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Morita equivalence based on Morita context for arbitrary semigroups

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Abstract

In this paper, we study the Morita context for arbitrary semigroups. We prove that, for two semigroups S and T, if there exists a Morita context (S, T, P, Q, τ, μ) (not necessary unital) such that the maps τ and μ are surjective, the categories US-FAct and UT-FAct are equivalent. Using this result, we generalize Theorem 2 in [2] to arbitrary semigroups. Finally, we give a characterization of Morita context for semigroups.

Keywords: semigroup, S-act, Morita context, functor, category.

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1. Introduction

Morita theory characterizes equivalences between module categories over rings with 1. Kyuno [5] studied Morita theory for rings without 1. Knauer [4] and Banschewski [1] independently generalized this theory to monoids. Banschewski [1] proved that for two semigroups S and T, if the two categories S-Act and T-Act are equivalent, then S is isomorphic to T. Talwar [8] extended Morita theory to semigroups with local units. He proved that for two semigroups with local units S and T, the two categories FS-Act and FT-Act are equivalent $\Longleftrightarrow$ there is a unitary Morita context (S, T, P, Q, τ, μ) such that the maps τ and μ are surjective, where $FS\text{-Act} = \{M \in S\text{-Act} | SM = M \text{ and } S \otimes \text{Hom}_S(S, M) \cong M\}$. In [7], Talwar investigated strong Morita equivalence for factorisable semigroups. He got that if there is a unitary Morita context (S, T, P, Q, τ, μ) such that the maps τ and μ are surjective, then S and T are strongly Morita equivalent. Chen and Shum [2] showed that, for factorisable semigroups S and T, if there exists a unitary Morita context (S, T, P, Q, τ, μ) such that the maps τ and μ are surjective, then the categories US-FAct and UT-FAct are equivalent.

In this paper, we mainly use the techniques of paper [5] to study the corresponding problems for arbitrary semigroups. The paper is constructed as follows: In Section 2, we recall some basic notions; In Section 3, we give the main results of the paper. We prove

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that, for two semigroups S and T, if there exists a Morita context (S, T, P, Q, τ, μ) (not necessary unital) such that the maps τ and μ are surjective, the categories US-FAct and UT-FAct are equivalent. Also, we extend Theorem 2 in [2] to arbitrary semigroups. In Section 4, we give a characterization of Morita contexts for semigroups.

2. Preliminaries

Let S be a semigroup. A set M is a left S-act if there is a function from $S \times M$ to M, denoted $(s, m) \to sm$, such that $(st)m = s(tm)$ $(\forall s, t \in S, m \in M)$. If M is a left S-act, we write ${}_SM$. A left S-act M is said to be unitary if $M = SM$. Similarly, we can define right acts over semigroups.

Let M and N be two S-acts. A map $f : M \to N$ is an S-morphism if f satisfies $f(sm) = sf(m)$, $(\forall m \in M, s \in S)$. Let $\mathrm{Hom}_S(M, N)$ denote the set of all S-morphisms from ${}_SM$ to ${}_SN$. Denote by $\mathrm{End}_S(M)$ the set of all S-morphisms from M to itself. Let S-Act denote the category of left acts over a semigroup S.

The unital left S-acts together with the S-morphisms form a full subcategory of S-Act, which we shall denote by US-Act.

Let S and T be two semigroups. An S-T-biact is a set M which is both left S-act and right T-act and $(sm)t = s(mt)$ for all $s \in S, t \in T$ and all $m \in M$. A biact is said to be unitary if it is left and right unitary. If M and N are S-T-biact, a map $f : M \to N$ is called biact morphism if f satisfies $f(sm) = sf(m)$ and $f(mt) = f(m)t$ for all $m \in M, s \in S, t \in T$.

Let S be a semigroup and $M \in S$-Act. An equivalence R on S is a congruence if for all $s, t, a \in S$,

$$(s, t) \in R \Rightarrow (as, at) \in R, (sa, ta) \in R.$$

An equivalence ρ on ${}_SM$ is a congruence if for all $s \in S, m, n \in M$,

$$(m, n) \in \rho \Rightarrow (sm, sn) \in \rho.$$

If ρ is a congruence on M, then M/ρ is also a left S-act. The act M/ρ is called a quotient act. Let ϵ be the identity congruence on M.

Let S be a semigroup and $M \in S$-Act. According to [2], we use the following notations.

$$\zeta_M = \{(x, y) \in M \times M | sx = sy, \forall s \in S\};$$

$$US\text{-FAct} = \{M \in US\text{-Act} | \zeta_M = \epsilon\}.$$

Obviously, ζ_M is a congruence on M.

For a right S-act A_S and a left S-act ${}_SB$, the tensor product $A \otimes_S B$ exists. In fact, $A \otimes_S B = (A \times B)/\sigma$, where σ is the equivalence on $A \times B$ generated by

$$\mathcal{R} = \{((xs, y), (x, sy)) | a \in A, b \in B, s \in S\}.$$

We denote the element $(x, y)\sigma$ of $A \otimes_S B$ by $x \otimes y$.

By Proposition 1.4.10 of [3], we have that $a \otimes b = c \otimes d \iff (a, b) = (c, d)$ or there is a sequence

$$(a, b) = (x_1, y_1) \to (x_2, y_2) \to \cdots \to (x_n, y_n) = (c, d)$$

such that either $((x_i, y_i), (x_{i+1}, y_{i+1})) \in T$ or $((x_{i+1}, y_{i+1}), (x_i, y_i)) \in T$, where $1 \leq i \leq n - 1$.

If A is a right S-act and B is an S-T-biact, then $A \otimes_S B$ is a right T-act with

$$(a \otimes b)t = a \otimes bt;$$

similarly, if A is a T-S-biact and B is a left S-biact, then $A \otimes_S B$ is a left T-act with

$$t(a \otimes b) = ta \otimes b$$

(Proposition 3.1, [8]).

3. Morita equivalence for semigroups

In this section, S and T are arbitrary semigroups. If there exists a Morita context (S,T,P,Q,τ,μ), we shall prove that the two categories $F: US\text{-FAct} \rightleftharpoons UT\text{-FAct}: G$ are equivalent. Furthermore, if (S,T,P,Q,τ,μ) is unital, we get that $F \cong (Q\otimes -)/\zeta_{(Q\otimes -)}$ and $G \cong (P\otimes -)/\zeta_{(P\otimes -)}$. This generalizes Theorem 2 in [2].

3.1. Definition. [8] Let S and T be two semigroups. If there exist sets P and Q, such that
1) P is an S-T-biact, Q is a T-S-biact;
2) there are biact morphisms $\tau: P\otimes_T Q \to S$ and $\mu: Q\otimes_S P \to T$ written correspondingly as

$$\tau(p\otimes q) = <p,q>, \quad \mu(q\otimes p) = [q,p]$$

such that

$$<p_1,q>\cdot p_2 = p_1\cdot[q,p_2], \quad [q_1,p]\cdot q_2 = q_1\cdot <p,q_2>$$

for each $p,p_1,p_2 \in P, q,q_1,q_2 \in Q$. Then (S,T,P,Q,τ,μ) is called a Morita context.

By Proposition 3.1 in [8], we have $\tau(p\otimes q)s = \tau((p\otimes q)s) = \tau(p\otimes qs)$, where $p \in P, q\in Q, s\in S$. We will use this fact in the proof of Lemma 3.2 and Lemma 3.4.

3.2. Lemma. *Let (S,T,P,Q,τ,μ) be a Morita context, where τ and μ are surjective. Then*
1) For all $M \in US$-FAct, set $U = Q\times M$. Then $\widetilde{(Q,M)} = (Q\times M)/\rho_{(Q\times M)} \in UT$-FAct, where $\rho_{Q\times M} = \{((q,m),(q',m')) \in U\times U \mid \tau(p\otimes q)m = \tau(p\otimes q')m', \forall p\in P\}$.
2) For all $N \in UT$-FAct, set $V = P\times N$. Then $\widetilde{(P,N)} = (P\times N)/\rho_{(P\times N)} \in UT$-FAct, where $\rho_{P\times N} = \{((p,n),(p',n')) \in V\times V \mid \mu(q\otimes p)n = \mu(q\otimes p')n', \forall q\in Q\}$.

Proof *1) i) Clearly, ρ_U is an equivalence on U. Set $\widetilde{(Q,M)} = U/\rho_U$. Denote by $\overline{(r,m)}$ the equivalence class $(r,m)\rho_U$. For $t\in T$, we can write $t = \mu(q\otimes p)$ since μ is surjective.*

For all $\overline{(q,m)} \in \widetilde{(Q,M)}, \mu(q'\otimes p') \in T$, define

$$\mu(q'\otimes p')\overline{(q,m)} = \overline{(q',\tau(p'\otimes q)m)}.$$

If $\overline{(q_1,m_1)} = \overline{(q_2,m_2)}$, for all $p\in P$, we have $<p,q_1>m_1 = <p,q_2>m_2$. Hence, the definition is independent of the choice of equivalence class representative.

If $\mu(q_1\otimes p_1) = \mu(q_2\otimes p_2)$, for all $x\in P$, we have

$$\begin{aligned} <x,q_1><p_1,q>m &= <x,q_1<p_1,q>>m = <x,[q_1,p_1]q>m \\ &= <x,[q_2,p_2]q>m = <x,q_2><p_2,q>m. \end{aligned}$$

Hence,

$$\overline{(q_1,<p_1,q>m)} = \overline{(q_2,<p_2,q>m)}.$$

Therefore, the definition is well-defined.

For all $\mu(q_1\otimes p_1), \mu(q_2\otimes p_2) \in T, \overline{(q,m)} \in \widetilde{(Q,M)}$, we have

$$(\mu(q_1\otimes p_1)\mu(q_2\otimes p_2))\overline{(q,m)} = \mu([q_1,p_1]q_2\otimes p_2)\overline{(q,m)} = \overline{([q_1,p_1]q_2,\tau(p_2\otimes q)m)}$$

and

$$\mu(q_1\otimes p_1)(\mu(q_2\otimes p_2)\overline{(q,m)}) = \mu(q_1\otimes p_1)\overline{(q_2,\tau(p_2\otimes q)m)} = \overline{(q_1,\tau(p_1\otimes q_2)\tau(p_2\otimes q)m)}.$$

Then $(\mu(q_1\otimes p_1)\mu(q_2\otimes p_2))\overline{(q,m)} = \mu(q_1\otimes p_1)(\mu(q_2\otimes p_2)\overline{(q,m)})$. This means that $\widetilde{(Q,M)}$ is a left T-Act.

ii) Suppose $(\overline{(q,m)},\overline{(q',m')}) \in \zeta_{\widetilde{(Q,M)}}$. For all $y\in Q, x\in P$, we have

$$\mu(y\otimes x)\overline{(q,m)} = \mu(y\otimes x)\overline{(q',m')}.$$

That is,
$$\overline{(y, \tau(x \otimes q)m)} = \overline{(y, \tau(x \otimes q')m')}.$$
This implies that
$$\tau(p \otimes y)\tau(x \otimes q)m = \tau(p \otimes y)\tau(x \otimes q^{'})m^{'},$$
for all $p \in P$*. Since* $M \in US$*-FAct, we have*
$$\tau(x \otimes q)m = \tau(x \otimes q^{'})m^{'}.$$
For arbitrary of x*, we get that* $\overline{(q, m)} = \overline{(q', m')}$.

iii) For all $m \in M$*, since* $M = SM$ *and* τ *is surjective, we have* $m = \tau(p \otimes q^{'})m^{'}$*, where* $m^{'} \in M$*. For all* $\overline{(q, m)} \in \widetilde{(Q, M)}$*, we have*
$$\overline{(q, m)} = \overline{(q, \tau(p \otimes q')m')} = \mu(q \otimes p)\overline{(q', m')} \in T\widetilde{(Q, M)}.$$
Hence, we get $T\widetilde{(Q, M)} = \widetilde{(Q, M)}$*. Therefore,* $\widetilde{(Q, M)} \in UT$*-FAct.*

2) For all $\overline{(p, n)} \in \widetilde{(P, N)}, \tau(p^{'} \otimes q^{'}) \in S$*, define*
$$\tau(p^{'} \otimes q^{'})\overline{(p, n)} = \overline{(p', \mu(q' \otimes p)n)}.$$
Similarly, we can prove $\widetilde{(P, N)} \in US$*-FAct.*

3.3. Theorem. *Let* S *and* T *be two semigroups. If* (S, T, P, Q, τ, μ) *is a Morita context with* τ *and* μ *surjective, then we have the category equivalence* $F : US\text{-FAct} \rightleftharpoons UT\text{-FAct} : G$*, where* $F = (Q \times -)/\rho_{(Q\times -)}$ *and* $G = (P \times -)/\rho_{(P\times -)}$.

Proof Let $f : M \longrightarrow N$ be an S-morphism, where $M, N \in US$-FAct. Define $\tilde{f} : \widetilde{(Q, M)} \longrightarrow \widetilde{(Q, N)}$ by
$$\tilde{f}(\overline{(q, m)}) = \overline{(q, f(m))}.$$
Suppose $\overline{(q, m)} = \overline{(q', m')}$. For all $p \in P$, we have $\tau(p \otimes q)m = \tau(p \otimes q^{'})m^{'}$. This implies that $f(\tau(p \otimes q)m) = f(\tau(p \otimes q^{'})m^{'})$. Since f is an S-morphism, it follows that $\tau(p \otimes q)f(m) = \tau(p \otimes q^{'})f(m^{'})$. Hence, $\overline{(q, f(m))} = \overline{(q', f(m'))}$. This proves that $\tilde{f}$ is well-defined.

It is easy to check that $\tilde{f}$ is a left T-morphism.

Let $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be two S-morphisms, where $U, V, W \in US$-FAct. Let $\tilde{f} : \widetilde{(Q, U)} \longrightarrow \widetilde{(Q, V)}$ and $\tilde{g} : \widetilde{(Q, V)} \longrightarrow \widetilde{(Q, W)}$ be T-morphisms determined by f and g respectively. Then $\widetilde{gf} = \tilde{g}\tilde{f}$. In fact, since $gf : U \longrightarrow W$ is an S-morphism, we have a T-morphism $\widetilde{gf} : \widetilde{(Q, U)} \longrightarrow \widetilde{(Q, W)}$. This implies that $\text{dom}(\widetilde{gf}) = \widetilde{(Q, U)} = \text{dom}(\tilde{g}\tilde{f})$. For all $\overline{(q, u)} \in \widetilde{(Q, U)}$, we have
$$\widetilde{gf}(\overline{(q, u)}) = \overline{(q, gf(u))} = \tilde{g}\overline{(q, f(u))} = \tilde{g}\tilde{f}\overline{(q, u)}.$$

Define $F : US\text{-FAct} \longrightarrow UT\text{-FAct}$ by $F(M) = (Q \times M)/\rho_{(Q\times M)} = \widetilde{(Q, M)}$ and $F(f) = \tilde{f}$, for all $M, N \in US\text{-FAct}, f \in \text{Hom}_S(M, N)$. Then F is a functor.

Similarly, for $U, V \in UT$-FAct, if $f : U \to V$ is a T-morphism, we can define S-morphism $\bar{f} : \widetilde{(P, U)} \longrightarrow \widetilde{(P, V)}$ by
$$\bar{f}(\overline{(p, u)}) = \overline{(p, f(u))}.$$
Also, for $U, V, W \in UT$-FAct, if $f : U \longrightarrow V$ and $g : V \longrightarrow W$ be two T-morphisms, then $\overline{gf} = \bar{g}\bar{f}$.

We can define a functor $G : UT\text{-FAct} \longrightarrow US\text{-FAct}$ by $G(N) = (P \times N)/\rho_{(P\times N)} = \widetilde{(P, N)}$ and $G(f) = \bar{f}$, for all $N \in UT\text{-FAct}, g \in \text{Hom}_T(M, N)$.

For $M \in US$-FAct, we have

$$GF(M) = G((\widetilde{Q, M})) = (P, \widetilde{\widetilde{(Q, M)}}).$$

Define $\eta_M : M \longrightarrow (P, \widetilde{\widetilde{(Q, M)}})$ by

$$\tau(p \otimes q)m \mapsto \overline{(p, \overline{(q, m)})}.$$

For all $p, p^{'} \in P, q, q^{'} \in Q, m, m^{'} \in M$, we have

$$\begin{aligned}
& \tau(p \otimes q)m = \tau(p^{'} \otimes q^{'})m^{'} \\
\Leftrightarrow \quad & \tau(x \otimes y)\tau(p \otimes q)m = \tau(x \otimes y)\tau(p^{'} \otimes q^{'})m^{'}, \\
& \text{for all } x \in P, y \in Q \text{ (since } M \in US\text{-FAct)} \\
\Leftrightarrow \quad & \overline{(y, \tau(p \otimes q)m)} = \overline{(y, \tau(p' \otimes q')m')}, \text{ for all } y \in Q \\
\Leftrightarrow \quad & \mu(y \otimes p)\overline{(q, m)} = \mu(y \otimes p^{'})\overline{(q', m)}, \text{ for all } y \in Q \\
\Leftrightarrow \quad & \overline{(p, \overline{(q, m)})} = \overline{(p', \overline{(q', m')})}.
\end{aligned}$$

This shows that η_M is well-defined and injective. It is obvious that η_M is surjective. For $m \in M$, write $m = \tau(p^{'} \otimes q^{'})m^{'}$, where $p^{'} \in P, q^{'} \in Q, m^{'} \in M$. For all $p \in P, q \in Q$, we have

$$\begin{aligned}
\eta_M(\tau(p \otimes q)\tau(p^{'} \otimes q^{'})m^{'}) &= \overline{(p, \overline{(q, \tau(p' \otimes q')m')})} = \overline{(p, \mu(q \otimes p')\overline{(q', m')})} \\
&= \tau(p \otimes q)\overline{(p', \overline{(q', m')})} = \tau(p \otimes q)\eta_M(\tau(p^{'} \otimes q^{'})m^{'}).
\end{aligned}$$

Hence, η_M is an S-isomorphism.

Let $f : M \longrightarrow N$ be an S-morphism. For $m = \tau(p \otimes q)m^{'} \in M$, we have

$$\begin{aligned}
GF(f)\eta_M(m) &= GF(f)\eta_M(\tau(p \otimes q)m^{'}) = GF(f)(\overline{(p, \overline{(q, m')})}) \\
&= \overline{(p, F(f)(\overline{(q, m')}))} = \overline{(p, \overline{(q, f(m'))})} = \eta_N f(m).
\end{aligned}$$

Hence, we have the following commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\eta_M \downarrow & & \downarrow \eta_N \\
GF(M) & \xrightarrow{GF(f)} & GF(N).
\end{array}$$

Therefore, $GF \cong 1_{US\text{-FAct}}$.

Similarly, we can prove that $FG \cong 1_{UT\text{-FAct}}$. This get the desired result. □

3.4. Lemma. *Let (S, T, P, Q, τ, μ) be a Morita context and $M \in US$-FAct. If $q_1 \otimes m_1 = q_2 \otimes m_2 \in Q \otimes M$, we have $\overline{(q_1, m_1)} = \overline{(q_2, m_2)}$.*

Proof 1) Suppose $((q_1, m_1), (q_2, m_2)) \in T$. Without loss of generality, We suppose $q_2 = q_1 s, m_1 = s m_2$, where $s \in S$. Then

$$\tau(p \otimes q_1)m_1 = \tau(p \otimes q_1)s m_2 = \tau(p \otimes q_1 s)m_2,$$

for all $p \in P$. Hence, we have $\overline{(q_1, m_1)} = \overline{(q_1 s, m_2)} = \overline{(q_2, m_2)}$.

2) If $q_1 \otimes m_1 = q_2 \otimes m_2$, By Proposition 1.4.10 of [3], we have that $(q_1, m_1) = (q_2, m_2)$ or for some positive integer $n > 1$, there is a sequence

$$(q_1, m_1) = (y_1, x_1) \to (y_2, x_2) \to \cdots \to (y_n, x_n) = (q_2, m_2)$$

in which, for each i in $\{1, 2, \cdots, n-1\}$, either $((y_i, x_i), (y_{i+1}, x_{i+1})) \in \mathcal{R}$ or $((y_{i+1}, x_{i+1}), (y_i, x_i)) \in \mathcal{R}$. By part 1), we can easily get that $\overline{(q_1, m_1)} = \overline{(q_2, m_2)}$. □

3.5. Definition. Let S and T be two semigroups. A Morita context (S, T, P, Q, τ, μ) is called unital, if P is a unital S-T-biact and Q is a unital T-S-biact.

3.6. Lemma. *Let (S,T,P,Q,τ,μ) be a unital Morita context and $M \in US\text{-}FAct$. Then we have a T-isomorphism $(Q\times M)/\rho_{(Q\times M)} \cong (Q\otimes M)/\zeta_{(Q\otimes M)}$.*

Proof Define a map $\varphi : (Q\times M)/\rho_{(Q\times M)} \to (Q\otimes M)/\zeta_{(Q\otimes M)}$ by $\varphi(\overline{(q,m)}) = (q\otimes m)\zeta$, where $(q\otimes m)\zeta$ represent the congruence class $(q\otimes m)\zeta_{(Q\otimes M)}$.

Suppose $\overline{(q_1,m_1)}, \overline{(q_2,m_2)} \in \widetilde{(Q,M)}$. If $\overline{(q_1,m_1)} = \overline{(q_2,m_2)}$, we have $\tau(p\otimes q_1)m_1 = \tau(p\otimes q_2)m_2$, for all $p\in P$. Then

$$\mu(y\otimes x)(q_1\otimes m_1) = \mu(y\otimes x)q_1\otimes m_1 = y\otimes\tau(x\otimes q_1)m_1 = y\otimes\tau(x\otimes q_2)m_2 = \mu(y\otimes x)(q_2\otimes m_2),$$

for all $y\in Q, x\in P$. This implies that $(q_1\otimes m_1)\zeta = (q_2\otimes m_2)\zeta$. Therefore, φ is well-defined. Obviously, φ is surjective.

If $(q_1\otimes m_1)\zeta = (q_2\otimes m_2)\zeta$, for all $x\in P, y\in Q$, we have

$$\mu(y\otimes x)(q_1\otimes m_1) = \mu(y\otimes x)(q_2\otimes m_2).$$

Since $y\otimes\tau(x\otimes q_1)m_1 = y\tau(x\otimes q_1)\otimes m_1 = \mu(y\otimes x)(q_1\otimes m_1)$, we get

$$y\otimes\tau(x\otimes q_1)m_1 = y\otimes\tau(x\otimes q_2)m_2.$$

By Lemma 3.4, we have

$$\overline{(y,\tau(x\otimes q_1)m_1)} = \overline{(y,\tau(x\otimes q_2)m_2)}.$$

For all $p\in P$, we have

$$\tau(\tau(p\otimes y)x\otimes q_1)m_1 = \tau(p\otimes y)\tau(x\otimes q_1)m_1 = \tau(p\otimes y)\tau(x\otimes q_2)m_2 = \tau(\tau(p\otimes y)x\otimes q_2)m_2.$$

Since P is unitary and τ is surjective, we get

$$\{\tau(p\otimes y)x | \text{for all } p,x\in P, q\in Q\} = SP = P.$$

Then $\overline{(q_1,m_1)} = \overline{(q_2,m_2)}$. This proves that φ is injective.

For all $\overline{(q,m)} \in \widetilde{(Q,M)}$, $\mu(y\otimes x)\in T$, we have

$$\begin{aligned}\varphi(\mu(y\otimes x)\overline{(q,m)}) &= \varphi(\overline{(y,\tau(x\otimes q)m)}) = (y\otimes\tau(x\otimes q)m)\zeta \\ &= (y\tau(x\otimes q)\otimes m)\zeta = (\mu(y\otimes x)q\otimes m)\zeta \\ &= \mu(y\otimes x)((q\otimes m)\zeta) = \mu(y\otimes x)\varphi(\overline{(q,m)}).\end{aligned}$$

Hence, φ is a T-isomorphism. That is, $(Q\times M)/\rho_{(Q\times M)} \cong (Q\otimes M)/\zeta_{(Q\otimes M)}$ as left T-act. □

By Theorem 3.3 and Lemma 3.6, we have the following theorem which generalizes Theorem 2 in paper [2].

3.7. Theorem. *Let S and T be two semigroups. If (S,T,P,Q,τ,μ) be a unital Morita context with τ and μ are surjective, then we have the category equivalence $F : US\text{-}FAct \rightleftharpoons UT\text{-}FAct : G$, where $F = (Q\otimes -)/\zeta_{(Q\otimes -)}$ and $G = (P\otimes -)/\zeta_{(P\otimes -)}$.*

4. Characterization of Morita context

In this section, we give an equivalent condition of Morita context in semigroup settings. Also, we give a characterization of Morita context for factorisable semigroups. Similar to Theorem 1 in [6], we have the following.

4.1. Theorem. *Let P and Q be two sets. We have the following equivalent conditions.*
1) There exist two semigroups S and T such that (S,T,P,Q,τ,μ) is a Morita context.
2) There exist maps $\Gamma : P\times Q\times P \to P$ and $\Delta : Q\times P\times Q \to Q$ such that
I) $\Gamma(\Gamma((p_1,q_1,p_2)),q_2,p_3) = \Gamma((p_1,\Delta((q_1,p_2,q_2)),p_3)) = \Gamma(p_1,q_1,\Gamma((p_2,q_2,p_3)))$;
II) $\Delta(\Delta((q_1,p_1,q_2)),p_2,q_3) = \Delta(q_1,\Gamma((p_1,q_2,p_2)),q_3) = \Delta(q_1,p_1,\Delta((q_2,p_2,q_3)))$.

Proof 1) $\Rightarrow$ 2) : Suppose that (S,T,P,Q,τ,μ) is a Morita context. Define $\Gamma : P \times Q \times P \to P$ and $\Delta : Q \times P \times Q \to Q$ by putting $\Gamma((p_1,q_1,p_2)) = \tau(p_1 \otimes q_1)\cdot p_2$ and $\Delta((q_1,p_1,q_2)) = \mu(q_1 \otimes p_1)\cdot q_2$. We can easily check that Γ and Δ satisfy the conditions I) and II).

2) $\Rightarrow$ 1) : Define $H_a : P \to P$ by putting $H_a(p) = \Gamma((a,p))$ and define $K_b : Q \to Q$ by putting $K_b(q) = \Delta((b,q))$, where $a \in P \times Q$ and $b \in Q \times P$.

We write $\mathcal{X} = \{H_a | a \in P \times Q\}$ and $\mathcal{Y} = \{K_b | b \in Q \times P\}$. For all $H_{(p_1,q_1)}, H_{(p_2,q_2)} \in \mathcal{X}$, for all $p \in P$, we have

$$H_{(p_1,q_1)}H_{(p_2,q_2)}(p) = \Gamma((p_1,q_1,\Gamma(p_2,q_2,p))) = \Gamma((\Gamma(p_1,q_1,p_2),q_2,p)) = H_{(\Gamma(p_1,q_1,p_2),q_2)}(p).$$

That is, $H_{(p_1,q_1)}H_{(p_2,q_2)} = H_{(\Gamma((p_1,q_1,p_2)),q_2)} \in \mathcal{X}$. Then we easily get that $\mathcal{X}$ is a subsemigroup of $\mathrm{End}(P)$. Similarly, we have that $\mathcal{Y}$ is a subsemigroup of $\mathrm{End}(Q)$.

For all $p \in P$, $H_a \in \mathcal{X}$, $K_b \in \mathcal{Y}$, define $H_a \cdot p = \Gamma((a,p))$ and $p \cdot K_b = \Gamma((p,b))$. Then P is a $\mathcal{X}$-$\mathcal{Y}$-biact. Similarly, for all $q \in Q$, we can define $K_b \cdot q = \Delta((b,q))$ and $q \cdot H_a = \Delta((q,a))$. This makes Q to be a $\mathcal{Y}$-$\mathcal{X}$-biact.

Now, we define $\alpha : P \otimes_{\mathcal{Y}} Q \to \mathcal{X}$ and $\beta : Q \otimes_{\mathcal{X}} P \to \mathcal{Y}$ by putting $\alpha(p \otimes q) = H_{(p,q)}$ and $\beta(q \otimes p) = K_{(q,p)}$, where $p \in P$ and $q \in Q$. It is easy to check that α and β are both biact morphisms. Then

$$\alpha(p_1 \otimes q)\cdot p_2 = H_{(p_1,q)}\cdot p_2 = \Gamma((p_1,q,p_2)) = p_1 \cdot K_{(q,p_2)} = p_1 \cdot \beta(q \otimes p_2).$$

Similarly, we have

$$\beta(q_1 \otimes p)q_2 = q_1\alpha(p \otimes q_2).$$

Then $(\mathcal{X},\mathcal{Y},P,Q,\alpha,\beta)$ is a Morita context. □

4.2. Definition. [7] A semigroup S is called factorisable if $S = S^2$.

4.3. Theorem. *Let P and Q be two sets. We have the following equivalent conditions.*

1) There exist two factorisable semigroups S and T such that (S,T,P,Q,τ,μ) is a unital Morita context and τ and μ are surjective.

In this case, $(Q \otimes -)/\zeta_{(Q\otimes -)} : US\text{-}Act \rightleftharpoons UT\text{-}Act : (P \otimes -)/\zeta_{(P\otimes -)}$ are equivalent functors.

2) There exist surjective maps $\Gamma : P \times Q \times P \to P$ and $\Delta : Q \times P \times Q \to Q$ satisfy the two conditions in part 2) of Theorem 4.1 and

III) For all $p, p' \in P$, $q \in Q$, there exist $p_1,p_2 \in P, q_1,q_2 \in Q$ such that

$$\Gamma(((p,q),p')) = \Gamma((\Gamma(p_1,q_1,p_2),q_2,p')).$$

IV) For all $p \in P$, $q,q' \in Q$, there exist $p_1,p_2 \in P, q_1,q_2 \in Q$ such that

$$\Delta(((q,p),q')) = \Delta((\Delta(q_1,p_1,q_2),p_2,q')).$$

Proof 1) $\Rightarrow$ 2) : Since S is factorisable and τ is surjective, for all $p \in P, q \in Q$, there exist $p_1,p_2 \in P, q_1,q_2 \in Q$ such that $\tau(p \otimes q) = \tau(p_1 \otimes q_1)\tau(p_2 \otimes q_2)$. Hence,

$$\begin{aligned}\Gamma(((p,q),p')) &= \tau(p\otimes q)p' = \tau(p_1 \otimes q_1)\tau(p_2 \otimes q_2)p' = \tau(p_1 \otimes q_1)\Gamma((p_2,q_2,p')) \\ &= \Gamma((p_1,q_1,\Gamma(p_2,q_2,p'))) = \Gamma((\Gamma((p_1,q_1,p_2)),q_2,p')).\end{aligned}$$

Therefore, the condition III) holds. Similarly, we can get IV).

By Theorem 3.7 or Theorem 2 in [2], we have the category equivalence $(Q\otimes -)/\zeta_{(Q\otimes -)} : US\text{-Act} \rightleftharpoons UT\text{-Act} : (P \otimes -)/\zeta_{(P\otimes -)}$.

2) $\Rightarrow$ 1) : For all $H_{(p,q)} \in \mathcal{X}, p' \in P$, by the condition III), we have

$$\Gamma(((p,q),p')) = \Gamma((\Gamma(p_1,q_1,p_2),q_2,p')).$$

This implies that

$$\begin{aligned} H_{(p,q)}(p^{'}) &= \Gamma(((p,q),p^{'})) = \Gamma((\Gamma(p_1,q_1,p_2),q_2,p^{'})) \\ &= \Gamma((p_1,q_1,\Gamma((p_2,q_2,p^{'})))) = H_{(p_1,q_1)}H_{(p_2,q_2)}(p^{'}). \end{aligned}$$

That is, $H_{(p,q)} = H_{(p_1,q_1)}H_{(p_2,q_2)}$. This proves that $\mathcal{X}$ is factorisable.

Similarly, we have that $\mathcal{Y}$ is a factorisable semigroup.

Since Γ and Δ are surjective, we obviously have that P and Q are unital as biacts and α and β are surjective. Hence, $(\mathcal{X}, \mathcal{Y}, P, Q, \alpha, \beta)$ is a unital Morita context. □

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