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A comment on "Generating matrix functions for Chebyshev matrix polynomials of the second type"

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Abstract

In this comment we will demonstrate that one of the main formulas given in Ref. [1] is incorrect.

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1. Introduction and motivation

It is well known that for a class of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ the so-called "generating functions" of this class are an useful tool for their study. A generating function is a function $F(x,t)$, analytic for some set $D \subset \mathbb{C}^2$, so that

$$F(x,t) = \sum_{n=0}^{\infty} \alpha_n P_n(x) t^n, (x,t) \in D.$$

For example, $F(x,t) = \exp 2xt - t^2$ is the generating function for Hermite polynomials $\{H_n(x)\}_{n\geq 0}$ because we can write:

$$F(x,t) = \exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n, \forall (x,t) \in \mathbb{C}^2.$$

The extension to the matrix framework for the classical families of Jacobi [3], Hermite [2], Gegenbauer [4], Laguerre [5] and Chebyshev [6] polynomials was made in recent years, and properties and applications of different classes for these matrix polynomials have been studied in several papers [7, 8, 10, 11, 9].

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In the matrix case, the importance of the generating function is similar to the scalar case, taking into account the possible spectral restrictions (for a matrix $A \in \mathbb{C}^{N\times N}$ we will denote by $\sigma(A)$ the spectrum of the matrix $\sigma(A) = \{z; z \text{ is a eigenvalue of } A\}$). For example:

- Laguerre matrix polynomials. If A is a matrix in $\mathbb{C}^{N\times N}$ such that $-k \notin \sigma(A)$ for every integer $k > 0$, and λ is a complex number with $\mathrm{Re}(\lambda) > 0$, the generating function [5] is given by:

$$(1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n, \forall\ x, t \in \mathbb{C}, |t| < 1.$$

- Hermite matrix polynomials. If A is a matrix in $\mathbb{C}^{N\times N}$ such that $\mathrm{Re}(z) > 0, \forall z \in \sigma(A)$, the generating function [2] is given by

$$e^{xt\sqrt{A} - t^2 I} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, A) t^n \ ,\ (x,t) \in \mathbb{R}^2$$

2. The detected mistake. An illustrative example

Recently, in Ref.[1], a generating matrix function for Chebyshev matrix polynomials of the second kind is presented. In theorem 2.1 of [1, p.27], the following formula (2.1) is established:

$$\sum_{n=0}^{\infty} U_n(x, A) t^n = \left(I - \sqrt{2A}xt + t^2 I\right)^{-1} \ ,\ |t| < 1, |x| < 1, \quad (2.1)$$

where I denotes the identity matrix of order N, matrix $A \in \mathbb{C}^{N\times N}$ satisfies $\mathrm{Re}(\lambda) > 0$ for all eigenvalue $\lambda \in \sigma(A)$ and $\left\|\sqrt{A}\right\| < 1/\sqrt{2}$. This formula (2.1) turns out to be the key for the development of the properties mentioned in the paper [1]. However, we will see that formula (2.1) is incorrect. For this, we only need to show that the matrix function $\left(I - \sqrt{2A}xt + t^2 I\right)$, regarded as a entire function of the complex variables x and t, is singular for some values of x and t under the previous hypotheses. For example, we consider $N = 2$, and the matrix

$$A = \begin{pmatrix} \frac{3}{16} + \frac{i}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \in \mathbb{C}^{2\times 2},$$

where $i^2 = -1$. Obviously, $\sigma(A) = \left\{\frac{3}{16} + \frac{i}{4}, \frac{1}{4}\right\}$ and A satisfies condition $\mathrm{Re}(\lambda) > 0$ for all eigenvalue $\lambda \in \sigma(A)$. It is easy to prove that

$$\sqrt{A} = \begin{pmatrix} \frac{1}{2} + \frac{i}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

which evidently satisfies $\sqrt{A}\sqrt{A} = A$, and condition $\left\|\sqrt{A}\right\| = \sqrt{5}/4 \approx 0.559017 < 1/\sqrt{2} \approx 0.707107$ holds.

It is easy to compute

$$\sqrt{2A} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which evidently satisfies $\sqrt{2A}\sqrt{2A} = 2A$, and then one gets:

$$I_{2\times 2} - \sqrt{2}Axt + t^2 I_{2\times 2} = \begin{pmatrix} 1 + t^2 - \frac{tx}{\sqrt{2}} - \frac{txi}{2\sqrt{2}} & 0 \\ 0 & 1 + t^2 - \frac{tx}{\sqrt{2}} \end{pmatrix}.$$

Taking, for example, the values

$$x = \frac{1}{2}, t = \frac{1}{16}\left(-\sqrt{-250+8i} + (2+i)\sqrt{2}\right) \approx 0.160967 - 0.89995i,$$

this choices meet the restrictions outlined ($|x| = \frac{1}{2} < 1, |t| = 0.914232 < 1$), but the term $1 + t^2 - \dfrac{tx}{\sqrt{2}} - \dfrac{txi}{2\sqrt{2}}$ is zero and matrix $I_{2\times 2} - \sqrt{2}Axt + t^2 I_{2\times 2}$ has a column of zeros, thus is singular. Thus (2.1) is meaningless. ‡

Therefore, I ask the authors of Ref. [1] to clarify the domain of choice for the variables x, t in formula (2.1) in order to guarantee the validity of the remaining formulas which are derived from (2.1) and used in the remainder of the aforementioned paper.

References

[1] Altin, A and Çekim, B. *Generating matrix functions for Chebyshev matrix polynomials of the second type*, Hacettepe Journal of Mathematics and Statistic **41 (1)**, 25-32, 2012.

[2] Jódar L. and Company R. *Hermite matrix polynomials and second order matrix differential equations*, Journal of Approximation Theory and Applications **12 (2)**, 20-30, 1996.

[3] Defez E., Jódar L. and Law A.G. *Jacobi matrix differential equation, polynomial solution and their properties*, Computers and Mathematics with Applications **48**, 789-803, 2004.

[4] Jódar L., Company R. and Navarro E. *Orthogonal matrix polynomials and second order differential equations*, Differential Equations and Dynamical Systems **3(3)**, 269-288, 1995.

[5] Jódar L., Company R. and Navarro E. *Laguerre matrix polynomials and systems of second order differential equations*, Applied Numeric Mathematics **15(1)**, 53-64, 1994.

[6] Defez E. and Jódar L., *Chebyshev matrix polynomials and second order matrix differential equations*, Utilitas Mathematica **61**, 107-123, 2002.

[7] Defez E. and Jódar L., *Some applications of the Hermite matrix polynomials series expansions*, Journal of Computational and Applied Mathematics **99**, 105-117, 1998.

[8] Defez E., Tung M. and Sastre J., *Improvement on the Bound of Hermite Matrix Polynomials*, Linear Algebra And its Applications **434**, 1910-1919, 2011.

[9] Durán A. J. and Grünbaum F. A. *A survey on orthogonal matrix polynomials satisfying second order differential equations*, Journal of Computational and Applied Mathematics **178**, 169-190, 2005.

[10] Sastre J., Ibáñez J., Defez E. and Ruiz P. *Efficient orthogonal matrix polynomial based method for computing matrix exponential*, Applied Mathematics and Computation **217**, 6451-6463, 2011.

[11] Sastre J., Ibáñez J., Defez E. and Ruiz P. *Computing matrix functions solving coupled differential models*, Mathematical and Computer Modelling **50**, 831-839, 2009.

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Of course, the choice of these values are not unique. For example, taking the values

$$x = 2i\sqrt{\frac{2}{8+\sqrt{2}}}, t = -i\left(\frac{\sqrt{9+\sqrt{2}}-1}{\sqrt{8+\sqrt{2}}}\right),$$

this choices satisfies the restrictions outlined ($|x| = 0.921835 < 1, |t| = 0.725853 < 1$), but $1 + t^2 - \frac{tx}{\sqrt{2}} = 0$ and then matrix function $I_{2\times 2} - \sqrt{2}Axt + t^2 I_{2\times 2}$ is singular.