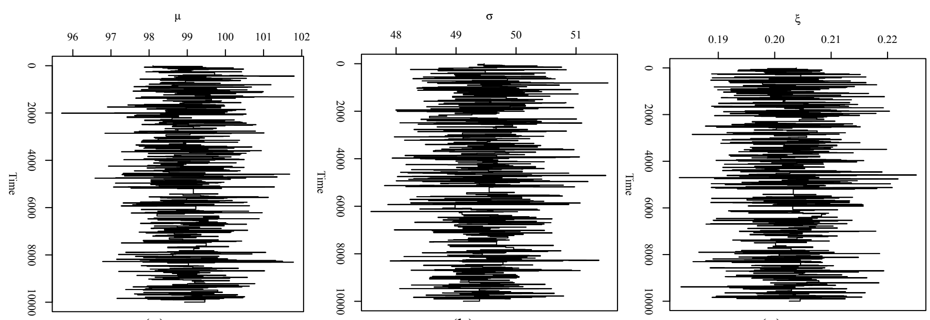
4. Simulation study

Simulations was performed in different configuration of the parameters, from the three extensions and the standard GEV distribution. The aim of this section if verify the ability of the estimation fits correctly the value of the parameters, in differents points of the generalization parameter. We performed all simulations with fixed (μ, σ, ξ) parameters at points $(100, 50, 0.2)$. For the δ of GGEV and α of EGEV, we used the values $(0.5, 2.0)$. For the λ of TGEV, we simulated points with $(-0.9, 0.9)$.

Table 1 shows the Bayes estimator with respect to quadratic loss (posterior mean), and credibility intervals of 95%. In all simulations, the posterior mean is near to the true value, and only for the parameter σ for the EGEV model, with $\alpha = 2.0$, the true value have been out of the credibility interval. The length of the intervals are similar between models, indicating that they have the same accuracy. For the GEV standard model, the credibility intervals was $(99.92; 101.97)$ for μ, $(49.78, 51.41)$ for σ and $(0.190, 0.218)$ for ξ. Figures 4-6 shows trace plots of the parameters for three simulations. In all cases, we observe that a stationary distribution whose central measure coincides with the true value.

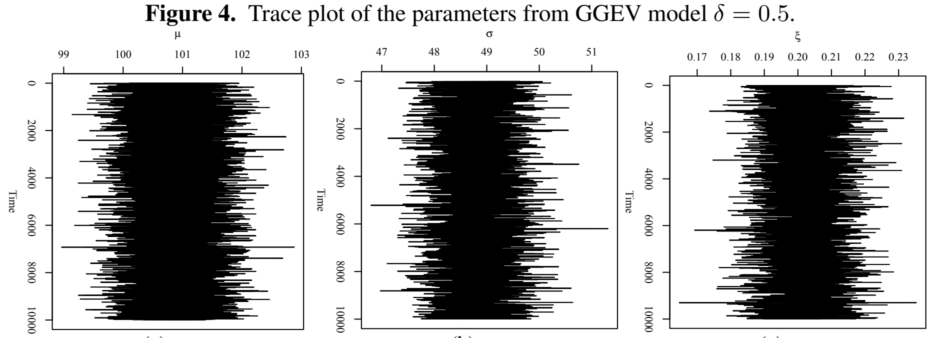
Table 1. Posterior means and 95% posterior credibility intervals for simulated data in the parameters models.

	EGEV, $\alpha = 0.5$		GGEV, $\delta = 0.5$		TGEV, $\lambda = -0.9$	
	Mean	95% C.I.	Mean	95% C.I.	Mean	95% C.I.
μ	99.4	(98.1; 100.7)	99.0	(97.6; 100.5)	100.4	(99.2; 101.4)
σ	49.7	(48.7; 50.7)	49.5	(48.4; 50.6)	50.6	(49.6; 51.7)
ξ	0.197	(0.182; 0.213)	0.203	(0.191; 0.214)	0.205	(0.189; 0.220)
	EGEV, $\alpha = 2.0$		GGEV, $\delta = 2.0$		TGEV, $\lambda = 0.9$	
	Mean	95% C.I.	Mean	95% C.I.	Mean	95% C.I.
μ	100.9	(99.9; 101.8)	100.3	(99.3; 101.3)	99.8	(99.0; 100.7)
σ	48.8	(47.8; 49.8)	50.1	(49.0; 51.2)	49.6	(48.8; 50.6)
ξ	0.202	(0.187; 0.218)	0.198	(0.176; 0.223)	0.181	(0.159; 0.203)



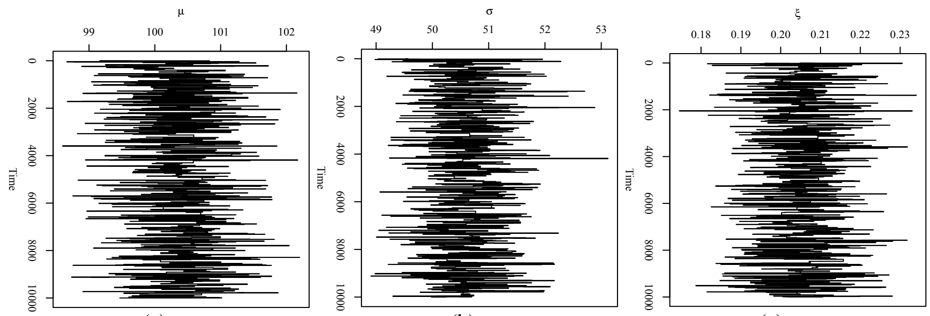
(a) (b) (c)

Figure 4. Trace plot of the parameters from GGEV model $\delta = 0.5$.



(a) (b) (c)

Figure 5. Trace plot of the parameters from EGEV model, $\alpha = 2.0$.



(a) (b) (c)

Figure 6. Trace plot of the parameters from TGEV model, $\lambda = 0.9$.

Figure 7 show the return level plot for standard GEV estimation. Altouthg this plot is common in literature of extremes, the objective of draw it in this work is to compare the GEV returns against it's generalizations.

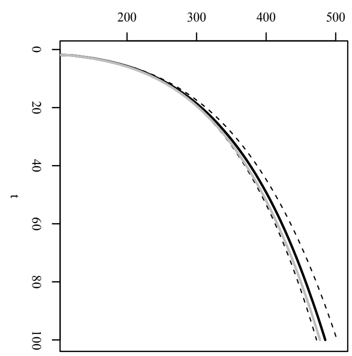


Figure 7. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated GEV model.

The Figures 8-10 show the posterior mean of the expected return levels, from $t = 2$ to 100, for each model simulated. From these figures, it is observed that for all simulations the true values of the returns are within the credibility interval, and the estimation is more accurate for GGEV model, where the line of the posterior mean returns is the nearest line of the true returns. The TGEV model is the less accurate model about the returns, which presents larger distance between the line of the mean and the true return, even so it is within the credibility interval. Comparing the three extensions proposed in this work with the returns of standard GEV, it is observed that increasing the parameter α in EGEV implies increasing values of returns, although even increasing $\alpha = 2.0$ we have similar results compared with standard GEV. About the GGEV returns, when δ decrease, we have a tail much more heavy than the standard GEV model. For TGEV model increase λ imply in lower returns values.

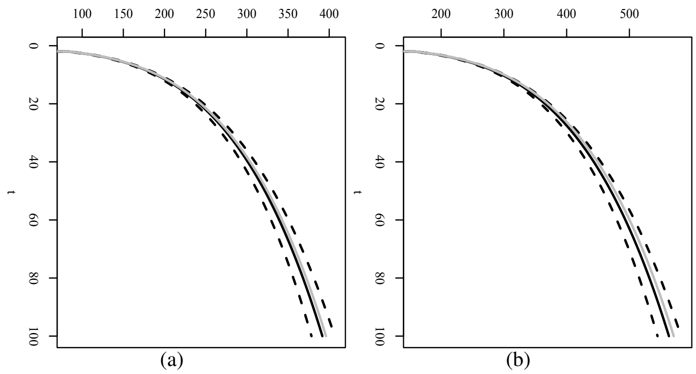


Figure 8. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated EGEV model with $\alpha = 0.5$ (a) and $\alpha = 2.0$ (b). The grey line is the true return of the model.

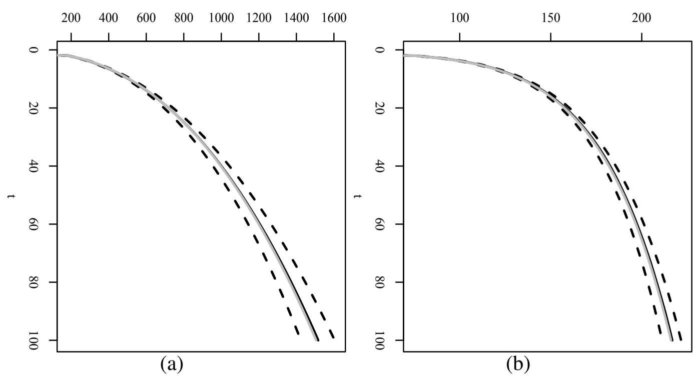


Figure 9. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated GGEV model with $\delta = 0.5$ (a) and $\delta = 2.0$ (b). The grey line is the true return of the model.

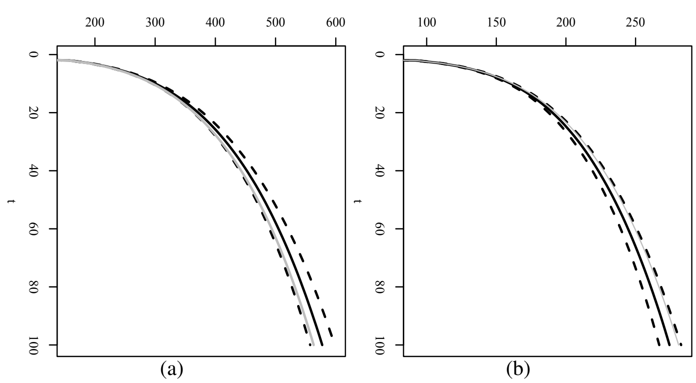


Figure 10. Posterior mean and 95% posterior credibility intervals of the return level plot for simulatated TGEV model with $\lambda = -0.9$ (a) and $\lambda = 0.9$ (b). The grey line is the true return of the model.

5. Applications to real data

We conduct two applications with maxima are analyzed of the three extensions to real data for illustrative purpose. The first example is a data set that consists of monthly maxima quota of Gurgueia River, located in the State of Piauí, Brazil. A river quota is the height of the water in the section relative to a given reference. Conventionally the quotas are measured in centimeters (cm). Large quota values can cause floods in the regions close to the rivers. Daily data was collected from 1975 to 2012. We analyse the maximum for each every 30 days. The second application consist to analyse rainfall data in Barcelos Station, located in the North of Portugal. The daily data was collected daily from 1931 to 2008, and we analysed the maxima of each 30 days.

In both the modeling was done using the GEV and its three generalizations proposed in this work. About the additional parameter, identifiability problems were detected in the estimation of the parameters. In this case, we created a grid of possible values for the additional parameter to estimate the other parameters using the Bayesian approach for each point of the grid, and choose the one grid point that has the lowest $-2\ell(\boldsymbol{\theta})$, that is the primitive measure to calculate BIC and DIC. After choosing the best point for each of the three generalizations, they are compared with the standard GEV, to decide what the best model that fits each of the applications. In both application, the dual gamma extension showed be the best model. Figure 11 shows the measure for a grid of points for the applications. In the Gurgueia river quota, the best fit was when $\delta = 0.06$, while for the Barcelos rainfall data, the best fit measure was when $\delta = 0.26$. For the exponentiated and transmuted generalizations, the best additional parameter in the grid of points was points near the standard GEV case.

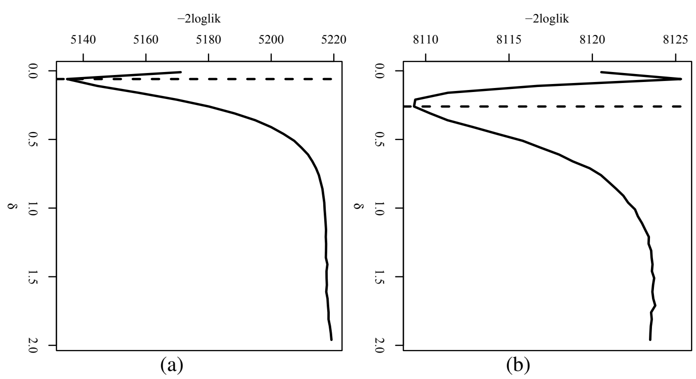


Figure 11. $-2\ell(\boldsymbol{\theta})$ for a grid of points to the δ of GGEV distribution. (a) Gurgueia river quota and (b) Barcelos rainfall.

Table 2 provides the BIC and DIC for the GEV and it's three generalizations proposed in this work. In general, the smaller the values of these statistics, the better the fit. We can note an advantage of the Dual-Gamma Generalization, followed by the standard GEV distribution. Table 3 shows de 95% credible interval for the parameters. We can verify a high value in location and scale parameter, and a negative value of the shape, indicating that the data has a lighted tail.

Figure 12 shows the return level plot for the applications. From this figure we can verify that, for gurgueia river quota, the returns of the model GGEV grow more slowly than the returns of the GEV, being more similar to the behavior of empirical returns. Based on the GGEV model, a return higher than 500cm once every $t = 20$ periods is expected. Each $t = 100$ periods of time, is waiting at least once a maximum higher than 604cm. For the Barcelos rainfall data, the return levels from the GGEV model is more similar than the return levels of the GEV model. Each $t = 20$ periods of time, is expected a return level higher than 76mm, while that for $t = 100$ periods of time, is expected that once the level would be equal or higher than 101mm. Figure 13 shows the predictive distribution for the applications based in GGEV model. We can verify a good fit for this model.

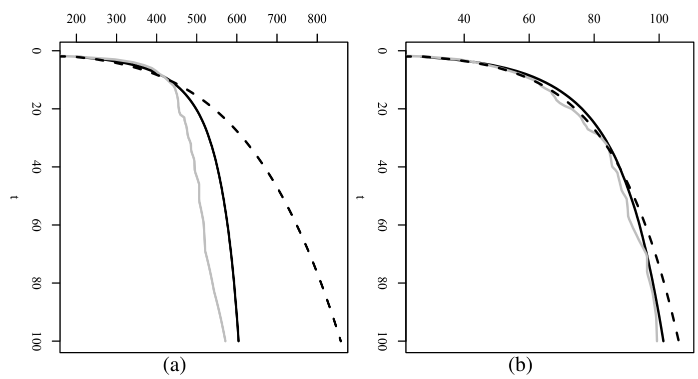


Figure 12. Return level plot for the applications. (a) Gurgueia river and (b) Barcelos Station, for the GGEV (full line), GEV (dotted Line:) and Empirical (grey line).

Table 2. BIC and DIC measures for applications

	River Quota at Gurgueia, Brazil				Rainfall at Barcelos, Portugal			
Model	GEV	EGEV	GGEV	TGEV	GEV	EGEV	GGEV	TGEV
DIC	5217	5215	5137	5218	8123	8122	8110	8123
BIC	5233	5239	5160	5241	8141	8148	8136	8149

Like the previous application, the best model pointed was the GGEV model, according to the Table 2. As the river data, in rainfall data, the model presents a lighted tail behavior, by the negative behavior of ξ (see Table 3).

Table 3. Mean ane 95% credibility intervals for parameters for the applications.

	River Quota at Gurgueia, Brazil		
Parameter	μ	σ	ξ
M (CI)	44.74 (38.35; 50.33)	14.37 (12.99; 16.08)	-0.020 (-0.023; -0.016)
	Rainfall at Barcelos, Portugal		
Parameter	μ	σ	ξ
M (CI)	5.21 (4.15; 6.20)	7.60 (7.13; 8.19)	-0.042 (-0.053; -0.031)

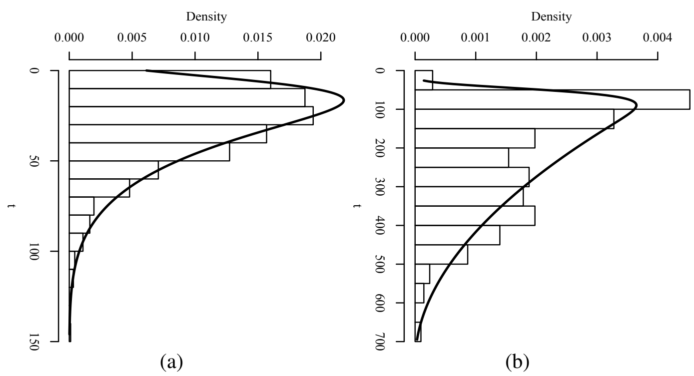


Figure 13. Predictive density for Barcelos station and Gurgueia river, respectively.

6. Concluding remarks

In this paper, we proposed three extensions to the GEV distribution, with an additional parameter which modifies the behavior of the distribution, composing as a alternative models for single maxima events. In each generalization, the GEV distribution appears as a particular case. We performed the modelling under a Bayesian approach and the estimation of the parameters was proposed using the MCMC algorithm. The results of simulations show that the proposed method is efficient in recovering the true values of the parameters of generalizations, which credibility intervals were obtained with great accuracy in relation to the true parameter estimation. In fact, the three generalizations can be used to fit real data, where in both applications, the best model according to the fit measure was the generalization of GGEV model. These generalizations can be applied to any kinds of environmental data that involves the analysis of maxima.

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Appendix

MCMC Algorithm. For the additional parameters (α for the exponentiated, δ for the dual Gamma, and λ for the transmuted, we propose a Grid of points, and perform the Bayesian estimation via MCMC for each point of the Grid. The point of the grid with best Goodness of fit is the choosen point, denoted α^*, δ^* and λ^* for α and δ, the Grid is from 0.01 to 2, with intervals of 0.05 (the case 1.00 is the standard GEV). For λ, the Grid is from -0.9 to 0.9, with intervals of 0.1 (the case 0 is the standard GEV).

After choose the best point in grid for each case, the parameters (μ, σ, ξ) are sampled using the Metropolis-Hastings algorithm. Details of the MCMC sampling scheme are given below. At iteration s, parameters are updated as follows:

Sampling $\Theta = (\mu, \sigma, \xi)$: Propose new values for these parameters where $\mu^* \sim N(\mu^{(s)}, V_\mu)$, $\xi^* \sim N(\xi^{(s)}, V_\xi) I_{-0.5,\infty}(\xi)$ and $\sigma^* \sim \text{Gamma}(\sigma^{2(s)}/V_\sigma, \sigma^{(s)}/V_\sigma)$. Accept the new values $\Theta^{(s+1)} = \Theta^*$ with probability α_Θ, where

$$\alpha_\Theta = \min\left\{1, \frac{\pi(\Theta^*|\mathbf{x}) f_N(\mu^{(s)} \mid \mu^*, V_\mu) f_N(\xi^{(s)} \mid \xi^*, V_\xi) f_G(\sigma^{(s)} \mid \sigma^{*2}/V_\sigma, \sigma^*/V_\sigma)}{\pi(\Theta^{(s)}|\mathbf{x}) f_N(\mu^* \mid \mu^{(s)}, V_\mu) f_N(\xi^* \mid \xi^{(s)}, V_\xi) f_G(\sigma^* \mid \sigma^{(s)2}/V_\sigma, \sigma^{(s)}/V_\sigma)}\right\},$$

where $\pi(\Theta^*|\mathbf{x})$ is the posterior density given by the combination with the likelihood and the prior distribution given in Section 3.

Proof of the propsition 2.2. Let $X \sim EGEV(\mu,\sigma,\xi,\delta)$. Then, for $\xi \neq 0$

$$\begin{aligned} F(x;\mu,\sigma,\xi,\alpha) &= \exp\left\{-\alpha[1+\xi(x-\mu)/\sigma]^{-1/\xi}\right\} = \exp\left\{-[\alpha^{-\xi}+\alpha^{-\xi}\xi(x-\mu)/\sigma]^{-1/\xi}\right\} \\ &= \exp\left\{-\left[1+\frac{\alpha^{-\xi}\xi\left[x-\mu+\frac{\sigma(\alpha^{-\xi}-1)}{\xi\alpha^{-\xi}}\right]}{\sigma}\right]^{-1/\xi}\right\} \\ &= \exp\left\{-\left[1+\frac{\xi}{\sigma/\alpha^{-\xi}}\left(x-\left(\mu+\frac{\sigma}{\xi}(\alpha^{\xi}-1)\right)\right)\right]^{-1/\xi}\right\} \end{aligned}$$

which is the cumulative distribution function of $GEV(\mu',\sigma',\xi)$, where $\mu' = \left(\mu+\frac{\sigma}{\xi}(\alpha^{\xi}-1)\right)$ and $\sigma' = \sigma/\alpha^{-\xi}$.

The proof for the case $\xi = 0$ is similar, where $EGEV(\mu,\sigma)$ is a $GEV(\mu',\sigma)$, where $\mu' = \mu + \sigma\log(\alpha)$.

Proof of the proposition 2.6. If f and F are respectively the density and cumulative distribution of TGEV distribution, they can be written in function of a standard GEV distribution with density g and cumulative function G, weighted by a λ parameter, as written in (1.6). Then, the shape parameter of TGEV distribution, for the case where $\xi > 0$ is given by

$$\begin{aligned} \lim_{x\to x^*}\frac{1-F(x;\mu,\sigma,\xi,\lambda)}{xf(x;\mu,\sigma,\xi,\lambda)} &= \lim_{x\to x^*}\frac{[1-(1+\lambda)\,G(x;\mu,\sigma,\xi)+\lambda\,G(x;\mu,\sigma,\xi)^2]}{xg(x;\mu,\sigma,\xi)\,(1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi))} \\ &= \lim_{x\to x^*}\frac{[1-G(x;\mu,\sigma,\xi)][1-\lambda G(x;\mu,\sigma,\xi)]}{xg(x;\mu,\sigma,\xi)\,[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} \\ &= \lim_{x\to x^*}\frac{1-G(x;\mu,\sigma,\xi)}{xg(x;\mu,\sigma,\xi)}\lim_{x\to x^*}\frac{[1-\lambda\,G(x;\mu,\sigma,\xi)]}{[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} \\ &= \xi\frac{\lim_{x\to x^*}[1-\lambda\,G(x;\mu,\sigma,\xi)]}{\lim_{x\to x^*}[1+\lambda-2\,\lambda\,G(x;\mu,\sigma,\xi)]} = \xi\frac{(1-\lambda)}{(1-\lambda)} = \xi. \end{aligned}$$

The proof for the case $\xi < 0$ is similar using (2.5).

Proof of the proposition 2.7. By simplicity of notation, consider $\Delta = \left[1+\frac{\xi(x-\mu)}{\sigma}\right]^{-1/\xi}$ for $\xi \neq 0$ and $\Delta = \exp\{-\frac{(x-\mu)}{\sigma}\}$ for $\xi = 0$. The cdf function of GEV distribution in (1.1) can be written as $G(x;\mu,\sigma,\xi) = \exp\{-\Delta\}$ and the cdf of EGEV in (2.3) can be written by $F(x;\mu,\sigma,\xi,\alpha) = \exp\{-\alpha\Delta\}$.

Then, for $\alpha > 1$, $G(x;\mu,\sigma,\xi) > F(x;\mu,\sigma,\xi,\alpha)$ and $\forall\ x$, $r_{GEV,t} < r_{EGEV,t}$. For $\alpha < 1$, $G(x;\mu,\sigma,\xi) < F(x;\mu,\sigma,\xi,\alpha)$ and $\forall\ x$, $r_{GEV,t} > r_{EGEV,t}$.

Proof of the proposition 2.8. The cdf of GGEV distribution in (2.1) can be rewritten as $F(x;\mu,\sigma,\xi,\delta) = 1 - F_G(\Delta;\delta,1)$, where F_G is the cdf of a Gamma distribution. Similarly, the GEV cdf in (1.1) can be rewritten as $G(x;\mu,\sigma,\xi) = 1 - F_G(\Delta;1,1)$.

For $\delta > 1$, $F_G(.;\delta,1) < F_G(.;1,1)$ and then $F(x;\mu,\sigma,\xi,\delta) > G(x;\mu,\sigma,\xi)$, which implies that $r_{GEV,t} > r_{GGEV,t}$. When $\delta < 1$, $F_G(.;\delta,1) > F_G(.;1,1)$ and then $F(x;\mu,\sigma,\xi,\delta) < G(x;\mu,\sigma,\xi)$, which implies that $r_{GEV,t} < r_{GGEV,t}$.

Proof of the proposition 2.9. Given the cdf $G(x;\mu,\sigma,\xi)$ of the GEV distribution in (1.1), the cdf of TGEV in (2.4) can be rewritten as

$$F(x;\mu,\sigma,\xi,\lambda) = G(x;\mu,\sigma,\xi) + \lambda\, G(x;\mu,\sigma,\xi)[1 - G(x;\mu,\sigma,\xi)].$$

Then, for $\lambda < 0$, $F(x;\mu,\sigma,\xi,\lambda) < G(x;\mu,\sigma,\xi)$ and then $r_{GEV,t} < r_{TGEV,t}$. When $\lambda > 0$, $F(x;\mu,\sigma,\xi,\lambda) > G(x;\mu,\sigma,\xi)$ and then $r_{GEV,t} > r_{TGEV,t}$.