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Q-Q plots with confidence for testing Weibull and exponential distributions

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Abstract

One of the basic graphical methods for assessing the validity of a distributional assumption is the Q-Q plot which compares quantiles of a sample against the quantiles of the distribution. In this paper, we focus on how a Q-Q plot can be augmented by intervals for all the points so that, if the population distribution is Weibull or exponential then all the points should fall inside the corresponding intervals simultaneously with probability $1 - \alpha$. These simultaneous $1 - \alpha$ probability intervals provide therefore an objective mean to judge whether the plotted points fall close to the straight line: the plotted points fall close to the straight line if and only if all the points fall within the corresponding intervals. The powers of five Q-Q plot based graphical tests and the most popular non-graphical Anderson-Darling and Cramér-von-Mises tests are compared by simulation. Based on this power study, the tests that have better powers are identified and recommendations are given on which graphical tests should be used in what circumstances. Examples are provided to illustrate the methods.

Keywords: Exponential distribution, Graphical methods, Hypotheses testing, Power; Q-Q plot, Simultaneous inference, Statistical simulation, Weibull distribution.

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1. Introduction

When a simple random sample $Y_1, \cdots, Y_n$ is drawn from a population, one important question is whether the population has a distribution of the form $F_0\left((y-\mu)/\sigma\right)$, where $F_0(\cdot)$ is a given cumulative distribution function (cdf), and $-\infty < \mu < \infty$ and $\sigma > 0$ are two unknown parameters. Note that μ is not necessarily the mean and σ is not necessarily the standard deviation of Y. One widely used graphical technique for dealing with this question is the Q-Q plot. In order to provide an objective judgement on whether the points $(z_k, Y_{[k]})$ fall close to a straight line and building on the work of Michael (1983). Chantarangsi et al. (2015) consider augmenting the normal probability plot by providing an interval for each $Y_{[k]}$ $(k = 1, \ldots, n)$ so that, if the population is normally distributed then all the $Y_{[k]}$ $(k = 1, \cdots, n)$ will fall into the corresponding intervals simultaneously with probability $1 - \alpha$. In this paper, the authors use the idea of Chantarangsi et al. (2015) on Q-Q plots to judge whether a sample is drawn from the Weibull or exponential distributions.

The exponential distribution $Exp(\mu, \sigma)$ is a location-scale family, but the Weibull distribution is not. Therefore, log-transformation is applied to the Weibull distribution to obtain the smallest extreme value distribution $SEV(\mu, \sigma)$, which is a location-scale family. A Q-Q plot consists of the n points $(q_k, Y_{[k]})$, $k = 1, \cdots, n$, where $Y_{[1]} \leq \cdots \leq Y_{[n]}$ are the ordered Y_k's and $q_1 < \ldots < q_n$ are a set of n reference values which represent the ordered values of a typical sample of size n from the distribution $F_0(y)$. There are several ways to choose the reference values $q_k = F_0^{-1}(p_k)$ where $F_0^{-1}(\cdot)$ is the inverse function of $F_0(\cdot)$. Various slightly different forms of p_k have been suggested in the statistical literature. See, e.g., Weibull (1939) [23], Blom (1958) [2] and Filliben (1975) [7]. Throughout this paper, we use $p_k = (k - 0.5)/n$ $(k = 1, \ldots, n)$, which are firstly given in Hazen (1914) [8] and used in the software packages R (when $n > 10$) and Matlab. Note that the choices of the p_k's do not affect the tests discussed in this paper.

If $Y_1, \cdots, Y_n$ have the distribution $F_0\left((y-\mu)/\sigma\right)$, then the n points $(q_k, Y_{[k]})$ should fall close to a straight line. In order to provide an objective judgement on whether the points $(q_k, Y_{[k]})$ fall close to a straight line, one can augment the Q-Q plot by providing an interval for each $Y_{[k]}$ $(k = 1, \cdots, n)$ so that, if the population follows the distribution $F_0\left((y-\mu)/\sigma\right)$, then all the $Y_{[k]}$ $(k = 1, \cdots, n)$ will fall inside the corresponding intervals simultaneously with probability $1 - \alpha$. Each of these n intervals can be depicted in the Q-Q plot as a vertical interval at the corresponding q_k. Therefore, if at least one point $(q_k, Y_{[k]})$ $(1 \leq k \leq n)$ does not fall within the corresponding interval then one can claim, with $1-\alpha$ confidence, that the population does not follow the distribution $F_0\left((y-\mu)/\sigma\right)$. This is in effect a size α test for the null hypothesis H_0: the population distribution is $F_0\left((y-\mu)/\sigma\right)$ for some $-\infty < \mu < \infty$ and $\sigma > 0$ against the alternative hypothesis H_a: H_0 is not true, but with a clear graphical interpretation on the Q-Q plot.

One way to construct the intervals is to use the Kolmogorov-Smirnov statistic

$$D = \max_{1 \leq k \leq n} \left| F_0\left((Y_{[k]} - \hat{\mu})/\hat{\sigma}\right) - (k - 0.5)/n \right| \tag{1.1}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the estimates of μ and σ, respectively. Note that D is sometimes also referred to as Lilliefors' (1967) statistic [11] when F_0 is the cdf of the standard normal distribution $\Phi(\cdot)$. Let c_D be a critical constant so that $P\{D \leq c_D\} = 1 - \alpha$ under H_0. This probability statement can be rewritten as

$$P\left\{ \ Y_{[k]} \in \hat{\mu} + \hat{\sigma} F_0^{-1}\left((k - 0.5)/n \pm c_D\right), \quad k = 1, \cdots, n \ \right\} = 1 - \alpha. \tag{1.2}$$

Hence, under H_0, each $Y_{[k]}$ should fall in the corresponding interval $\hat{\mu} + \hat{\sigma} F_0^{-1}\left((k - 0.5)/n \pm c_D\right)$ simultaneously for $k = 1, \cdots, n$ with probability $1 - \alpha$.

The second set of intervals is due to Michael (1983) [16] and based on the statistic

$$(1.3)\qquad D_m = \max_{1\le k\le n}\left|(2/\pi)\arcsin\sqrt{F_0\left((Y_{[k]}-\hat{\mu})/\hat{\sigma}\right)} - (2/\pi)\arcsin\sqrt{(k-0.5)/n}\right|.$$

Let c_{D_m} be a critical constant so that $P\{D_m \le c_{D_m}\} = 1-\alpha$ under H_0. This probability statement can be rewritten as

$$(1.4)\qquad P\Big\{Y_{[k]} \in \hat{\mu}+\hat{\sigma}F_0^{-1}\big(\sin^2[\arcsin\sqrt{(k-0.5)/n}\pm\frac{\pi}{2}c_{D_m}]\big) \text{ for } k=1,\dots,n\Big\} = 1-\alpha.$$

The purpose of this paper is to propose three new graphical tests and to compare the powers of these graphical tests in order to identify the one having larger overall power.

The layout of the paper is as follows. Section 2 presents the methods of parameter estimation for Weibull and exponential distributions. Section 3 then constructs graphical tests for testing Weibull and exponential distributions based on the tests proposed in Chantarangsi et al. (2015) [5]. The powers of these graphical and two non-graphical tests are then compared in a simulation study in order to identify the tests that have overall good power in Section 4. An illustrative example is presented in Section 5.

2. Distribution function and Parameter estimation

2.1. Weibull distribution. A random variable X is said to have the Weibull distribution, $Wbl(a,b,c)$, if its cdf is given by

$$(2.1)\qquad F(x|a,b,c) = 1-\exp\Big\{-\Big[\frac{x-a}{b}\Big]^c\Big\},\quad x>a,\ b>0,\ c>0$$

where a is called the location parameter, b the scale parameter and c the shape parameter. In this paper, it is assumed a is known and so $Y=\ln(X-a)$ has the so-called **smallest extreme value (SEV) distribution**. The cdf of Y is given by

$$(2.2)\qquad F(y|\mu,\sigma) = 1-\exp\Big(-\exp\Big(\frac{y-\mu}{\sigma}\Big)\Big),\quad -\infty<y<\infty$$

where $-\infty<\mu=\ln b<\infty$ is the location parameter and $\sigma=1/c>0$ is the scale parameter. In short, $Y\sim SEV(\mu,\sigma)$. The original null hypothesis $H_0: X_1,\dots,X_n$ come from $Wbl(a,b,c)$, where a is known, is therefore the same as $H_0: Y_1=\ln(X_1-a),\dots,Y_n=\ln(X_n-a)$ are from $SEV(\mu,\sigma)$ for some unknown parameters μ and σ.

Note that the $\boldsymbol{p^{th}}$ **quantile** of the distribution $SEV(0,1)$ is given by $F^{-1}(p) = \ln(-\ln(1-p))$. Hence a Q-Q plot contains the n points $(\ln(-\ln(1-p_k)), Y_{[k]})$, $k=1,\dots,n$ where $p_k=\frac{k-0.5}{n}$.

Since both the location and scale parameters of $SEV(\mu,\sigma)$ are unknown, they have to be estimated. We consider three popular estimators proposed in the statistical literature: the maximum likelihood estimators (MLE), the best linear unbiased estimators (BLUE) and the best linear invariant estimators (BLIE). They are studied to see which one gives better power. The MLEs are given (cf. Krishnamoorthy (2006) [9]) by

$$(2.3)\qquad \tilde{\mu} = \tilde{\sigma}\ln\Big(\frac{1}{n}\sum_{k=1}^{n}\exp\Big(\frac{Y_k}{\tilde{\sigma}}\Big)\Big),$$

$$(2.4)\qquad \tilde{\sigma} = -\bar{Y}+\frac{\sum_{k=1}^{n}Y_k\exp\left(\frac{Y_k}{\tilde{\sigma}}\right)}{\sum_{k=1}^{n}\exp\left(\frac{Y_k}{\tilde{\sigma}}\right)}.$$

Pirouzi–Fard and Holmquist (2013) [19] considered the statistic D_m in which the BLUEs of μ and σ in $SEV(\mu,\sigma)$ are obtained by the generalised least squares (GLS) method. Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered values of a sample of size n from $SEV(0,1)$ with

$$\mu_k = \mathrm{E}(Z_{[k]}),\ k = 1, \ldots, n \tag{2.5}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}),\ \ k = 1, \ldots, n. \tag{2.6}$$

Pirouzi–Fard and Holmquist (2007) [17] propose the approximations

$$\mu_k \approx \begin{cases} -\ln(n) - \gamma, & \text{for } k = 1, \\ \ln(-\ln(1 - [\frac{k-0.4866}{n+0.1840}])), & \text{for } k = 2, \ldots, n \end{cases} \tag{2.7}$$

where $\gamma \approx 0.577215665$ is Euler's constant. Pirouzi–Fard and Holmquist (2008) [18] propose the approximations

$$\sigma_{rk}^2 \approx \begin{cases} \frac{\pi^2}{6}, & \text{for } r = k = 1, \\ \frac{(k-0.469)([n+0.831-k][n+0.073])^{-1}}{\ln(\frac{n+0.831-k}{n+0.356})\ln(\frac{n+0.779-k}{n+0.356})}, & \text{for } 1 \leq r \leq k \leq n \end{cases} \tag{2.8}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2$ is the covariance of the $Z_{[r]}$ and $Z_{[k]}$ and so, if $r = k$, $\sigma_{rk}^2 = \sigma_k^2$. Let $\boldsymbol{\mu} = [\mu_1 \ldots \mu_n]'$, $V = (\sigma_{rk}^2)_{n\times n}$ and $\mathbf{Y} = [Y_{[1]} \ldots Y_{[n]}]'$. Then $Y_{[k]} = \mu + \sigma Z_{[k]}$ and $E(Y_{[k]}) = \mu + \sigma E(Z_{[k]}) = \mu + \sigma\mu_k$.

Consider the regression model

$$Y_{[k]} = \mu + \sigma\mu_k + \varepsilon_k,\ k = 1, \ldots, n, \tag{2.9}$$

with $\mathrm{Cov}(Y_{[r]}, Y_{[k]}) = \sigma^2 \mathrm{Cov}(Z_{[r]}, Z_{[k]}) = \sigma^2\sigma_{rk}^2$. Since the $Y_{[k]}$'s are heteroscedastic and autocorrelated, the unknown $\boldsymbol{\beta} = [\mu, \sigma]'$ in (2.9) can be estimated by using the GLS method, which result in the BLUEs $\dot{\boldsymbol{\beta}} = (\mathbf{X}'V^{-1}\mathbf{X})^{-1}\mathbf{X}'V^{-1}\mathbf{Y}$ where $\mathbf{X} = [\mathbf{1}, \boldsymbol{\mu}]$ and $V = (\sigma_{rk}^2)_{n\times n}$. Lloyd (1952) [14] is the first to apply the GLS method for estimating the parameters of a location–scale distribution.

Although BLUEs have some very nice properties, they often have larger mean square errors than some other linear estimators. The BLIEs are given in Mann (1969) [15] by

$$\ddot{\mu} = \dot{\mu} - \dot{\sigma}\Big(\frac{E_{12}}{1 + E_{22}}\Big),\ \ddot{\sigma} = \frac{\dot{\sigma}}{1 + E_{22}} \tag{2.10}$$

where $\dot{\mu}$ and $\dot{\sigma}$ are the BLUEs of μ and σ and $\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} = \mathbf{X}'V^{-1}\mathbf{X}$.

2.2. Exponential distribution. The cdf of the two-parameter exponential distribution with the location parameter μ and the scale parameter σ is given by

$$F(y|\mu, \sigma) = 1 - \exp\Big(-\frac{y-\mu}{\sigma}\Big), \quad y > \mu,\ \sigma > 0. \tag{2.11}$$

Specifically, the $\boldsymbol{p^{th}}$ **quantile** of a random variable $Y \sim Exp(\mu, \sigma)$ is given by

$$F^{-1}(p) = \mu - \sigma\ln(1 - p). \tag{2.12}$$

In particular, the p^{th} quantile of the random variable $\frac{Y-\mu}{\sigma} \sim Exp(0,1)$ is $-\ln(1-p)$.

2.2.1. *Parameter estimation.* Again the three popular estimators MLE, BLUE and BLIE are investigated in order to find the estimator that gives good overall powers.

The MLEs of μ and σ are given (cf. Krishnamoorthy, 2006 [9]) by

$$(\tilde{\mu}, \tilde{\sigma}) = \Big(Y_{[1]}, \frac{1}{n}\sum_{k=1}^{n}(Y_k - Y_{[1]})\Big) = \big(Y_{[1]}, \bar{Y} - Y_{[1]}\big). \tag{2.13}$$

Let $Z_{[1]} \leq \ldots \leq Z_{[n]}$ be the ordered sample from $Exp(0,1)$. Then we have (cf. Ahsanullah and Hamedani, 2010 [1])

$$\mu_k = \mathrm{E}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{n-i+1}, \; k = 1, \ldots, n \tag{2.14}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}, \; k = 1, \ldots, n \tag{2.15}$$

$$\sigma_{rk}^2 = \mathrm{Cov}(Z_{[r]}, Z_{[k]}) = \sum_{i=1}^{k} \frac{1}{(n-i+1)^2}, \; 1 \leq r \leq k \leq n. \tag{2.16}$$

where $\sigma_{rk}^2 = \sigma_{kr}^2 = \mathrm{Cov}(Z_{[r]}, Z_{[k]})$. Similar to the case of Weibull distribution, the BLUEs of (μ, σ) can be obtained by the generalised least squares method and are given by

$$\dot{\mu} = \frac{nY_{[1]} - \bar{Y}}{n-1}, \tag{2.17}$$

$$\dot{\sigma} = \frac{n(Y_{[1]} - \bar{Y})}{n-1}. \tag{2.18}$$

See, e.g., Ahsanullah and Hamedani (2010) [1] for details.

The BLIEs are given in Mann (1969)[15] by

$$\ddot{\mu} = (1 + \frac{1}{n})Y_{[1]} - \frac{\bar{Y}}{n}, \tag{2.19}$$

$$\ddot{\sigma} = \bar{Y} - Y_{[1]}. \tag{2.20}$$

3. The tests

The five graphical tests considered in this paper include the two existing tests D, D_m mentioned in the introduction and the three new tests D_e, D_{be} and D_{bi} based on those in Chantarangsi et al.(2015) [5] for testing normality. The $(\hat{\mu}, \hat{\sigma})$ in each test can therefore be substituted by $(\tilde{\mu}, \tilde{\sigma})$, $(\dot{\mu}, \dot{\sigma})$ or $(\ddot{\mu}, \ddot{\sigma})$. In this section, we assume H_0 is true and provide all the tests of size α. The D and D_m tests using $(\tilde{\mu}, \tilde{\sigma})$ have been considered by Kimber (1985) [10]. The D and D_m tests using $(\tilde{\mu}, \tilde{\sigma})$ have been studied in Coles (1989) [3], which shows that the $(\dot{\mu}, \dot{\sigma})$ gives better powers than the $(\tilde{\mu}, \tilde{\sigma})$.

Recall that $Z_1, ..., Z_n$ denote a simple random sample drawn from $SEV(0,1)$ or $Exp(0,1)$ and $Z_{[1]} \leq ... \leq Z_{[n]}$ be the ordered values. The expected values and variances of $Z_{[k]}$ for $k = 1, \ldots, n$ are given by

$$\mu_k = \mathrm{E}(Z_{[k]}), \tag{3.1}$$

$$\sigma_k^2 = \mathrm{Var}(Z_{[k]}) = \mathrm{E}(Z_{[k]}^2) - \mu_k^2 \tag{3.2}$$

where $f_k(z)$ is the probability density function of $Z_{[k]}$ and is defined by

$$f_k(z) = \frac{n!}{(k-1)!(n-k)!} \left(F_Z(z)\right)^{k-1} \left(1 - F_Z(z)\right)^{n-k} f_Z(z) \; , \; -\infty \leq z \leq \infty.$$

First, we consider testing the Weibull distribution. Recall that $Z_1, ..., Z_n$ denote a simple random sample from $SEV(0,1)$, $Z_{[1]} \leq ... \leq Z_{[n]}$ are the ordered values, and $\mu_k = \mathrm{E}(Z_{[k]})$, $\sigma_k^2 = \mathrm{Var}(Z_{[k]})$. It is clear that $(Y_{[1]}, ..., Y_{[n]})$ have the same joint distribution as $(\mu + \sigma Z_{[1]}, ..., \mu + \sigma Z_{[n]})$. In particular, we have $\mathrm{E}(Y_{[k]}) = \mu + \sigma\mu_k$ and $\mathrm{Var}(Y_{[k]}) = \sigma^2\sigma_k^2$.

The test D_e uses the test statistic

$$D_e = \max_{1 \leq k \leq n} \left| \frac{Y_{[k]} - (\hat{\mu} + \hat{\sigma}\mu_k)}{\hat{\sigma}\sigma_k} \right|, \tag{3.3}$$

where $(\hat{\mu}, \hat{\sigma})$ is the estimator of (μ, σ) and can be any one of the three estimators MLE $(\tilde{\mu}, \tilde{\sigma})$, BLUE $(\dot{\mu}, \dot{\sigma})$ and BLIE $(\ddot{\mu}, \ddot{\sigma})$ considered in Section 2.

It is clear from expression (3.3) that the distribution of D_e does not depends on the unknown parameters μ and σ^2. The critical constant c_e, which satisfies $P\{D_e \leq c_e\} = 1 - \alpha$ under H_0, can easily be computed accurately by using a large number of simulations, as in Chantarangsi et al. (2015) [5]. See Edwards and Berry (1987) [6] and Liu et al. (2005) [13] for ways to assess the accuracy of this approach. It is noteworthy that simulation methods are also used to compute the critical constants of the D and D_m tests; see, e.g., Michael (1983) [16] and Scott and Stewart (2011) [20].

The probability statement $P\{D_e \leq C_e\} = 1-\alpha$ produces the following simultaneously probability intervals for $Y_{[1]}, ..., Y_{[n]}$:

$$P\left\{\ Y_{[k]} \in [\hat{\mu} + \hat{\sigma}\mu_k \pm c_e\hat{\sigma}\sigma_k] \ \text{ for } \ k = 1, \cdots, n\ \right\} = 1 - \alpha. \tag{3.4}$$

The D_{be} test is constructed in the following steps. Let $F_0(\cdot)$ denote the cdf of $SEV(0,1)$. Note that, under H_0, $U_k = F_0\Big(\dfrac{Y_k - \mu}{\sigma}\Big)$, $k = 1, \ldots, n$ has a uniform distribution on the interval $(0,1)$ and the order statistic $U_k = F_0\Big(\dfrac{Y_k - \mu}{\sigma}\Big)$ has the beta distribution with parameters k and $n - k + 1$.

- **Step 1**. Construct p^* level highest-density probability interval $[L(p^*, k, n), U(p^*, k, n)]$ for $U_{[k]}$, which is the shortest probability interval for $U_{[k]}$ among all the p^* level probability intervals for $U_{[k]}$.
- **Step 2**. Find p^* so that

$$K(p^*) \equiv P\Big\{F_0^{-1}(L(p^*, k, n)) \leq \frac{Y_{[k]} - \hat{\mu}}{\hat{\sigma}} \leq F_0^{-1}(U(p^*, k, n)) \ \text{ for } k = 1, ..., n\Big\} \quad = \quad 1 - \alpha.$$

  Such a p^* can be found by simulation and a standard numerical searching algorithm in a similar way as in Chantarangsi et al. (2015) [5].
- **Step 3**. Under H_0, the simultaneous $1 - \alpha$ probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ are therefore given by

$$\hat{\mu} + \hat{\sigma}F_0^{-1}(L(p^*, k, n)) \leq Y_{[k]} \leq \hat{\mu} + \hat{\sigma}F_0^{-1}(U(p^*, k, n)), \quad k = 1, \ldots, n.$$

Hence test D_{be} rejects H_0 if and only if at least one $Y_{[k]}$ is not included in its corresponding interval $[\hat{\mu} + \hat{\sigma}F_0^{-1}(L(p^*, k, n)), \hat{\mu} + \hat{\sigma}F_0^{-1}(U(p^*, k, n))]$.

The D_{bi} test uses statistic

$$D_{bi} \ = \ \max_{1 \leq k \leq n} \frac{\big|F_0\big((Y_{[k]} - \hat{\mu})/\hat{\sigma}\big) - (k - 0.5)/n\big|}{\sqrt{(k - 0.5)(n - k + 0.5)/n^3}}.$$

Let c_{bi} be a critical constant so that $P\Big\{D_{bi} < c_{bi}\Big\} = 1 - \alpha$, under H_0, which can be determined by using simulation as before. The simultaneous $1 - \alpha$ probability intervals for $Y_{[1]} \leq ... \leq Y_{[n]}$ are therefore given by

$$Y_{[k]} \in \hat{\mu} + \hat{\sigma}F_0^{-1}\Big(\frac{k - 0.5}{n} \pm c_{bi}\sqrt{\frac{(k - 0.5)(n - k + 0.5)}{n^3}}\Big) \text{ for } k = 1, \ldots, n. \tag{3.5}$$

The test D and D_m are specified in (1.1), (1.2) and (1.3), (1.4), respectively, but with $F_0(\cdot)$ being the cdf of $SEV(0,1)$.

The non-graphical Anderson-Darling (AD) test rejects H_0 if and only if AD $> c$ where

$$\text{AD} = -\sum_{k=1}^{n} \Big[\frac{(2k - 1)\{\ln(F_0(Y_{[k]})) + \ln(1 - F_0(Y_{[n+1-k]}\}}{n}\Big] - n. \tag{3.6}$$

The critical constant c, which satisfies $P\{\text{AD} < c\} = 1-\alpha$ under H_0, can be determined by simulation as before.

The non-graphical Cramér-von Mises (CvM) test rejects H_0 if and only if CvM $> c$ where

$$\text{CvM} = \sum_{k=1}^{n}\left[F_0(Y_{[k]}) - \frac{2k-1}{2n}\right]^2 + \frac{1}{12n} \tag{3.7}$$

The critical constant c, which satisfies $P\{\text{CvM} < c\} = 1-\alpha$ under H_0, can again be determined by simulation.

For testing the Exponential distribution $Exp(\mu, \sigma)$, the five graphical and two non-graphical tests for testing the Weibull distribution given above are easily modified by simply assuming that $F_0(\cdot)$ is the cdf of $Exp(0,1)$ and that $Z_1, \ldots, Z_n$ are a simple random sample from $Exp(0,1)$ to give the five graphical and two non-graphical tests also denoted as D, D_m, D_e, D_{be}, AD and CvM.

Our focus is on the five graphical tests D, D_m, D_e, D_{bi} and D_{be}, each providing a set of simultaneous $1-\alpha$ probability intervals for the $Y_{[k]}$'s. These intervals can be used in the Q-Q plot to objectively judge whether the n points $(q_k, Y_{[k]})$ fall close to a straight line. We also want to compare the powers of the five graphical and the two non-graphical tests.

From many simulation studies on power comparison published in statistical literature (cf. Littell et al. (1979) [12] and Sürücü (2008) [21]), the AD and CvM tests usually have larger power than other tests, for testing Weibull or Exponential distributions. This is the reason why AD and CvM tests are included in our power comparison study.

4. Power comparisons

The power of a test is evaluated by simulation as the proportion of times the null hypothesis H_0 is rejected by the test for a given alternative distribution. In our simulation study, each critical constant c is based on 30,000 simulations and each power value is based on 10,000 simulations. The powers of the seven tests are computed for all possible combinations of $\alpha = 0.01, 0.05, 0.1$, the three estimators (MLE, BLUE, BLIE), sample size n from a set of values, and the alternative distribution from a set of distributions. The set of alternative distributions includes many of the distributions used in several published studies on power comparison of tests for Weibull or Exponential distributions (cf. Littell, et al. (1979) [12], Kimber (1985) [10], Coles (1989) [3], Tiku and Singh (1981) [22], Castro-Kusiss (2011) [4], Pirouzi-Fard and Holmquist et al. (2013) [19]).

4.1. For Weibull distribution. The alternatives are divided into the following three groups. The **first** group of seven distributions are asymmetric on the support $(0, \infty)$ and includes $\chi^2(1)$, $\chi^2(3)$, $\chi^2(4)$, $\chi^2(6)$, $\chi^2(10)$, $LogN(0,1)$ and Half-normal(0,1) ($HN(0,1)$). The **second** group of seven distributions are on the interval $(0,1)$ and includes $U(0,1)$, $beta(2,2)$, $beta(2,5)$, $beta(5,1.5)$, $beta(0.5,0.5)$, $beta(0.5,3)$ and $beta(1,2)$. The **third** group of seven distributions are symmetric on the support $(-\infty, \infty)$ and include $Laplace(0,1)$, $logistic(0,1)$, $N(0,1)$, $t(1)$, $t(3)$, $t(4)$ and $t(6)$.

Sample sizes $n = 10$, 25, 40 ,100, 150, 200, 250, 300, 350, 400 and 500 are used for the alternative distributions from Group I and Group II. For the alternative distributions from Group III, the considered sample sizes are $n = 5(5)30$, 40, 50, 100, 150, and 200 since the powers are very close to 100% already at sample size $n = 200$.

From the results of our study, which one of the three estimators is used has little effect on the powers of the seven tests. Hence any one of the three estimators can be used with any one of the seven tests. Tables 1-3 give the powers of the tests when BLUE is used.

From the power results in Table 1 for **the first group** of alternative distributions, the following observations can be made. The D_{be} test has good power, even relative to the non-graphical tests AD and CvM, against the alternatives in Group I except $\chi^2(1)$ and $HN(0,1)$. D_m and D_{be} have similar powers. Overall D and D_e tend to be less powerful than the other tests. While D_{bi} has better power than D and D_e on many cases, it is less powerful than D_{be} and D_m overall.

From the power results in Table 2 for **the second group** of alternative distributions, the D_{bi} test often has the best powers and is more powerful than the non-graphical AD and CvM tests on most occasions whereas the D and D_e tests generally have least powers. However, when $n \leqslant 40$, D_e seems to have greater powers than all the other tests. Additionally, the powers of D_m and D_{be} are close to each other. All tests have little power in detecting the departure from the Weibull distribution of $beta(2,5)$. Also, the D_{bi} test is more powerful than the non-graphical AD and CvM tests on most occasions.

From the power results in Table 3 for **the third group** of alternative distributions, the AD and CvM tests are overall more powerful than the other tests. Nevertheless, for $N(0,1)$, the D_{bi} test is more powerful than the AD and CvM tests. The D, D_e and D_{bi} tests have low power over the distributions in Group III and the powers of D_{bi} are less than those of D and D_e for larger sample sizes. Among the graphical tests, the D_m and D_{be} tests are more powerful overall.

4.2. For exponential distribution. The **first** group of nine distributions are asymmetric on the support $(0,\infty)$ includes $\chi^2(1)$, $\chi^2(3)$, $\chi^2(4)$, $\chi^2(6)$, $\chi^2(10)$, $LogN(0,1)$, $HN(0,1)$, $Wbl(0,0.5,0.5)$ and $Wbl(0,2,2)$. The **second** and the **third** groups of the distributions are the same as the second and third groups, respectively, given in Section 4.1

From our simulation study, the BLIE often gives the best power, even though the power differences between BLIE and BLUE are often small. Hence BLIE is recommended for testing Exponential distribution.

From the power results given in Table 4 for **the first group** of alternative distributions, the following observations can be made. The two non–graphical AD and CvM tests are the most powerful against all alternative distributions exception $LogN(0,1)$. Interestingly, the powers of the D test are as good as those of the others. Moreover, the D_e test is the best choice against $LogN(0,1)$. On the other hand, it has low powers in comparison with the other tests in this group. Also, the D_{bi} test is the best choice against $HN(0,1)$; however, it has the least power among $\chi^2(1)$, $LogN(0,1)$ and $Wbl(0,0.5,0.5)$. For the other alternative distributions, powers of the D_{bi} test is slightly better than those of D_m and D_{be}.

From the power results given in Table 5 for **the second group** of the alternative distributions, we can observe that the D_{bi} test shows good power, even relative to the non-graphical AD and CvM tests, except for $beta(0.5,3)$. The D_e test has the worst power among all the tests except for $beta(0.5,3)$. The powers of the D_m and D_{be} tests are not as high as those of the D_{bi}, AD and CvM tests in many cases, but they perform quite well overall the alternative distributions generally.

From the power results given in Tables 6 for **the third group** of the alternative distributions, the powers of all tests are very similar. Nevertheless, the CvM test is slightly more powerful than the other tests.

The overall conclusions from this power study for both Weibull and Exponential distributions are as follows. Although not completely dominated, the D and D_e are less powerful than the other three graphical tests in most scenarios and so not recommended. Therefore, the graphical tests D_m, D_{be} and D_{bi} are recommended for use with Q-Q plot.

Table 1. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group I

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$\chi^2(1)$	10	5.88	5.27	**8.26**	5.35	3.72	5.96	6.01
	25	7.92	5.28	**12.20**	4.69	3.62	8.90	7.60
	40	8.59	6.08	**15.10**	5.50	4.14	10.54	9.45
	100	15.12	11.76	**21.55**	11.83	8.97	20.02	17.65
	150	18.26	17.09	28.31	18.26	13.23	**30.18**	22.92
	200	24.79	23.03	32.11	24.50	15.90	**36.20**	30.45
	250	28.68	29.80	36.00	31.73	20.39	**44.26**	36.02
	300	33.80	35.66	40.64	39.01	23.22	**52.24**	43.40
	350	40.00	42.05	44.83	44.21	27.60	**59.38**	50.78
	400	44.94	48.46	49.11	49.36	31.80	**64.56**	56.43
	500	53.12	57.82	54.76	59.67	37.64	**75.64**	66.21
$\chi^2(3)$	10	4.95	5.56	4.26	5.30	**6.35**	5.22	5.36
	25	5.19	6.95	3.49	6.86	**7.90**	4.90	5.45
	40	5.69	8.05	3.41	7.80	**8.61**	5.22	5.92
	100	6.42	10.25	4.15	**10.42**	10.25	7.72	7.23
	150	7.60	11.63	5.42	**12.32**	10.91	10.08	9.00
	200	8.90	13.49	6.17	**14.26**	12.48	11.82	10.62
	250	9.53	15.12	6.08	**15.46**	13.48	12.28	11.15
	300	10.46	16.67	6.61	**17.40**	13.29	15.54	12.76
	350	11.89	18.54	7.58	**18.94**	14.85	17.26	14.84
	400	12.59	19.61	8.46	**20.36**	15.34	18.60	15.64
	500	14.85	23.04	9.59	**23.24**	16.16	22.08	18.56
$\chi^2(4)$	10	5.38	6	3.61	5.94	**7.15**	5.10	5.78
	25	5.86	8.51	3.28	7.82	**9.94**	5.76	6.65
	40	6.52	10.8	3.69	10.56	**12.73**	7.30	7.25
	100	9.46	16.89	5.73	**16.46**	16.36	12.88	11.57
	150	11.94	**21.11**	8.48	21.04	19.32	18.50	15.73
	200	15.81	26.15	10.32	**26.42**	21.74	22.88	19.73
	250	18.3	31.00	11.59	**32.24**	25.62	26.9	23.20
	300	20.29	35.16	14.34	**36.96**	26.09	34.22	27.74
	350	23.49	39.30	15.93	**40.20**	29.91	38.24	31.85
	400	25.85	42.15	18.41	**45.44**	30.54	42.80	34.45
	500	31.83	49.39	21.69	**52.28**	35.57	51.98	42.92
$\chi^2(6)$	10	5.27	6.31	3.25	7.16	**8.16**	5.64	6.11
	25	6.89	11.60	3.55	10.16	**13.83**	8.02	8.22
	40	8.68	14.66	4.67	14.64	**16.47**	10.06	10.47
	100	14.87	**28.75**	9.88	27.98	27.93	22.66	19.81
	150	20.91	38.06	15.08	**39.04**	33.85	34.52	28
	200	27.7	47.71	19.75	**48.26**	40.11	43.62	37.24
	250	33.44	56.15	23.16	**56.50**	45.65	52.74	45.45
	300	38.48	62.91	28.89	**65.20**	49.38	64.00	53.22
	350	44.89	69.20	33.12	**71.16**	55.36	69.82	60.17
	400	49.49	74.85	39.38	75.80	59.42	**76.5**	65.55
	500	59.44	82.92	46.73	84.48	66.89	**85.54**	76.69
$\chi^2(10)$	10	6.05	7.42	3.25	7.62	**9.90**	6.58	6.94
	25	8.45	15.36	4.53	15.18	**18.09**	10.70	10.44
	40	12.10	20.96	6.79	20.46	**23.40**	15.80	14.34
	100	22.95	**44.68**	16.41	43.98	42.34	37.64	31.43
	150	32.45	58.86	25.65	**60.14**	52.66	54.08	45.41
	200	43.97	70.82	32.42	**72.00**	62.19	68.92	58.92
	250	52.61	80.28	40.01	**81.68**	70.69	78.58	69.77
	300	60.13	86.40	49.10	**87.44**	75.20	86.82	77.76
	350	69.08	91.21	57.34	91.68	81.42	**91.76**	84.74
	400	74.35	94.05	65.87	**94.66**	85.55	94.38	88.95
	500	84.01	97.62	76.72	98.10	91.39	**98.30**	95.04
$LogN(0,1)$	10	9.34	12.40	3.97	12.43	**16.88**	10.36	11.25
	25	18.62	34.49	11.94	33.89	**40.13**	28.40	25.46
	40	28.12	52.20	20.64	51.58	**56.83**	43.92	38.57
	100	63.39	**91.66**	51.29	91.59	90.10	86.94	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.18	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.56	98.54
	250	97.35	**100**	94.65	99.98	99.89	99.90	99.74
	300	98.93	**99.98**	98.58	**99.98**	99.95	99.98	99.94
	350	99.71	99.99	99.60	99.99	**100**	99.98	99.97
	400	99.88	**100**	99.88	**100**	**100**	**100**	**100**
	500	99.99	**100**	99.99	**100**	**100**	**100**	**100**
$HN(0,1)$	10	5.93	5.43	**8.56**	5.25	4.00	6.62	5.99
	25	7.15	5.28	**12.18**	4.51	3.05	8.64	7.45
	40	8.75	5.91	**14.58**	5.38	4.06	10.40	9.28
	100	14.39	11.21	**21.20**	10.78	9.02	19.24	16.33
	150	18.87	17.01	27.84	17.83	13.15	**28.12**	23.71
	200	25.35	23.58	32.23	25.03	16.05	**36.26**	31.01
	250	30.30	29.43	35.25	30.53	20.49	**43.74**	37.25
	300	34.27	35.62	40.75	38.35	23.16	**52.94**	44.61
	350	40.56	41.92	45.11	44.39	27.38	**59.48**	51.30
	400	44.67	47.59	48.83	49.75	30.69	**65.56**	56.10
	500	53.89	58.04	55.33	59.65	37.40	**75.70**	66.90

The bolded number is the highest power among the seven tests for each sample size.

Table 2. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group II

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$U(0,1)$	10	14.21	12.00	**20.92**	9.97	5.44	19.68	15.94
	25	32.05	31.54	42.77	25.61	36.30	**48.62**	39.43
	40	49.17	66.52	57.26	60.33	**77.42**	72.12	60.46
	100	91.16	99.99	92.79	99.93	**100**	99.32	97.21
	150	98.66	100	99.86	**100**	**100**	**100**	99.89
	200	99.88	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	10	6.59	5.50	**9.59**	4.80	3.14	7.52	6.88
	25	9.56	6.47	**14.47**	5.53	4.15	13.38	10.62
	40	12.84	9.65	18.01	8.36	10.64	**18.16**	15.15
	100	28.22	46.57	29.62	39.38	**56.22**	49.62	38.56
	150	41.99	76.58	39.77	70.98	**84.94**	70.94	56.61
	200	56.74	92.79	47.98	90.94	**96.39**	83.74	71.93
	250	68.90	98.58	57.59	97.30	**99.50**	92.04	82.95
	300	75.78	99.76	70.69	99.55	**99.92**	97.06	90.43
	350	84.35	99.98	81.58	99.89	**99.99**	98.90	94.79
	400	89.27	**100**	89.49	99.99	**100**	99.46	97.29
	500	94.82	**100**	97.48	**100**	**100**	99.96	99.25
$beta(2,5)$	10	4.62	4.41	4.78	4.28	4.39	**4.86**	4.85
	25	**5.20**	4.62	5.16	3.64	4.27	4.34	5.10
	40	**4.57**	3.98	4.56	3.96	4.30	4.54	4.65
	100	5.37	4.85	3.94	4.23	**6.14**	5.50	5.64
	150	5.67	5.27	4.07	5.30	6.67	**7.26**	6.04
	200	6.24	6.88	4.16	6.22	**8.42**	7.28	7.01
	250	6.71	8.16	3.29	6.39	**10.59**	8.06	7.99
	300	6.57	8.53	3.49	8.11	**11.78**	9.66	8.18
	350	7.58	10.32	3.54	9.15	**14.76**	10.70	9.57
	400	8.96	12.58	3.90	10.45	**16.46**	11.28	10.85
	500	9.14	15.62	3.29	12.60	**20.97**	14.32	11.99
$beta(5,1.5)$	10	8.01	6.43	**11.80**	5.54	3.14	9.92	8.34
	25	15.01	11.07	**22.14**	8.52	8.50	21.30	17.66
	40	20.65	21.21	29.10	17.81	27.54	**35.22**	26.04
	100	52.08	87.37	52.56	82.54	**92.98**	79.70	66.78
	150	72.15	99.11	70.78	98.45	**99.75**	94.98	86.69
	200	86.56	99.98	86.87	99.96	**100**	99.22	95.78
	250	93.99	**100**	95.62	**100**	**100**	99.78	98.62
	300	97.42	**100**	98.47	**100**	**100**	**100**	99.68
	350	99.16	**100**	99.93	**100**	**100**	**100**	99.93
	400	99.59	**100**	**100**	**100**	**100**	**100**	99.98
	500	99.96	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	10	33.39	30.47	42.47	25.68	19.10	**46.44**	38.96
	25	73.77	85.45	79.02	80.91	**92.48**	90.24	82.42
	40	92.10	99.60	93.11	99.37	**99.84**	99.30	97.09
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	250	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	300	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	350	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	400	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	500	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	10	7.64	6.09	**10.92**	5.59	3.40	8.78	7.67
	25	11.56	8.25	**18.65**	6.62	4.76	17.96	12.59
	40	15.57	12.17	**24.85**	10.12	10.80	21.82	18.44
	100	34.31	44.73	43.08	40.37	46.97	**53.34**	44.78
	150	48.78	68.59	55.84	67.44	72.02	**74.86**	62.26
	200	62.84	86.65	65.45	85.04	**88.69**	86.86	76.82
	250	74.08	95.30	73.12	93.69	**96.18**	93.42	87.03
	300	81.81	98.42	81.59	97.86	**98.85**	97.84	93.06
	350	88.79	99.57	88.39	99.41	**99.71**	98.88	96.52
	400	92.53	99.81	92.62	99.82	**99.94**	99.74	98.00
	500	97.01	100	97.31	99.97	**100**	99.98	99.68
$beta(1,2)$	10	6.89	5.53	**10.12**	4.95	3.19	9.56	6.88
	25	11.17	8.10	**17.98**	6.36	5.07	15.70	12.68
	40	15.92	12.97	**23.00**	10.83	13.73	22.46	18.94
	100	35.17	56.25	39.37	50.40	**65.34**	57.34	46.89
	150	50.68	84.91	53.03	81.43	**91.37**	79.70	66.85
	200	67.18	97.10	64.43	95.51	**98.58**	90.42	81.10
	250	77.27	99.60	72.79	99.35	**99.83**	96.64	89.98
	300	86.15	99.93	84.35	99.88	**99.96**	98.92	95.14
	350	90.96	**100**	92.33	99.99	**100**	99.66	97.74
	400	94.92	**100**	97.08	**100**	**100**	99.96	99.07
	500	98.28	**100**	99.71	**100**	**100**	99.98	99.89

The bolded number is the highest power among the seven tests for each sample size.

Table 3. Powers (in %) for testing Weibull distribution with BLUE and $\alpha = 0.05$ against the alternative distributions from Group III

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$Laplace(0,1)$	5	9.73	9.71	8.24	10.60	**11.44**	8.42	9.82
	10	19.56	24.02	12.10	24.88	**27.00**	23.03	24.12
	15	28.93	34.3	21.59	35.23	34.12	35.57	**35.92**
	20	37.33	43.57	32.12	46.18	41.05	**47.03**	46.43
	25	46.92	52.22	40.26	54.51	46.24	**57.06**	56.08
	30	55.42	58.61	48.28	60.53	51.40	**65.73**	65.04
	40	67.43	68.97	61.76	70.24	57.72	**78.08**	76.93
	50	76.62	75.98	71.28	77.85	62.10	**85.56**	84.58
	100	96.73	95.10	94.00	95.95	79.88	**98.80**	98.60
	150	99.68	98.97	98.98	99.27	89.72	**99.93**	99.91
	200	99.91	99.84	99.84	99.91	95.01	**99.98**	99.98
$logistic(0,1)$	5	6.92	6.96	6.11	7.49	**8.39**	6.00	6.92
	10	12.14	16.66	6.23	16.95	**21.01**	15.21	15.6
	15	17.08	24.35	11.14	24.59	**28.35**	22.95	22.02
	20	21.99	32.40	17.64	33.89	**36.11**	31.65	29.29
	25	27.67	40.51	23.56	**41.59**	41.50	39.05	36.56
	30	33.29	46.99	29.12	**47.51**	46.99	47.12	43.69
	40	41.67	57.45	39.37	57.19	55.02	**58.72**	54.4
	50	49.23	64.32	47.57	65.94	60.65	**67.92**	62.79
	100	80.52	89.51	77.33	90.48	81.39	**93.46**	90.59
	150	93.54	96.73	92.68	97.40	91.26	**98.8**	98.01
	200	98.02	99.20	97.08	99.43	95.98	**99.8**	99.58
$N(0,1)$	5	6.12	6.14	5.35	6.35	**7.10**	5.19	6.33
	10	9.34	12.40	3.97	12.31	**16.88**	10.66	11.25
	15	11.99	19.50	6.00	19.18	**25.58**	15.97	15.54
	20	15.08	26.60	9.04	26.15	**33.13**	22.62	20.79
	25	18.62	34.49	11.94	34.17	**40.13**	28.44	25.46
	30	21.20	40.80	14.26	40.64	**46.48**	33.81	29.35
	40	28.12	52.20	20.64	51.58	**56.83**	44.20	38.57
	50	34.00	63.23	25.66	62.38	**66.57**	54.84	47.11
	100	63.39	91.66	51.29	**91.65**	90.10	87.06	79.81
	150	81.86	98.38	73.18	**98.48**	97.35	97.11	94.17
	200	93.03	99.80	87.30	**99.80**	99.33	99.59	98.54
$t(1)$	5	29.51	29.54	27.90	30.42	**31.14**	29.09	30.53
	10	57.46	59.16	52.82	60.89	58.12	60.83	**61.63**
	15	74.7	75.48	71.96	76.67	72.05	78.63	**78.87**
	20	84.57	84.78	82.93	86.41	79.99	**88.29**	88.27
	25	90.87	90.96	89.64	91.88	85.82	**93.61**	93.57
	30	94.48	94.24	94.23	**98.23**	89.47	96.59	96.57
	40	98.48	98.15	98.16	98.04	95.13	99.24	**99.25**
	50	99.55	99.39	99.17	99.46	97.38	**99.82**	99.81
	100	99.99	99.99	99.99	99.99	99.95	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	10.22	10.22	9.11	11.07	**11.78**	9.47	10.68
	10	20.44	24.62	14.82	25.56	**26.67**	24.35	24.79
	15	29.91	35.99	25.70	36.78	**37.79**	37.32	36.57
	20	38.28	46.30	36.23	47.67	44.65	**48.31**	46.71
	25	44.87	52.07	44.71	54.12	48.93	**55.70**	53.75
	30	53.21	59.70	51.88	61.79	55.00	**64.49**	62.31
	40	65.06	70.69	65.83	71.58	61.89	**76.47**	74.48
	50	73.21	77.30	73.89	78.96	67.17	**84.37**	82.35
	100	94.82	94.91	95.01	95.87	84.33	**98.24**	97.79
	150	99.17	99.05	99.08	99.25	92.34	**99.81**	99.78
	200	99.90	99.82	99.79	99.91	96.78	**99.99**	**99.99**
$t(4)$	5	8.42	8.34	7.19	9.38	**10.01**	7.86	9.03
	10	16.23	20.52	10.06	21.19	**24.02**	19.26	19.71
	15	24.30	31.57	18.27	31.69	**34.57**	31.35	30.48
	20	29.66	39.30	**27.64**	**41.26**	39.82	40.30	38.41
	25	36.57	46.94	34.70	48.43	44.96	**48.65**	46.28
	30	43.92	54.04	42.72	55.34	50.89	**57.18**	54.20
	40	54.16	63.65	55.10	65.51	57.76	**68.83**	65.64
	50	63.98	71.89	64.72	73.25	63.75	**78.45**	75.41
	100	89.52	92.18	90.06	93.15	80.28	**96.31**	95.22
	150	97.67	98.05	97.77	98.56	89.32	**99.62**	99.50
	200	99.71	99.67	99.44	99.69	95.12	**99.98**	99.94
$t(6)$	5	7.36	7.44	6.39	8.06	**8.94**	6.60	8.00
	10	13.12	17.30	7.71	17.84	**21.37**	15.56	16.08
	15	19.79	27.33	13.72	27.05	**31.09**	25.84	25.03
	20	24.37	34.85	20.20	35.55	**37.02**	34.02	32.06
	25	28.99	41.57	25.69	**42.88**	42.01	41.23	38.45
	30	35.57	48.13	32.68	**49.40**	47.53	48.97	45.51
	40	45.09	59.16	44.56	60.42	55.11	**61.44**	57.07
	50	52.67	66.73	53.43	67.79	60.94	**70.89**	66.24
	100	82.40	89.59	81.38	90.57	80.02	**94.11**	91.83
	150	94.14	96.94	94.13	97.61	89.50	**98.94**	98.26
	200	98.59	99.17	98.32	99.32	94.79	**99.84**	99.74

The bolded number is the highest power among the seven tests for each sample size.

Table 4. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group I

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$\chi^2(1)$	5	3.88	3.93	6.43	4.22	3.92	**6.91**	4.38
	10	10.66	10.95	11.78	10.29	5.98	**17.95**	12.18
	15	19.85	21.14	15.45	19.33	8.04	**31.62**	22.77
	20	29.58	31.39	19.63	29.14	10.39	**44.49**	34.35
	25	39.12	42.05	23.29	41.13	13.56	**55.70**	44.23
	30	48.89	52.44	28.02	52.94	16.41	**66.29**	55.15
	40	65.77	70.48	38.14	68.94	25.37	**81.21**	71.65
	50	77.49	81.98	47.76	81.63	32.17	**90.29**	82.33
	100	98.44	99.19	91.11	99.25	77.25	**99.76**	99.16
	150	99.8	99.95	99.04	99.98	95.65	99.97	99.95
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(3)$	5	6.71	6.68	6.03	6.5	6.72	6.55	**7.13**
	10	9.14	9.38	8.09	9.31	**9.99**	8.35	9.85
	15	11.41	12.33	10.13	12	**13.27**	11.11	12.82
	20	12.79	13.49	11.58	14.08	**15.26**	12.9	14.53
	25	14.86	16.54	14.33	17.05	**18.98**	15.67	17.36
	30	17.48	20.55	18.15	20.5	**23.20**	19.07	20.93
	40	22.36	27.02	23.75	26.2	**29.30**	26.17	27.44
	50	28.63	33.3	29.57	33.12	32.83	32.79	**33.31**
	100	55.11	62.31	57.49	62.72	55.78	**66.06**	64.9
	150	75.82	81.42	77.27	80.94	73.85	**85.92**	84.15
	200	88.34	91.63	88.78	91.78	84.15	**94.96**	93.84
$\chi^2(4)$	5	7.53	7.52	6.83	**8.40**	7.55	7.31	8.38
	10	12.49	12.71	10.84	12.52	13.65	12.47	**14.34**
	15	17.87	18.81	15.87	19.37	20.18	18.21	**20.34**
	20	22.79	24.45	21.38	25.45	26.9	24.29	**27.25**
	25	28.83	31.47	28.2	31.62	**34.64**	32.04	34.48
	30	35.45	38.87	35	39.77	41.44	39.55	**42.07**
	40	47.26	52.1	48.1	52	53.31	54.19	**55.67**
	50	59.32	64.21	59.55	63.68	62.12	67.96	**68.35**
	100	91.96	93.2	90.83	93.39	89.79	**96.13**	95.92
	150	98.9	99.05	98.33	98.97	97.78	**99.76**	99.67
	200	99.86	99.85	99.71	99.88	99.48	**99.97**	99.94
$\chi^2(6)$	5	9.01	8.76	7.76	9.82	9.09	8.82	**9.91**
	10	18.46	18.81	15.91	18.81	20.24	18.83	**21.69**
	15	28.86	29.97	25.94	30.47	31.66	30.47	**33.50**
	20	39.04	40.25	36.76	40.41	42.5	43.67	**46.71**
	25	49.3	52.05	47.95	51.09	54.19	55.2	**57.70**
	30	59.48	62.14	58.25	62.56	63.71	66.24	**68.38**
	40	75.66	78.24	74.9	76.43	78	82.77	**83.69**
	50	86.15	86.82	83.99	86.31	85.44	91.07	**91.42**
	100	99.54	99.47	99.07	99.45	98.92	99.8	**99.81**
	150	99.99	99.99	99.99	**100**	99.96	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$\chi^2(10)$	5	11.25	10.8	9.24	11.59	11.31	10.15	11.63
	10	25.03	25.23	21.74	25.71	26.69	26.66	**29.69**
	15	40.73	41.76	37.59	42.23	42.87	45.36	**48.66**
	20	55.33	55.86	51.79	55.47	57.69	61.25	**63.83**
	25	67.44	68.15	64.6	67.48	69.71	73.69	**75.90**
	30	77.25	77.87	74.37	78.26	78.83	83.11	**84.53**
	40	89.81	89.72	87.86	89.06	89.67	93.82	**94.33**
	50	95.93	96.11	94.85	95.74	95.32	98.07	**98.18**
	100	99.97	99.96	99.91	99.98	99.89	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$LogN(0,1)$	5	4.94	4.86	5.46	5.07	4.92	**5.60**	4.98
	10	6.6	6.45	**10.54**	6.48	6.05	8.55	7.11
	15	8.41	8.25	**13.68**	8.24	7.28	10.84	9.22
	20	9.98	9.73	**17.84**	9.63	8.53	12.6	11.09
	25	11.5	10.76	**20.89**	10.67	8.9	13.91	12.43
	30	13.09	12.49	**23.86**	12.69	10.04	16.17	14.93
	40	16.56	15.92	**29.88**	15.67	11.7	19.52	18.62
	50	19.22	18.29	**34.85**	18.23	12.62	22.86	21.12
	100	34.56	36.09	**59.85**	38.55	23.17	41.63	40.03
	150	48.77	55.58	**76.67**	57.09	37.67	62.01	57.11
	200	62.53	71.6	**87.66**	74.74	50.32	77.89	71.49
$HN(0,1)$	5	7.71	7.56	6.47	**8.05**	7.73	7.14	8.01
	10	11.95	11.51	9.81	11.89	11.87	11.56	**13.63**
	15	15.4	14.68	12.49	14.9	15.01	15.28	**17.56**
	20	18.66	17.42	15.07	17.37	19.53	19.5	**21.97**
	25	22.44	20.66	17.79	20.32	23.72	23.84	**26.89**
	30	26.36	24.34	20.92	24.39	28.17	28.53	**31.49**
	40	33.68	31.05	25.74	29.35	37.97	38.52	**42.03**
	50	41.27	38.31	29.78	35.99	44.46	47.83	**50.45**
	100	71.14	70.53	51.8	68.05	74.89	80.67	**82.66**
	150	87.36	87.97	68.69	85.55	90.08	94.22	**94.95**
	200	95.33	95.82	80.58	95.21	95.53	98.39	**98.51**
$Wbl(0,2,2)$	5	12.13	11.97	10.16	13.08	12.18	11.35	**13.10**
	10	26.38	26.68	23.32	27.15	27.95	28.51	**31.66**
	15	42	41.66	37.49	42.47	42.88	46.8	**49.91**
	20	56.21	55.18	51.39	56.12	58.68	63.61	**66.42**
	25	69.1	67.8	63.63	68.26	71.4	76.72	**78.60**
	30	79.46	78.94	75.08	78.38	81.81	86.19	**87.49**
	40	90.65	90.41	86.89	89.52	92.1	95.26	**95.75**
	50	96.17	96.11	93.6	95.57	96.25	98.61	**98.75**
	100	99.99	**100**	99.95	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Wbl(0,0.5,0.5)$	5	8.09	8.08	14.47	8.58	8.09	**15.83**	9.42
	10	34.64	35.48	33.86	33.55	23.84	**48.41**	38.89
	15	58.36	59.64	48.38	57.67	39.07	**72.17**	64.11
	20	75.35	76.39	59.78	75.66	51.32	**86.42**	80
	25	85.75	86.82	70.47	86.84	62.04	**93.62**	89.27
	30	92.51	93.55	79.13	93.13	72.78	**97.25**	94.6
	40	97.97	98.55	91.08	98.43	86.55	**99.45**	98.89
	50	99.52	99.64	97.03	99.67	93.63	**99.93**	99.71
	100	**100**	**100**	**100**	**100**	99.96	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bolded number is the highest power among the seven tests for each sample size.

Table 5. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group II

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$U(0,1)$	5	15.67	15.55	14.28	16.24	15.71	16.46	**17.66**
	10	31.33	29.25	27.39	28.75	27.13	37.18	**38.87**
	15	45.01	40.55	37.26	39.61	47.03	55.29	**56.27**
	20	56.22	52.96	46.61	50.21	**75.36**	69.99	70.11
	25	66.49	72.22	55.16	64.58	**91.51**	81.01	80.71
	30	76.06	87.6	63.65	80.84	**97.90**	88.46	88.06
	40	86.99	98.51	76.27	96.24	**99.93**	96.62	96.14
	50	93.95	99.94	83.99	99.64	**100**	99.18	98.96
	100	99.95	**100**	99.03	**100**	**100**	**100**	**100**
	150	**100**	**100**	99.97	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,2)$	5	17.46	17.34	15.33	17.35	17.48	17.36	**19.58**
	10	39.35	37.79	34.76	38.38	37.23	45.71	**48.30**
	15	59.77	56.97	53.34	56.85	59.87	69.63	**71.17**
	20	74.61	72.36	68.37	71.85	82.53	85.9	**86.51**
	25	85.39	85.76	79.74	83.82	93.97	93.73	**94.01**
	30	92.35	94.41	87.41	92.36	**98.52**	97.28	97.43
	40	97.95	99.47	95.49	98.83	**99.92**	99.64	99.57
	50	99.66	99.96	98.94	99.96	**99.99**	99.96	99.96
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(2,5)$	5	11.54	11.21	9.58	11.68	11.55	10.35	12.07
	10	22.41	22.2	19.41	22.78	23.13	23.4	**26.49**
	15	34.2	33.89	29.72	35.67	34.7	38.87	**41.73**
	20	46.85	46.9	42.85	46.37	50.73	55.33	**58.30**
	25	59.29	59.51	54.71	58.35	65.09	69.04	**70.72**
	30	69.49	69.56	64.45	68.86	74.73	78.33	**79.83**
	40	83.18	84.2	78.88	82.76	88.58	91.46	**92.18**
	50	91.68	92.59	87.17	91.88	94.65	**96.61**	96.51
	100	99.89	99.95	99.62	99.97	99.97	**99.99**	**99.99**
	150	**100**	**100**	99.99	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(5,1.5)$	5	29.41	29.25	26.77	30.17	29.43	32.5	**34.46**
	10	67.79	65.79	63.03	66.24	63.23	77.22	**78.49**
	15	88.88	87.03	84.94	86.69	91.03	94.62	**94.72**
	20	96.36	96.32	94.22	95.58	**99.06**	98.85	98.88
	25	99.04	99.53	97.9	99.24	**99.93**	99.83	99.83
	30	99.77	99.94	99.41	99.92	**100**	99.96	99.97
	40	99.99	**100**	99.99	**100**	**100**	**100**	**100**
	50	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,0.5)$	5	15.37	17.09	17.51	15.55	15.02	**19.08**	18.55
	10	25.68	23.86	24.3	23.7	17.77	**33.07**	30.86
	15	34.68	30.67	29.02	29.62	39.82	**46.82**	42.86
	20	42.05	43.44	33.96	37.41	**67.97**	58.65	52.89
	25	51.63	64.34	39.1	53.45	**85.32**	69.81	63.54
	30	60.5	82.04	46.19	72.82	**94.24**	79.64	73.03
	40	73.47	96.83	56.8	93.4	**99.34**	91.34	85.51
	50	84.76	99.67	66.83	99.02	**99.97**	97.12	93.44
	100	99.58	**100**	95.72	**100**	**100**	**100**	99.94
	150	**100**	**100**	99.8	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$beta(0.5,3)$	5	4.2	4.29	5.74	4.68	4.22	**6.20**	4.43
	10	6.89	7.12	5.54	6.57	3.41	**11.14**	7.24
	15	11.39	12.59	6.15	11.45	3.69	**19.21**	12.77
	20	16.4	18.64	5.8	18.03	4.06	**27.30**	18.89
	25	21.82	25.79	6.1	25.21	5.23	**35.77**	25.36
	30	29.42	35.96	7.52	34.63	7.15	**46.15**	33.48
	40	41.8	50.45	10.36	49.5	10.79	**61.25**	47.45
	50	53.19	63.21	14.96	62.88	13.39	**73.41**	58.37
	100	88.7	95.48	62	95.76	46.13	**97.09**	91.98
	150	98.05	99.38	90.74	99.59	79.03	**99.79**	98.87
	200	99.75	99.94	98.74	**99.99**	94.15	**99.99**	99.85
$beta(1,2)$	5	9.27	9	7.91	9.37	9.29	8.25	**9.51**
	10	13.61	13.05	11.31	13.28	12.79	13.94	**16.16**
	15	18.9	17.48	15.04	17.5	17.18	19.96	**21.98**
	20	22.78	20.16	18.07	20.65	24.65	26.83	**28.78**
	25	27.42	24.72	20.68	24.74	34.97	33.24	**35.41**
	30	33.8	31.55	25.3	28.85	**46.92**	39.23	41.61
	40	42.05	45	30.49	37.17	**68.24**	51.83	53.67
	50	50.05	59.11	33.95	51.04	**82.03**	61.94	62.77
	100	82.37	98.78	58.94	96.71	**99.93**	92.94	92.8
	150	95.21	**100**	77.37	99.97	**100**	99.26	99.11
	200	99.06	**100**	89.27	**100**	**100**	99.94	99.93

The bolded number is the highest power among the seven tests for each sample size.

Table 6. Powers (in %) for testing exponential distribution with BLIE and $\alpha = 0.05$ against the alternative distributions from Group III

Alternatives	n	D	D_m	D_e	D_{be}	D_{bi}	AD	CvM
$Laplace(0,1)$	5	25.61	24.7	21.48	26.16	25.67	25.04	**27.65**
	10	60.42	60.3	56.88	61.58	60.62	62.09	**65.41**
	15	83.94	83.34	81.32	82.93	82.41	85.7	**87.23**
	20	93.38	92.83	91.86	92.78	92.59	94.33	**95.17**
	25	97.35	97.23	96.71	97.19	96.87	98.02	**98.31**
	30	98.95	98.74	98.56	99.07	98.51	99.26	**99.39**
	40	99.92	99.89	99.85	99.83	99.75	99.95	**99.96**
	50	99.99	99.97	99.97	99.98	99.98	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$logistic(0,1)$	5	21.7	20.94	18.16	22.39	21.76	21.18	23.72
	10	53	52.19	48.92	53.43	52.87	56.92	**60.18**
	15	77.37	75.92	72.94	75.67	76.04	81.49	**83.42**
	20	89.05	88.33	86.86	88.07	88.63	92.34	**93.20**
	25	94.98	94.46	93.29	94.59	94.89	96.95	**97.39**
	30	97.64	97.47	**99.51**	97.89	97.6	98.76	99
	40	99.75	99.65	**99.92**	99.57	99.62	99.88	99.92
	50	99.97	99.96	**100**	99.95	99.92	99.98	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$Cauchy(0,1)$	5	36.91	36.15	34.7	37.87	36.95	37.42	**38.16**
	10	71.92	72.42	71.82	72.93	73.18	72.74	**73.67**
	15	89.01	90.12	89.82	89.63	90.32	90.3	**90.66**
	20	95.56	96.1	96.2	96	96.2	96.62	**96.64**
	25	98.37	98.53	98.63	98.44	98.43	**98.96**	98.84
	30	99.26	99.48	99.53	99.57	99.33	**99.61**	99.59
	40	99.92	99.93	99.93	99.91	99.9	**99.97**	**99.97**
	50	99.98	99.96	99.98	99.98	99.95	**99.99**	99.98
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$N(0,1)$	5	19.96	19.38	16.72	20.56	20	19.36	**21.68**
	10	48.77	48.2	44.51	48.66	47.92	53.85	**56.93**
	15	72.26	70.56	67.53	70.91	70.84	78.64	**80.52**
	20	86.74	85.46	83.21	85.28	86.97	91.58	**92.48**
	25	93.69	93.11	91.48	92.8	94.22	96.61	**97.10**
	30	97.4	96.97	95.94	96.93	97.79	98.89	**98.97**
	40	99.53	99.6	99.36	99.54	99.7	99.87	**99.87**
	50	99.9	99.93	99.74	99.9	99.97	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(1)$	5	36.95	36.49	35.15	37.38	36.99	37.63	38.10
	10	71.96	72.62	71.82	73.13	73.33	72.87	**73.95**
	15	88.84	89.8	89.55	88.73	90.01	90.29	**90.42**
	20	95.55	95.99	95.99	95.8	95.96	96.28	**96.40**
	25	98.1	98.47	98.38	98.38	98.5	98.73	**98.75**
	30	99.44	99.57	99.54	99.55	99.46	**99.70**	99.64
	40	99.94	99.94	99.94	99.91	99.89	**99.96**	**99.96**
	50	**100**	**100**	**100**	99.99	**100**	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(3)$	5	24.99	24.26	21.54	25.92	25.03	24.09	**26.51**
	10	57.86	57.82	54.6	59.09	58.19	60.54	**63.44**
	15	79.54	79.53	77.17	79.71	79.48	81.94	**83.82**
	20	91.11	90.99	89.6	90.34	90.63	92.85	**93.56**
	25	96.16	96.11	95.66	96.26	95.85	97.23	**97.55**
	30	98.54	98.27	98.01	98.24	97.99	99.02	**99.15**
	40	99.78	99.71	99.69	99.74	99.66	99.84	**99.87**
	50	99.92	99.95	99.86	99.94	99.88	**99.96**	**99.96**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(4)$	5	23.11	22.34	19.63	24.39	23.22	22.4	24.80
	10	56.85	56.46	53.18	56.42	56.33	59.36	**62.35**
	15	77.96	77.74	75.06	78.02	77.67	80.79	**82.63**
	20	89.87	89.29	88.05	89.12	89.5	92.19	**92.92**
	25	96.04	95.82	95.24	95.62	95.75	97.18	**97.59**
	30	98.27	98	97.55	98.3	97.75	98.93	**99.11**
	40	99.67	99.62	99.54	99.61	99.54	99.88	**99.91**
	50	**100**	99.98	99.95	99.94	99.94	**100**	**100**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**
$t(6)$	5	22.14	21.71	18.71	22.41	22.19	21.57	**24.02**
	10	54.25	53.75	50.25	54.45	53.79	57.52	**60.85**
	15	76.8	76.15	73.28	75.82	75.88	80.59	**82.37**
	20	89.28	88.48	86.95	88.44	88.94	91.94	**92.83**
	25	95.62	95.19	94.29	95.04	95.05	97.1	**97.37**
	30	98.15	98.08	97.46	97.75	97.92	98.83	**99.06**
	40	99.72	99.64	99.52	99.72	99.6	99.83	**99.87**
	50	99.98	99.98	99.95	99.94	99.94	99.99	**99.99**
	100	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	150	**100**	**100**	**100**	**100**	**100**	**100**	**100**
	200	**100**	**100**	**100**	**100**	**100**	**100**	**100**

The bolded number is the highest power among the seven tests for each sample size.

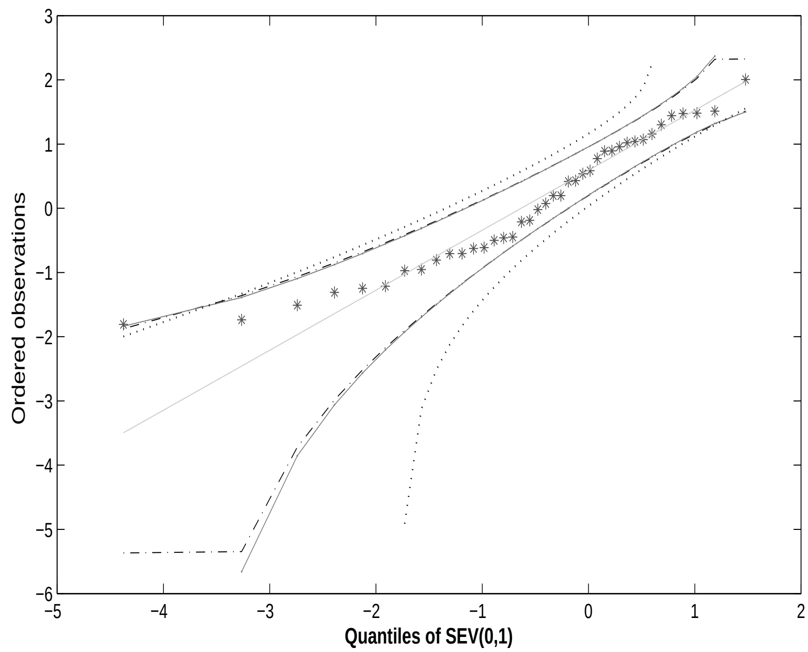

Figure 1. The Q-Q plot and simultaneous probability intervals of D_m (dot dashed lines), D_{be} (solid lines) and D_{bi} (dotted lines) for testing the $SEV(\mu, \sigma)$ distribution by using BLUE at $\alpha = 0.05$

5. Illustrative examples

Example 1 The following sample of $n = 40$ observations is available: 0.1638, 0.176, 0.2208, 0.2697, 0.2872, 0.2976, 0.3782, 0.3851, 0.4464, 0.4934, 0.4946, 0.5341, 0.5413, 0.6063, 0.631, 0.6395, 0.8083, 0.829, 0.9798, 1.0765, 1.2162, 1.2174, 1.5189, 1.539, 1.7137, 1.7962, 2.1652, 2.4304, 2.445, 2.6073, 2.772, 2.8333, 2.9133, 3.1765, 3.6735, 4.2328, 4.3731, 4.4028, 4.5422 and 7.4225. We want to test whether the population from which the sample is taken has the distribution $SEV(\mu, \sigma)$ for some unknown μ and $\sigma > 0$. Following the recommendations in the last section, we can apply any one of the D_m, D_{be} and D_{bi} tests at $\alpha = 0.05$. The Q-Q plot together with the corresponding intervals for the $Y_{[k]}$'s of D_m, D_{be} and D_{bi} are given in Figure 1. Since $Y_{[1]}$ is outside the corresponding interval of each of D_m, D_{be} and D_{bi} tests, H_0 is rejected by each of D_m, D_{be} and D_{bi} tests, i.e. we can claim that the sample does not follow a $SEV(\mu, \sigma)$ distribution.

For the non-graphical tests AD and CvM, the tests statistics of AD and CvM are 0.5111 and 0.0783, respectively. Also, the corresponding critical values of AD and CvM at $\alpha = 0.05$ are 0.7529 and 0.1240, respectively. Hence the hypothesis H_0 is not rejected by either AD or CvM in this case.

Example 2 The following sample of $n = 40$ observations is available: 3.4966, 3.6591, 4.2103, 4.7391, 4.9138, 5.0151, 5.7313, 5.7879, 6.264, 6.6019, 6.6103, 6.8772, 6.9248, 7.3341, 7.4817, 7.5318, 8.4437, 8.5465, 9.2456, 9.6554, 10.2049, 10.2092, 11.2582, 11.3228, 11.8595, 12.0993, 13.0847, 13.72, 13.7534, 14.1161, 14.4676, 14.5948, 14.7573, 15.2703, 16.1591, 17.0582, 17.2696, 17.3136, 17.5179 and 20.9488. We want to test that H_0 the population from which the sample is taken has the distribution $Exp(\mu, \sigma)$ for some unknown μ and $\sigma > 0$.

The usual Q-Q plot with the corresponding intervals for the $Y_{[k]}$'s of D_{bi} with $\alpha = 0.05$ are given in Figure 2. Since several points are outside the corresponding intervals of D_{bi}, e.g., $Y_{[3]}$, $Y_{[4]}$, $Y_{[39]}$ and $Y_{[40]}$, the null hypothesis H_0 is rejected by D_{bi}.

For the non-graphical tests, the test statistics AD and CvM are 1.5627 and 0.2773, respectively. Also, the critical values at $\alpha = 0.05$ and $n = 40$ are 1.1755 and 0.2107, respectively. Hence the null hypothesis H_0 is also rejected by AD or CvM.

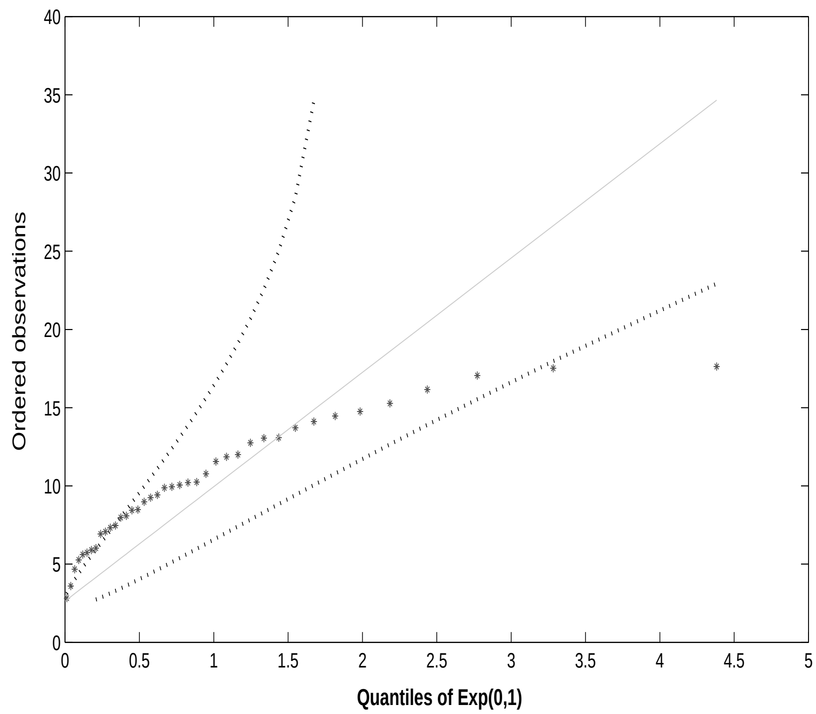

Figure 2. The Q-Q plot and simultaneous probability intervals of D_{bi} (dotted lines) for testing $Exp(\mu, \sigma)$ distribution by using BLIE at $\alpha = 0.05$

6. Conclusions

Generally, the Kolmogorov-Smirnov test (D test) has a very low power. Although the Anderson-Darling and Craḿer-von-Mises tests are non-graphical, they may not be more powerful than the graphical tests. According to Wanpen et al.(2015), the D_{bi} and D_e tests should be used for testing normality based on a simple random sample. For testing the Weibull and exponential distributions, the D_{be}, D_{bi} and D_{sp} tests should be used. Although the D_e test is one of the graphical tests recommended for testing normality when a simple random sample is considered, it is a bad choice for testing Weibull and exponential distributions.

Specifically, we obtain the simultaneous $1-\alpha$ probability intervals suitable for Q-Q plots on testing the Weibull and exponential distributions. They become the objective judgement on Q-Q plots for practitioners who want to use the graphical test.

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