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EQUITABLE EDGE CHROMATIC NUMBER OF $P_m \otimes S_n^0$ AND $S_m^0 \otimes S_n^0$

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Abstract. The equitable edge chromatic number is the minimum number of colors required to color the edges of a graph G and satisfies the criterion, if for each vertex $v \in V(G)$, the number of edges of any one color incident with v differs from the number of edges of any other color incident with v by atmost one. In this paper, the equitable edge chromatic number of tensor product of Path P_m coupled with Crown S_n^0 and also two Crown graphs S_m^0 along with S_n^0 are obtained.

1. Introduction

The graphs considered in this paper are finite and undirected graphs without loops. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of a graph G by $\Delta(G)$. An edge coloring of a graph G is simply an assignment of colors to the edges of G such that the adjacent edges incident with a vertex receives a different color.

Given an edge-coloring of graph G with k colors $1, 2, \ldots, k$ for all $v \in V(G)$, let $c_i(v)$ denote the number of edges incident with v colored i. The chromatic index of a graph G, denoted $\chi'(G)$, is the minimum number of different colors required for a proper edge coloring of graph G. The graph G is k-edge-chromatic if $\chi'(G) = k$. Obviously $\chi'(G) \geq \Delta(G)$. In 1916, König has proved that every bipartite graph can be edge colored with exactly $\Delta(G)$ colors, that is $\chi'(G) = \Delta(G)$. In 1964, Vizing [7] proved that $\chi'(G) \leq \Delta(G) + 1$.

In 1973, Meyer [4] presented the concept of equitable coloring and equitable chromatic number. Later in 1994, a fundamental work on the concept of equitable edge coloring was first studied by Hilton and de Werra [3]. The tensor product of graphs was introduced by Paul .M. Weichsel[8] in 1961. It is also called as

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Kronecker, direct, cross or categorical product. The graph products are engrossing that the purpose is not to construct a complex graph, but to decompose it into simple graphs. The geometric nature of the tensor product of two graphs may sometimes be connected or disconnected.

Many real life situations can be modeled as a graph coloring problem, some of them are planning and scheduling problems, timetabling and map coloring. Since graph coloring problem is a NP-hard problem, until now there are not known deterministic methods as a whole that can solve such problems. So various algorithms and deterministic approaches and have been built to solve it. The main objective of this paper is to establish the tensor product of Path with Crown graph and two Crown graphs together, then equitably coloring the edges of these product graphs and thus finding its equitable edge chromatic number.

2. Preliminaries

Definition 1. [6] *A path is a simple graph whose vertices can be rearranged into a linear sequence in such a way that two adjacent vertices are always consecutive in the sequence and vice versa, non-adjacent vertices are non-consecutive. Such a graph on n vertices is often denoted P_n.*

Definition 2. *A crown graph on $2n$ vertices is a graph with an edge from u_i to v_j whenever $i \neq j$. The crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed and is denoted by S_n^0.*

Definition 3. [9] *The tensor product of G and H is the graph, denoted as $G \otimes H$, whose vertex set is $V(G) \otimes V(H) = V(G \otimes H)$, and for each vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus*

$$\begin{aligned} V(G \otimes H) &= \{(g,h) / g \in V(G) \text{ and } h \in V(H)\} \\ E(G \otimes H) &= \{(g,h)(g',h') / gg' \in E(G) \text{ and } hh' \in E(H)\}. \end{aligned}$$

Theorem 4. [8] *Let G and H be connected graphs. $G \otimes H$ is connected if and only if either G or H contains an odd cycle.*

Corollary 5. [8] *If G and H are connected graphs with no odd cycles ,then $G \otimes H$ has exactly two connected components.*

Definition 6. [5] *For k-proper edge coloring f of graph G, if $||E_i| - |E_j|| \leq 1$, $i, j = 0, 1, 2, \ldots, k-1$, where $E_i(G)$ is the set of edges of color i in G, then f is called a k-equitable edge coloring of graph G, and*

$$\chi'_{=}(G) = \min\{k : \text{there exists a } k\text{-equitable edge-coloring of graph } G\}$$

is called the equitable edge chromatic number of graph G.

Lemma 7. [1] *For any simple graph $G(V, E)$, $\chi'_{=}(G) \geq \Delta(G)$.*

Theorem 8. [7] *Let G be a graph.Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Lemma 9. [10] *Let G be a graph and let $k \geq 2$. If $k \nmid d(v)$ for each $v \in V(G)$, then G has an equitable edge-coloring with k colors.*

Lemma 10. [10] *Let G be a graph and let $k \geq 2$. If the k-core of G is a set of isolated vertices, then G has an equitable edge-coloring with k colors.*

For additional graph theory terminologies not defined in this paper can be found in [1, 2].

3. Main Results

Theorem 11. $\chi'_{=}\left(P_m \otimes S_n^0\right) = \Delta$, *for all m, n.*

Proof. The vertices and edges of the path P_m be $V\left(P_m\right) = \{u_i : 1 \leq i \leq m\}$, $E\left(P_m\right) = \{(u_i, u_j) : 1 \leq i < j \leq m\}$ and the crown graph S_n^0 be

$$V\left(S_n^0\right) = \begin{cases} v_i : 1 \leq i \leq n \\ v_j : n+1 \leq j \leq 2n \end{cases}$$

and

$$E\left(S_n^0\right) = \left\{(v_i, v_j) : 1 \leq i \leq n, n+1 \leq j \leq 2n, i \neq j(\mathrm{mod} n)\right\}.$$

The edges of S_n^0 are denoted by $e_k, 1 \leq k \leq (n-1)n$ and these edges incident with vertices $v_i : 1 \leq i \leq n$ is obtained by

$$e_k^{(i)}, k = (i-1)n + (r-i+1), 1 \leq r \leq n-1, 1 \leq i \leq n.$$

The edges adjacent to other edges of S_n^0 are obtained as follows:
Case (i) For $1 \leq k \leq n$

$$e_k^{(j)}, k = \begin{cases} sn - (s-k), & \text{if } s < k \\ (s+1)n - (s-k), & \text{if } s \geq k, \text{where} 1 \leq s \leq n-2. \end{cases}$$

Case (ii) For $k > n$. In this case the edges of S_n^0 also have the same adjacency pattern with the edges obtained from case (i). The adjacent edges of $k > n$ is obtained from

$$e_k = e_t, \text{where } t = \left[k(\mathrm{mod} n) + l\right](\mathrm{mod} n), (ln+1) \leq k \leq (l+1)n, 1 \leq l \leq n-2.$$

and $t = n$, if $t \equiv 0(\mathrm{mod} n)$ then proceed with case (i).

Let η be the numbering of the edges of e_k, so that $\sum \eta(e_k^{(i)})$ denote the sum of the numbers of edges adjacent to e_k with respect to one end vertex v_i. Similarly $\sum \eta(e_k^{(j)})$ denote the sum of the numbers of edges adjacent to e_k incident with other end vertex v_j. The vertices and edges of tensor product of path and crown graphs are

$$V\left(P_m \otimes S_n^0\right) = \{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq 2n\}$$

and

$$E\left(P_m \otimes S_n^0\right) = \{[(u_i, v_j), (u_k, v_l)] : 1 \leq i \leq m, 1 \leq k \leq m, 1 \leq j \leq 2n, 1 \leq l \leq 2n\}.$$

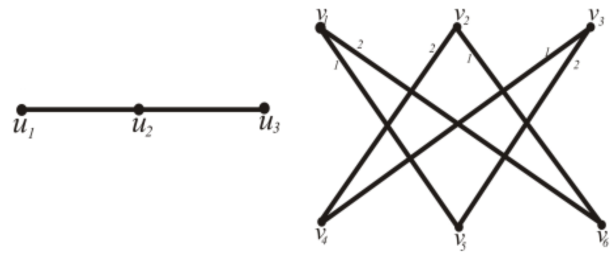

FIGURE 1. P_3 and S_3.

Since $P_m \otimes S_n^0$ is isomorphic to $S_n^0 \otimes P_m$. Without loss of generality, we assume $m \leq n$ for all cases of m and n. We prove that the upper bound for equitable edge coloring of graph $P_m \otimes S_n^0$ is Δ. For the purpose of proof, color the edges of S_n^0. Clearly $\chi'\left(S_n^0\right) = \Delta$.

Define the function π as follows for the egde coloring of S_n^0, $\pi : S \to C$ where $S = V\left(S_n^0\right) \cup E(S_n^0)$ and $C = \{1, 2 \ldots, \Delta\}$.
For $1 \leq k \leq (n-1)n$

$$\pi\left(e_k\right) = \left[i + j + k + \sum \eta\left(e_k^{(i)}\right) + \sum \eta\left(e_k^{(j)}\right)\right] \left(\bmod \Delta\left(\ S_n^0\right)\right).$$

By this pattern, we obtain the edge colored graph of S_n^0. Let $G = P_m \otimes S_n^0$, the resulting graph of G consists of two connected components which are disjoint. Based on this observation the first and second component of G can be represented as G_1 and G_2 respectively (see Figure 2).

Now the equitable edge coloring of edges in G is obtained as follows, define a function $f : G' \to C'$ where $G' = V\left(G\right) \cup E(G)$ and $C' = \{1, 2 \ldots, \Delta\}$.
Case 1. m is odd.The coloring method of the two components are as follows:

$$For\ G_1, f\left[\left(u_i, v_j\right), \left(u_k, v_l\right)\right] = \begin{cases} C(v_i, v_j), \text{ if } i < k, j < l \\ C(v_i, v_j) + \Delta\left(S_n^0\right), \text{ if } i < k, j > l \end{cases}$$

$$For\ G_2, f\left[\left(u_i, v_j\right), \left(u_k, v_l\right)\right] = \begin{cases} C(v_i, v_j) + \Delta\left(S_n^0\right), \text{if} i < k, j > l \\ C(v_i, v_j), \text{ if } i < k, j < l \end{cases}$$

Case 2. m is even.The coloring pattern of the two components G_1 and G_2 are as follows:

$$For\ G_1, f\left[\left(u_i, v_j\right), \left(u_k, v_l\right)\right] = \begin{cases} C(v_i, v_j), \text{ if } i < k, j < l \\ C(v_i, v_j) + \Delta\left(S_n^0\right), \text{ if } i < k, j > l \end{cases}$$

$$For\ G_2, f\left[\left(u_i, v_j\right), \left(u_k, v_l\right)\right] = \begin{cases} C(v_i, v_j) + \Delta\left(S_n^0\right), \text{ if } i < k, j > l \\ C(v_i, v_j), \text{ if } i < k, j < l \end{cases}$$

Clearly f is an equitable edge coloring of $P_m \otimes S_n^0$. Let $E(P_m \otimes S_n^0) = \{E_1, E_2, \ldots$

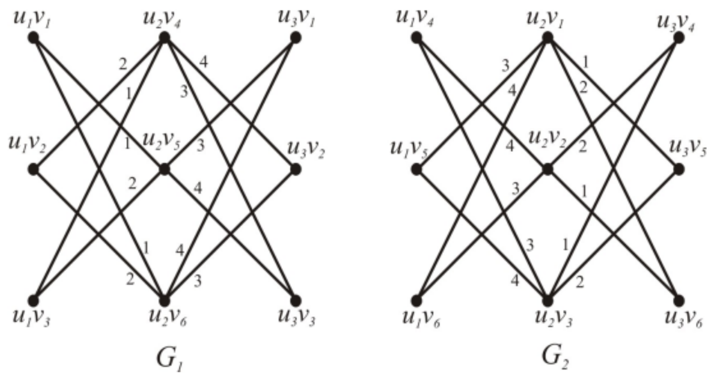

FIGURE 2. P_3 and S_3.

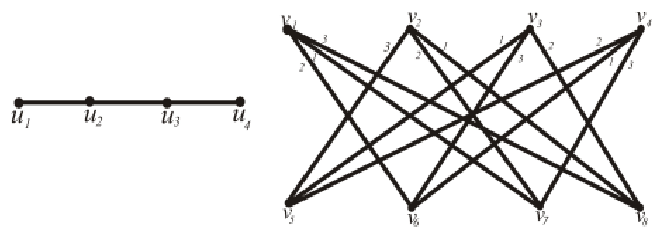

FIGURE 3. P_4 and S_4.

$\ldots, E_\Delta\}$, where E_i denotes the i^{th} color class of f. Since $|E_1| = |E_2| = \cdots = |E_\Delta| = (m-1)n$ (See Figure 2 & Figure 4). We have $||E_i| - |E_j|| \leq 1$, $1 \leq i < j \leq n-1$ for all m and n. Therefore there exists an equitable edge coloring of $P_m \otimes S_n^0$ with upper bound Δ. Hence $\chi'_{=}\left(P_m \otimes S_n^0\right) = \Delta$. □

Theorem 12. *For $m, n \geq 3$, $\chi'_{=}\left(S_m^0 \otimes S_n^0\right) = \Delta$.*

Proof. Consider two Crown graphs S_m^0 and S_n^0 whose vertices and edges are defined as $V\left(S_m^0\right) = \{u_i : 1 \leq i \leq 2m\}$, $E\left(S_m^0\right) = \{(u_i, u_k) : 1 \leq i \leq m, m+1 \leq k \leq 2m\}$ and $V\left(S_n^0\right) = \{v_j : 1 \leq j \leq 2n\}$, $E\left(S_n^0\right) = \{(v_j, v_l) : 1 \leq j \leq n, n+1 \leq l \leq 2n\}$ where S_m^0 consists of $2m$ vertices and $(m-1)$ edges and S_n^0 consists of $2n$ vertices and $n(n-1)$ edges respectively.

By the tensor product of S_m^0 and S_n^0, we obtain

$$V\left(S_m^0 \otimes S_n^0\right) = \{u_i v_j : 1 \leq i \leq 2m; 1 \leq j \leq 2n\}$$

and

$$E\left(S_m^0 \otimes S_n^0\right) = \{(u_i v_j, u_k v_l) : 1 \leq i \neq k \leq 2m; 1 \leq j \neq l \leq 2n\}.$$

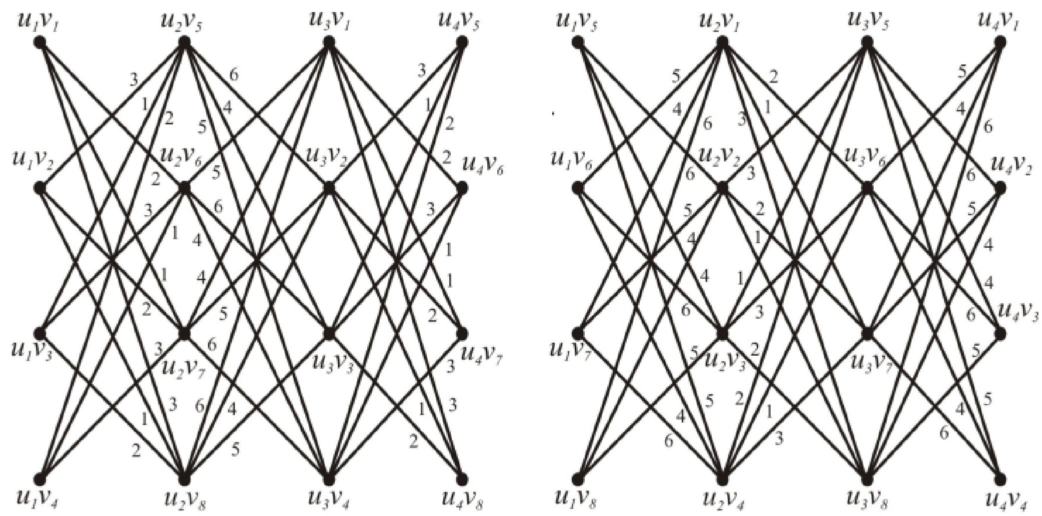

FIGURE 4. $P_4 \otimes S_4$.

Let $e_{(i)(j)(k)(l)}$ be the edge connecting the vertices u_iv_j and u_kv_l of $S_m^0 \otimes S_n^0$ and is denoted by e_r, $r = 1, 2, \cdots 2\,(m-1)\,(n-1)\,mn$.

Since $S_m^0 \otimes S_n^0$ is isomorphic to $S_n^0 \otimes S_m^0$. Without loss of generality, we assume $m \le n$ for all cases of m and n. Let $G = S_m^0 \otimes S_n^0$ and f be a function defined by $f : S \to C$ where $S = V\,(G) \cup E\,(G)$ and $C = \{1, 2, \ldots, \Delta\,(G)\}$.

The equitable edge coloring of G is obtained as follows:

$$f\,(u_iv_j, u_kv_l) = 2i + j + \Delta\,(k + l) + r\,(\mathrm{mod}\Delta)$$

for

$$1 \le i \ne k \le 2m; 1 \le j \ne l \le 2n$$

By using this pattern of coloring the graph G is equitably edge colored. For example consider the case of $m = n = 3$ (See Figure 5) the edge coloring of its tensor product graph requires only 4 colors which is equal to Δ of $S_3^0 \otimes S_3^0$. Clearly, it is evident that f is a Δ-equitable edge coloring of $S_m^0 \otimes S_n^0$. Let $E\left(S_m^0 \otimes S_n^0\right) = \{E_1, E_2, \ldots, E_\Delta\}$, where E_i denotes the i^{th} color class of f. Since $|E_1| = |E_2| = \cdots = |E_\Delta| = 2mn$. We have $||E_i| - |E_j|| \le 1$,$1 \le i < j \le n - 1$ for all m and n. We see that there exists an Δ-equitable edge coloring of $S_m^0 \otimes S_n^0$. Hence $\chi'_{=}\left(S_m^0 \otimes S_n^0\right) = \Delta$. □

4. CONCLUSION

In this paper, the equitable edge coloring of tensor product of Path with Crown graph and together with two Crown graphs are proved, the proofs establishes an

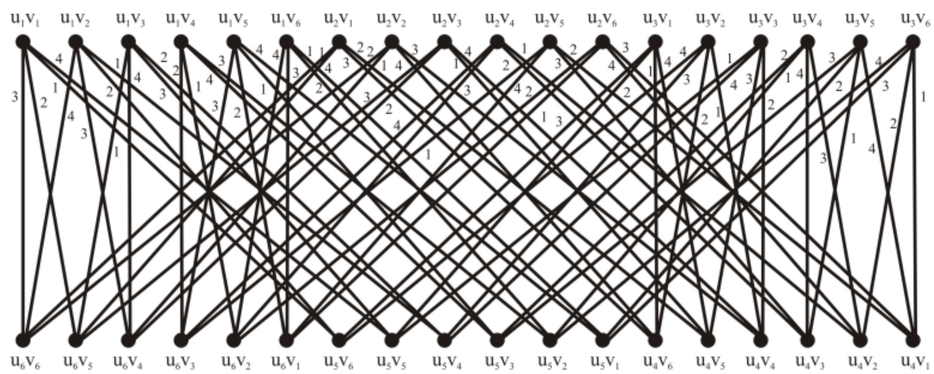

FIGURE 5. $S_3^0 \otimes S_3^0$.

optimal solution. Many research works has been done on the area of chromatic number of graph products. But the idea of equitable coloring in the product of graphs is a recent approach. We are most interested in linking tensor product with equitable edge colorings of graphs. It would be further interesting to determine the bounds of equitably edge coloring of other products of graphs.

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