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Soft topological space and topology on the Cartesian product

Matejdes Milan*

Abstract

The paper deals with a soft topological space which is defined over an initial universe set U with a fixed set of parameters E. The main goal is to point out that any soft topological space is homeomorphic to a topological space $(E \times U, \tau)$ where τ is an arbitrary topology on the product $E \times U$, consequently many soft topological notions and results can be derived from general topology. Furthermore, in many papers some notions are introduced by different ways and it would be good to give a unified approach for a transfer of topological notions to a soft set theory and to create a bridge between general topology and soft set theory.

Keywords: Soft set, Soft open (closed) set, Soft interior (closure) of soft set, Soft topological space, Separation axioms, Soft continuity, Soft e-continuity.

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1. Introduction

In 1999, Molodtsov [18], [19], [20] introduced a soft set theory as a new tool for investigation of uncertainties where we can find a large range of applications of soft sets in many different fields. There has been a rapid growth of interest in soft set theory, its applications and its connection with another mathematical branches [1], [2], [4], [5], [7], [8], [12], [13], [14], [15], [16], [23]. Moreover, there are many papers devoted to soft topological spaces [3], [6], [9] ,[10], [11], [17], [21], [22]. The basic topological notions such as the soft open and soft closed sets, soft subspace, soft closure and soft interior, soft boundary, soft separation axioms, soft continuity have been introduced and the investigation of their basic properties has been established.

We continue investigating the soft topological theory based on a corresponce between set valued mappings and binary relations. Their close connection shows that both definitions of a soft set by a set valued mapping and by a relation are equivalent and there is

*Department of Mathematics and Computer Science, Faculty of Education, Trnava University in Trnava, Priemyselná 4, 918 43 Trnava, Slovakia. Email: milan.matejdes@truni.sk

only a formal difference between them. Furthemore, the binary relation view is very comfortable and many results concerning the properties of the operations on soft sets follow from the set theory. On the other hand, the set valued mapping view gives possibilities for a further investigation of the soft set theory in many directions, since the theory of set valued mappings is strong and has many applications in mathematics (general topology, generalized continuities, linear and dynamic programming, differential inclusions, fixe point theory, statistics, economics and so on).

This paper shows that many results concerning soft topological spaces follow from general topology. In particular, some notions introduced in soft topological spaces and their consequences (the properties of soft open (closed) sets, interior and closure of soft set, soft cluster points) are identical with corresponding notions known from general topology. Some of them are different (soft separation axioms, soft continuity) and they are introduced by different ways. The main goal of this paper is to give a unified view for a further development of soft topological spaces based on the results of general topology.

2. Relations, set valued mappings and their correspondence

Any subset S of a Cartesian product $A \times U$ is called a binary relation from a set A to a set U. By $\mathbf{R}(A, U)$, we denote a set of all binary relations from A to U and $S[a] := \{u \in U : [a, u] \in S\}$. The operations of sum $S \cup T$, $\cup_{t \in T} S_t$, intersection $S \cap T$, $\cap_{t \in T} S_t$, complement S^c and difference $S \setminus T$ of relations are defined in the obvious way as in the set theory.

By $F : A \to 2^U$ we denote a set valued mapping from A to power set 2^U of U. The set of all set valued mappings from A to 2^U is denoted by $\mathbf{F}(A, U)$. If F, G are two set valued mappings, then $F \subset G$ $(F = G)$ means $F(a) \subset G(a)$ $(F(a) = G(a))$ for any $a \in A$.

A graph of F is a set $Gr(F) := \{[a, u] \in A \times U : u \in F(a)\}$ and it is a subset of $A \times U$, hence $Gr(F) \in \mathbf{R}(A, U)$. So, any set valued mapping F determines a relation from $\mathbf{R}(A, U)$ denoted by $R_F := \{[a, u] \in A \times U : u \in F(a)\} = Gr(F)$.

On the other hand, any relation $S \in \mathbf{R}(A, U)$ determines a set valued mapping F_S from A to 2^U where $F_S(a) = S[a]$. So, there is one-to-one correspondence between a relation S from $\mathbf{R}(A, U)$ and a set valued mapping G from $\mathbf{F}(A, U)$, i.e.,

$$S \mapsto F_S,\ F_S(a) = S[a],\quad G \mapsto R_G,\ R_G[a] = G(a),$$

$$F_{R_G} = G,\ R_{F_S} = S.$$

2.1. Remark. For $H, G, F_t \in \mathbf{F}(A, U)$, $t \in T$, we define the following obvious set valued mapping operations and we give also their binary relation equivalents.

(1) Sum: $\cup_{t \in T} F_t : A \to 2^U$
$(\cup_{t \in T} F_t)(a) = \cup_{t \in T} F_t(a) = \cup_{t \in T} R_{F_t}[a] = (\cup_{t \in T} R_{F_t})[a]$, $a \in A$,

(2) Intersection: $\cap_{t \in T} F_t : A \to 2^U$
$(\cap_{t \in T} F_t)(a) = \cap_{t \in T} F_t(a) = \cap_{t \in T} R_{F_t}[a] = (\cap_{t \in T} R_{F_t})[a]$, $a \in A$,

(3) Complement: $H^c : A \to 2^U$
$(H^c)(a) = U \setminus H(a) = U \setminus R_H[a] = R_H^c[a]$, $a \in A$,

(4) Difference: $H \setminus G : A \to 2^U$
$(H \setminus G)(a) = H(a) \setminus G(a) = R_H[a] \setminus R_G[a] = (R_H \setminus R_G)[a]$, $a \in A$.

The next lemma is a consequence of Remark 2.1.

2.2. Lemma. *For $H, G, F_t \in \mathbf{F}(A, U)$ and $S, P, R_t \in \mathbf{R}(A, U)$, $t \in T$, the following equations hold.*

(1) $R_{\cup_{t \in T} F_t} = \cup_{t \in T} R_{F_t}$, $F_{\cup_{t \in t} R_t} = \cup_{t \in T} F_{R_t}$,

(2) $R_{\cap_{t\in T}F_t} = \cap_{t\in T}R_{F_t}$, $F_{\cap_{t\in t}R_t} = \cap_{t\in T}F_{R_t}$,
(3) $R_{H^c} = R_H^{\ c}$, $F_{S^c} = F_S^{\ c}$,
(4) $R_{H\setminus G} = R_H \setminus R_G$, $F_{S\setminus P} = F_S \setminus F_P$.

3. Set valued mapping and binary relation representation of soft set

In this section we will consider soft sets over a common initial universe set U and a fixed set of parameters E and a definition of a soft set is introduced by a set valued mapping (see references).

3.1. Definition. If $F : E \to 2^U$ is a set valued mapping, then a pair (F, E) is called a soft set over U with respect to a set of parameters E. The family of all soft sets over U with respect to a set of parameters E is denoted by $SS(E, U)$.

As we said above there is no difference between the graph of a set valued mapping F and a relation $Gr(F) \subset E \times U$, which is a member of $\mathbf{R}(E, U)$. So, a soft set can be defined equivalently as follows.

3.2. Definition. A soft set over U with respect to a set E is any subset of $E \times U$. So, in this case a soft set is a member of $\mathbf{R}(E, U)$.

From Definition 3.2 we can see a benefit of both equivalent interpretations of a soft set. Any operation known from a set theory setting can be used for a soft set (soft sets) from $\mathbf{R}(E, U)$. In this case we deal with the soft sets as subsets of $E \times U$ and it is not necessary to use the different notations (symbols) for operations and many proofs can be omitted. For example, the next operations on the soft sets form $\mathbf{R}(E, U)$, $R \subset S$, $R = S$, $R \setminus S$, $R \cap S$, $R \cup S$, $\cup_{t\in T}R_t$, $\cap_{t\in T}R_t$, R^c are the set theory ones and all known properties from set theory hold in the soft set setting (for example associativity, commutativity, distributivity, de Morgan laws and so on).

Equivalently, if a soft set is understood as a pair (F, U) $(F \in \mathbf{F}(E, U))$, we can define standard operations on the set valued mappings (sum, intersection, complement, difference) which have equivalent binary relation forms, as we see from Remark 2.1 and Lemma 2.2.

3.3. Lemma. *Let $S \in \mathbf{R}(E, U)$, $G \in \mathbf{F}(E, U)$ and $(H_1, E), (H_2, E) \in SS(E, U)$. Then*

(1) *$G(a) \subset S[a]$ for all $a \in E$ iff $G \subset F_S$ iff $R_G \subset S$ iff a soft set (G, E) is a soft subset of (F_S, E),*
(2) *$S[a] \subset G(a)$ for all $a \in E$ iff $S \subset R_G$ iff $F_S \subset G$ iff a soft set (F_S, E) is a soft subset of (G, E),*
(3) *$G(a) = S[a]$ for all $a \in E$ iff $G = F_S$ iff $R_G = S$ iff a soft set (F_S, E) is equal to a soft set (G, E),*
(4) *a soft set (H_1, E) is a soft subset of (H_2, E) iff $R_{H_1} \subset R_{H_1}$ iff $H_1 \subset H_2$,*
(5) *a soft set (H_1, E) is equel to a soft set (H_2, E) iff $R_{H_1} = R_{H_1}$ iff $H_1 = H_2$.*

4. Special soft sets, their notation and terminology

Let $A \subset E$, $X \subset U$. Then $A \times X$ is called a rectangle soft set. It represents a constant soft set (a constant set valued mapping F with values $F(a) = X$ if $a \in A$ and $F(a) = \emptyset$ if $a \notin A$) denoted also $c(A, X)$. Maximal (minimal) rectangle soft set with respect to the set inclusion is $E \times U$ $(\emptyset \times \emptyset)$ called a full soft set (a null soft set). For the special cases of a constant soft set $c(A, X) \in \mathbf{R}(A, U)$ we introduce the next notation and terminology. Let $A \subset U$, $X \subset U$, $e \in E$, $x \in U$.

(1) $c(A,x) := A \times \{x\}$ - a horizontal x-line on A,
(2) $c(E,x) := E \times \{x\}$ - a full horizontal x-line,
(3) $c(e,X) := \{e\} \times X$ - a vertical e-line on X,
(4) $c(e,U) := \{e\} \times U$ - a full vertical e-line,
(5) $c(e,x) := [e,x]$ - a point, denoted also $P[e,x]$ or briefly P.

4.1. Lemma. *Let $S \in \mathbf{R}(E,U)$, $G \in \mathbf{F}(E,U)$. Then*

(1) $S = \cup_{e \in E}\big(\{e\} \times S[e]\big) = \cup_{e \in E}[S \cap c(e,U)] = \cup_{x \in U}[S \cap c(E,x)]$,
(2) $R_G = \cup_{e \in E}\big(\{e\} \times G(e)\big) = \cup_{e \in E}[R_G \cap c(e,U)] = \cup_{x \in U}[R_G \cap c(E,x)]$.

5. Soft topological space

By [9],[10],[21] a soft topological space is a triplet (E,U,τ), where $\tau \subset SS(E,U)$ is a topology. So, τ is represented by a family of set valued mappings F from $\mathbf{F}(E,U)$ each of them has a binary relation representation R_F from $\mathbf{R}(E,U)$. Put $\tau_{E\times U} := \{R \in \mathbf{R}(E,U) : (F_R, E) \in \tau\}$.

On the other hand, if $(E \times U, \tau_{E\times U})$ is a topological space, then $(E,U,\tau_{E,U})$ is a soft topological space, where $\tau_{E,U} = \{(G,E) \in SS(E,U) : R_G \in \tau_{E\times U}\}$. Then a soft topological space can be characterized (defined) as follows:

5.1. Proposition. *A triplet $(E,U,\tau_{E,U})$ is a soft topological space if and only if $(E \times U, \tau_{E\times U})$ is a topological space. For brevity we will denote $\tau_{E\times U}$ as well as $\tau_{E,U}$ by τ and the difference between both topologies is clear from notation (E,U,τ) (a soft topological space, $\tau \subset SS(E,U)$) and $(E \times U, \tau)$ (a topological space, $\tau \subset \mathbf{R}(E,U)$).*

Again, there is a one-to-one correspondence between the soft topological spaces and the topological spaces and any soft topological space can be considered as an arbitrary topological space on the product of two sets. The members from τ are called open sets in a topological space $(E \times U, \tau)$ or soft open sets in a soft topological space (E,U,τ). A complement of an open set (a soft open set) is called a closed set (a soft closed set). It can be formulated by the following lemma.

5.2. Lemma. *A soft set (G,E) is soft open (soft closed) in a soft topological space (E,U,τ) iff R_G is open (closed) in a topological space $(E \times U, \tau)$ and S is open (closed) in a topological space $(E \times U, \tau)$ iff (F_S, E) is soft open (soft closed) in a soft topological space (E,U,τ).*

Any topological notion known from general topology on the product of two sets can be introduced (formulated) also for a soft topological space by direct reformulation. From Proposition 5.1 we can see that many results in a soft topological space follow from general topology, provided they are directly reformulated. It is not necessary to prove the properties of the soft open and soft closed sets, the properties of the soft interior and soft closure operators, soft cluster points, soft interior points, soft topological subspaces, soft boundary sets, the soft separation axioms and so on, provided they are defined by the same way as in topological spaces. But some notions are defined by a different way and we will discuss them below.

On the other hand, many interesting and important results follow from general topology, provided τ is the Tychonoff (product) topology on $E \times U$. But if a topology τ on $E \times U$ is not Tychonoff, many results do not hold. So, there are many open problems in the soft topology setting and before researchers is a huge challenge.

6. Comparison of some topological notions and soft topological ones

In this section we show a few correspondences between some topological notions and soft topological ones. For example, our expectation is that the soft closure of a soft set (G, E) in a soft topological space (E, U, τ) agrees with the closure of R_G in a topological space $(E \times U, \tau)$.

Recall a few basic topological notions. Let $(E \times U, \tau)$ be a topological space. A set $H \in \tau$ is an open neighborhood of a point $P[a, x] \in E \times U$, if $P[a, x] \in H$ and $P[a, x]$ is a cluster point of $S \subset E \times U$, if any open neighborhood of $P[a, x]$ intersects S. The set of all cluster points of S in $(A \times U, \tau)$ is equal to the closure of S denoted by $cl(S)$, which is the smallest closed set containing S or it is the intersection of all closed sets containing S. A point $P[a, x] \in E \times U$ is an interior point of S, if there is an open set $H \in \tau$ such that $P[a, x] \in H \subset S$ and S is open, if any its point is an interior point of S. The sum of all open subsets of S is called the interior of S denoted by $int(S)$.

1. Open (closed) sets and interior (cluster) points of a set versus soft open (closed) sets and a-interior (a-cluster) points of a soft set

By [9], an a-soft open neighborhood of x is any open soft set (G, E) such that $x \in G(a)$, equivalently R_G is open and $x \in R_G[a]$ or $P[a, x] \in R_G$. A point $x \in U$ is said to be an a-cluster point of $(H, E) \in SS(E, U)$ if for every a-soft open neighborhood (G, E) of x, (H, E) and (G, E) are not soft disjoint (there is $e \in E$ such that $H(e) \cap G(e) \neq \emptyset$). The set of all a-cluster points of (H, A) is denoted by $cl(H, a)$, see [9]. Similarly, $int(H, a)$ is a set of all a-interior points of (H, E), see [9].

6.1. Lemma.
(1) *$x \in cl(H, a)$ if and only if $P[a, x] \in cl(R_H)$, so $cl(H, a) = cl(R_H)[a]$,*
(2) *$x \in int(H, a)$ if and only if $P[a, x] \in int(R_H)$, so $int(H, a) = int(R_H)[a]$.*
Consequently, $cl(H, a)$ ($int(H, a)$) is a set of all cluster (interior) points of R_H in $(E \times U, \tau)$ from the full vertical a-line ($cl(H, a) = cl(R_H) \cap c(a, U)$ ($int(H, a) = int(R_H) \cap c(a, U)$)).

Proof. (1)
"$\Rightarrow$" Let $x \in cl(H, a)$ and $P[a, x] \in S \in \tau$. Then $x \in F_S(a)$. That means (F_S, E) is a-open neighborhood of x, so (H, E) and (F_S, E) are not soft disjoint. Hence, there are $e \in E, y \in U$ such that $y \in H(e) \cap F_S(e)$ or $P[e, y] \in R_H \cap S$. That means $R_H \cap S \neq \emptyset$ or $P[a, x] \in cl(R_H)$.

"$\Leftarrow$" Let $P[a, x] \in cl(R_H)$ and (G, E) be a-soft open neighborhood of x. Then $P[a, x] \in R_G \in \tau$ and $R_H \cap R_G \neq \emptyset$. So, there are $e \in E$ and $y \in U$ such that $y \in R_H[e] \cap R_G[e]$, hence (H, E) and (G, E) are not soft disjoint, so $x \in cl(H, a)$.
Item (2) is similar.

□

6.2. Lemma. *Let $cl(G, E)$, $int(G, E)$ be a soft closure, a soft interior of a soft set (G, E), respectively (see [9]). Then*
(1) *$cl(G, E) = (F_{cl(R_G)}, E)$,*
(2) *$int(G, E) = (F_{int(R_G)}, E)$.*

Proof. (1) By the definition of the soft closure ([9]) and by Lemma 5.2, $cl(G, E)$ is the intersection of all soft closed supersets $(G_t, E), t \in T$ of (G, E) if and only if $\cap_{t \in T} R_{G_t}$ is the intersection of all closed (in $(E \times U, \tau)$) supersets R_{G_t} of R_G. That means $\cap_{t \in T} R_{G_t} =$

$cl(R_G)$ is the graph of a multivalued mapping H for which $cl(G,E)=(H,E)$, so $F_{cl(R_G)}=H$.

(2) is similar.

□

Since $(F_{cl(R_G)},E)=cl(G,E)$, $cl(G,E)$ is a soft set given by a set valued mapping with the values $cl(R_G)[a]$, $a\in E$ which is equal to a set valued mapping $R_{G,E}$ defined in [9] as $R_{G,E}(a)=G(a)\cup cl(G,a)=cl(G,a)=cl(R_G)[a]$ (see Lemma 6.1). So, $cl(G,E)=(R_{G,E},E)$. Similarly, $R_{G,E}(a)=G(a)\cap int(G,a)=int(R_G)[a]$ for $int(G,E)=(R_{G,E},E)$ see [9]. So Proposition 3.9 and 3.12 of [9] are clear.

2. Separation axioms

In the literature ([11], [17], [21]) we can see notation $x\in(F,E)$, where F is a set valued mapping from E to 2^U and $x\in U$. It means $x\in F(e)$ for any $e\in E$. So, the notation $x\in(F,E)$ is in fact the inclusion $c(E,x)\subset R_F$. It was used in the definitions of soft separation axioms in a soft topological space. In general topology, the separation axioms separate two different points or a closed set and a point or two disjoint closed sets. For example, by [21], (E,U,τ) is called soft T_2, if for every distinct points x,y of U there are two soft open sets (F,E) and (G,E) such that $x\in(F,E)$, $y\in(G,E)$ and (F,E) and (G,E) are soft disjoint. That means it separates two full horizontal lines $c(E,x)$ and $c(E,y)$. Further, if (F,E) is a soft closed set and $x\notin(F,E)$, a soft regularity introduced in [21] separates two sets, namely $c(E,x)$ and R_F which need not be disjoint. It is a very strict definition as we see from the next lemma.

6.3. Lemma. *Let (E,U,τ) be a soft topological space and (F,E) be a soft closed set. If there are $e_1,e_2\in E$ and a point $y\in U$ such that $y\in F(e_1)$ and $y\notin F(e_2)$ (it is sufficient (E,U,τ) is not indiscrete), then (E,U,τ) is not soft regular (in the sense of [21]). Consequently, if some soft topological space over U is soft regular, then any soft closed set (F,E) is constant, i.e., there is a set $X\subset U$ such that $F(e)=X$ for any $e\in E$.*

Proof. Suppose (E,U,τ) is soft regular. It is clear $y\notin(F,E)$. Then there are two soft open and soft disjoint sets (G,E) containing y and (H,E) containing (F,E), but $y\in G(e_1)\cap H(e_1)$, a contradiction.

□

The next theorem shows that soft regularity in the sense of [21] seems to be a rather strong definition.

6.4. Theorem. *If a non indiscrete soft topological space is soft regular, then any soft closed set is a constant soft set (it is of the form $c(E,X)$).*

In [10], the authors introduced other soft separation notions, namely T_0, T_1, T_2, T_3. We recall only two of them (for further see [10]). A soft topological space (E,U,τ) is called soft T_2, if for any distinct points x and y of U and for every $a\in E$ there exist two soft open sets (G,E) and (H,E) such that $x\in_a(G,E)$, $y\in_a(H,E)$ and $G(a)\cap H(a)=\emptyset$ ($z\in_a(G,E)$ means that $z\in G(a)$ and $z\notin_a(G,E)$ means that $z\notin G(a)$). In this case we separate a couple of the points $P[a,x]$ and $P[a,y]$ from full vertical a-line $c(a,U)$ by two soft open sets "disjoint at a". This is a different definition of a soft T_2-space introduced in [21]. Further by [10], a soft topological space (E,U,τ) is called a soft T_3-space if for every point $x\in U$, for every $a\in E$ and for every soft closed set (Q,E) such that $x\notin_a(Q,E)$ there exist two soft open sets (G,E) and (H,E) such that $x\in_a(G,E)$, $Q(a)\subset H(a)$, and $G(a)\cap H(a)=\emptyset$. In this case the sets $G(a)\cap c(a,U)$ and $H(a)\cap c(a,U)$ are two subsets of $E\times U$ which are open in a subspace $(c(a,U),\tau_a)$ of $(A\times U,\tau)$.

Generally, for any full vertical line $c(e,U)$, topology τ induces a relative topology τ_e on $c(e,U)$, so also on U (for different e_1, e_2 the induced topological spaces (U,τ_{e_1}) and (U,τ_{e_2}) can be different). So we have the next theorem (which does not hold generally. " $\Rightarrow$ " follows from hereditary of $T_i, i = 0,1,2,3$ and " $\Leftarrow$ " follows from a character of the definitions of the soft separation axioms in [10]).

6.5. Theorem. *A soft topological space (E,A,τ) is soft T_i (in the sense of [10]) if and only if the topological space (U,τ_e) is T_i $(i = 0,1,2,3)$ for any $e \in E$.*

Any subset S of $(E \times U, \tau)$ induces relative topology τ_S on S. Since the properties T_i $(i = 0,1,2,3)$ are hereditary, Proposition 3.13 of [10] holds for any $S \subset E \times U$ not only for $E \times Y$ (see Definition 3.12 in [10] or [21]).

3. Soft e-continuity of f and continuity of $e \times f$,
soft continuity of Φ_{ef}

In [3], [9], [10], [13], [22] for two functions $e : E_1 \to E_2$, $f : U_1 \to U_2$, a function Φ_{ef} from $SS(E_1,U_1)$ to $S(E_2,U_2)$ was defined (denoted also f_{pu} in [22], φ_ϕ in [3]). We will show a connection between Φ_{ef} and the product $e \times f : E_1 \times U_1 \to E_2 \times U_2$, defined as $(e \times f)([e_1,x_1]) = [e(e_1), f(x_1)]$. Define two soft mappings:
$\mathbf{S}_{e\times f} : SS(E_1,U_1) \to SS(E_2,U_2)$ as $\mathbf{S}_{e\times f}((H,E_1)) = (F_{(e\times f)(R_H)}, E_2)$,
$\mathbf{S}^{-1}_{e\times f} : SS(E_2,U_2) \to SS(E_1,U_1)$ as $\mathbf{S}^{-1}_{e\times f}((G,E_2)) = (F_{(e\times f)^{-1}(R_G)}, E_1)$.

6.6. Theorem. *Let $(H,E_1) \in SS(E_1,U_1)$ and $(G,E_2) \in SS(E_2,U_2)$. Then*
(1) $\mathbf{S}_{e\times f} = \Phi_{ef}$, *i.e.,* $\Phi_{ef}((H,E_1)) = (F_{(e\times f)(R_H)}, E_2)$,
(2) $\mathbf{S}^{-1}_{e\times f} = \Phi^{-1}_{ef}$, *i.e.,* $\Phi^{-1}_{ef}((G,E_2)) = (F_{(e\times f)^{-1}(R_G)}, E_2)$.

Proof. Let $H \in \mathbf{F}(E_1,U_1)$ and R_H be a corresponding relation, so $R_H[a] = H(a)$. Then, by Lemma 4.1 (1),

$$(e \times f)(R_H) = (e \times f)\big(\cup_{a\in E_1} (\{a\} \times R_H[a])\big) =$$
$$= \cup_{a\in E_1}(e \times f)\big(\{a\} \times R_H[a]\big) = \cup_{a\in E_1}\big(e(a) \times f(R_H[a])\big).$$

That means $(e \times f)(R_H)$ is a subset of $E_2 \times U_2$ and corresponding set valued mapping denoted by $G : E_2 \to U_2$ has its values given by $\big[(e \times f)(R_H)\big][p_2]$ for any $p_2 \in E_2$.

$$G(p_2) = \big[(e \times f)(R_H)\big][p_2] = \big[\cup_{a\in E_1}\ \big(e(a) \times f(R_H[a])\big)\big][p_2] =$$
$$= \cup_{a\in E_1}\big[e(a) \times f(R_H[a])\big][p_2] = \cup\{f(R_H[a]) : e(a) = p_2\} =$$
$$= \cup\{f(H(a)) : a \in e^{-1}(p_2)\}.$$

So, (G,E_2) is the image of (H,E_1) under Φ_{ef} as it was defined in [10]. So, $\Phi_{ef}((H,E_1)) = (G,E_2) = (F_{(e\times f)(R_H)}, E_2)$ (see Lemma 3.3 (3)) or $\Phi_{ef} = \mathbf{S}_{e\times f}$.

Similarly we can show (see [10]) that $\Phi^{-1}_{ef}((G,E_2)) = (D,E_1)$,
where $D(p_1) = f^{-1}(G(e(p_1))) = (e\times f)^{-1}(R_G)[p_1]$. So, $\Phi^{-1}_{ef}((G,E_2)) = (F_{(e\times f)^{-1}(R_G)}, E_1)$ or $\Phi^{-1}_{ef} = \mathbf{S}^{-1}_{e\times f}$.

□

Proposition 2.18 and 2.19 of [9] (Proposition 2.8 of [10]) follow from the properties of the image and the inverse image, which hold generally for any function.

In [10], for two soft topological spaces (E_1,U_1,τ) and (E_2,U_2,σ) a definition of a soft e-continuity of f was introduced by the following way.

6.7. Definition. Let (E_1, U_1, τ) and (E_2, U_2, σ) be two soft topological spaces and $x \in U_1$, $e : E_1 \to E_2$. A map $f : U_1 \to U_2$ is called soft e-continuous at the point x if for every $a \in E_1$ and every $e(a)$-soft open neighborhood (G, E_2) of $f(x)$ in (E_2, U_2, σ) there exists an a-soft open neighborhood (H, E_1) of x in (E_1, U_1, τ) such that $\Phi_{ef}((H, E_1))$ is a soft subset of (G, E_2). If the map f is soft e-continuous at any point $x \in E_1$, then we say that the map f is soft e-continuous.

Now we reformulate the definition above in the corresponding topological spaces $(E_1 \times U_1, \tau)$ and $(E_1 \times U_2, \sigma)$.

6.8. Definition. Let $(E_1 \times U_1, \tau)$ and $(E_1 \times U_2, \sigma)$ be two topological spaces, $x \in U_1$, $e : E_1 \to E_2$. A map $f : U_1 \to U_2$ is called soft e-continuous at the point x if for every $a \in E_1$ (i.e., for every $[a, x] \in c(E_1, x)$) and every open set $G \in \sigma$ containing $[e(a), f(x)]$ there exists an open set $H \in \tau$ containing $[a, x]$ such that $(e \times f)(H) \subset G$. If the map f is soft e-continuous at any point $x \in U_1$, then we say that the map f is soft e-continuous.

Since $\Phi_{ef}((H, E_1)) = (F_{(e\times f)(R_H)}, E_2)$ (see Theorem above), $\Phi_{ef}((H, E_1))$ is a soft subset of (G, E_2) iff $(e \times f)(R_H) \subset R_G$. That means the soft e-continuity of f at x means that the set of all continuity points (in the general topology sense) of $e \times f : (E_1 \times U_1, \tau) \to (E_2 \times U_2, \sigma)$ contains a full horizontal x-line $c(E_1, x)$. Since $\Phi_{ef} = \mathbf{S}_{e\times f}$ and $\Phi_{ef}^{-1} = \mathbf{S}_{e\times f}^{-1}$, the next theorem is clear and Propositions 2.18 and 2.19 of [10] follows from standard equivalent conditions of continuity.

6.9. Theorem. *The next conditions are equivalent*

(1) *A function f is soft e-continuous (in the sense of [10]),*
(2) *$e \times f : (E_1 \times U_1, \tau) \to (E_2 \times U_2, \sigma)$ is continuous (in the topological sense),*
(3) *$\Phi_{ef}^{-1}((G, E_2)) \in \tau$ for any $(G, E_2) \in \sigma$.*

Finally, we recall a notion of a soft set point mentioned in [22]. A soft point, denoted by e_F is a soft set for which $F(e) \neq \emptyset$ and $F(a) = \emptyset$ for all $a \in E \setminus \{e\}$ and $e_F \in (G, E)$ means $F(a) \subset G(a)$ for all $a \in E$. So, a soft point is in fact any vertical e-line $c(e, X) = \{e\} \times X$ on X, for $X \neq \emptyset$. By [22], Φ_{ef} $(= f_{pu})$ is soft continuous (soft pu-continuous see [22]) at a soft point e_F if for any soft open set (G, E_2) containing $\Phi_{ef}(e_F)$ there is a soft open set (H, E_1) containing e_F such that $\Phi_{ef}((H, E_1))$ is a soft subset of (G, E_2) and Φ_{ef} is soft continuous if it is so at any soft point. Since a point $P[e, x]$ is also a soft point (namely e_F where $F(a) = \emptyset$ for $a \neq e$ and $F(e) = \{x\}$), soft continuity of Φ_{ef} in the sense of [22] implies a topological continuity of $e \times f$ at any point $P[e, x] \in E \times U$. The opposite implication also holds, as we prove in the next theorem.

6.10. Theorem. *A function Φ_{ef} $(= f_{pu})$ is soft continuous (in the sense of [22]) if and only if $e \times f$ is continuous (in the topological sense). Consequently, the soft continuity is equivalent to the soft e-continuity.*

Proof. It is sufficient to prove "$\Leftarrow$". Let g_K (i.e., $K(g) \neq \emptyset$ and $K(e) = \emptyset$ for $e \neq g$) be a soft point and (G, E_2) be a soft open superset of $\Phi_{ef}(g_K)$. Since R_G is an open set in $E_2 \times U_2$, R_G is open neighborhood of a point $[e(g), f(x)]$ for any $x \in K(g)$. Since $e \times f$ is continuous at $[g, x]$, for any $x \in K(g)$ there is an open subset H_x of $E_1 \times U_1$ containing $[g, x]$ such that $(e \times f)(H_x) \subset R_G$. Put $H := \cup_{x\in K(g)} H_x \supset R_{g_K}$. Then H is open in $E_1 \times U_1$ and $(e \times f)(H) \subset R_G$, so $(F_{(e\times f)(H)}, E_2)$ is a soft subset of (G, E_2) (see Lemma 3.3 (2)). Since (F_H, E_1) is a soft open set containing g_K (see Lemma 3.3 (1)), $\Phi_{ef}(F_H, E_1) = (F_{(e\times f)(H)}, E_2)$ (see Theorem 6.6 (1)) is a soft subset of (G, E_2).

□

7. Conclusion

This paper deals with the study of the theory of soft topological spaces and the main result is to show a deep connection between a soft topology τ on $SS(E,U)$ and a topology $\tau_{E\times U} = \{R \subset E \times U : (F_R, E) \in \tau\}$ on the product $E \times U$. Any soft topological space (E,U,τ) can be considered as a topological space on $E \times U$ and any topological space (U,τ) can be considered as a soft topological space over U with respect to a singleton $E = \{e\}$. From this correspondence between (E,U,τ) and $(E \times U, \tau_{E\times U})$, it follows that many results from soft set theory are consequences of the topological results. In fact, (E,U,τ) and $(E \times U, \tau_{E\times U})$ are homeomorphic. A homeomorphism $h : (E \times U, \tau) \to (E,U,\tau)$ is given by $h([a,x]) = (F_{c(a,x)}, E)$, where $F_{c(a,x)}$ is given by $F_{c(a,x)}(e) = \{x\}$ for $e = a$ and $F_{c(a,x)}(e) = \emptyset$ for $e \neq a$.

Similarly, if $E = \{e\}$, then (E,U,τ) and (U,τ) are homeomorphic and a homeomorphism $h : (U,\tau) \to (E,U,\tau)$ is defined by $h(x) = (F_{c(e,x)}, E), x \in U$. This homeomorphism is a very good tool for finding a soft topological space which has a soft property P_1 (for example soft T_i) and it has not a soft property P_2 (soft T_j, $i < j$). Then, it is sufficient to find a topological space which has the property P_1 (T_i) and it has not the property P_2 (T_j, $i < j$, $i,j = 1,2,3,4$).

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