

Complete q th moment convergence of weighted sums for arrays of row-wise extended negatively dependent random variables

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Abstract

In this paper, the complete q th moment convergence of weighted sums for arrays of row-wise extended negatively dependent (abbreviated to END in the following) random variables is investigated. By using Hoffmann-Jørgensen type inequality and truncation method, some general results concerning complete q th moment convergence of weighted sums for arrays of row-wise END random variables are obtained. As their applications, we extend the corresponding result of Wu (2012) to the case of arrays of row-wise END random variables. The complete q th moment convergence of moving average processes based on a sequence of END random variables is obtained, which improves the result of Li and Zhang (2004). Moreover, the Baum-Katz type result for arrays of row-wise END random variables is also obtained.

Keywords: END random variables; Weighted sums; Complete moment convergence; Complete convergence.

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1. Introduction and Lemmas

The concept of complete convergence was given by Hsu and Robbins[1] in the following way. A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant θ if for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \epsilon) < \infty.$$

In view of the Borel-Cantelli lemma, the above result implies that $X_n \rightarrow \theta$ almost surely. Hence the complete convergence is very important tool in establishing almost sure convergence. When $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables, Baum and Katz[2] proved the following

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remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \epsilon n^{1/p})$ for any $\epsilon > 0$, where $S_n = \sum_{i=1}^n X_i$.

1.1. Theorem. $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables, $r > 1/2$ and $p > 1$. Then

$$\sum_{n=1}^{\infty} n^{p-2} P(|S_n| > \epsilon n^r) < \infty \text{ for all } \epsilon > 0,$$

if and only if $E|X|^{p/r} < \infty$, where $EX = 0$ whenever $1/2 < r \leq 1$.

Many useful linear statistics based on a random sample are weighted sums of independent and identically distributed random variables, see, for example, least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. However, in many stochastic model, the assumption that random variables are independent is not plausible. Increases in some random variables are often related to decreases in other random variables, so an assumption of dependence is more appropriate than an assumption of independence. The concept of END random variables was firstly introduced by Liu[3] as follows.

1.2. Definition. Random variables $\{X_i, i \geq 1\}$ are said to be END if there exists a constant $M > 0$ such that both

$$(1.1) \quad P\left(\bigcap_{i=1}^n (X_i \leq x_i)\right) \leq M \prod_{i=1}^n P(X_i \leq x_i)$$

and

$$(1.2) \quad P\left(\bigcap_{i=1}^n (X_i > x_i)\right) \leq M \prod_{i=1}^n P(X_i > x_i)$$

hold for each $n \geq 1$ and all real numbers x_1, x_2, \dots, x_n .

In the case $M = 1$ the notion of END random variables reduces to the well-known notion of so-called negatively dependent (ND) random variables which was introduced by Lehmann[4]. Recall that random variables $\{X_i, i \geq 1\}$ are said to be positively dependent (PD) if the inequalities (1.1) and (1.2) hold both in the reverse direction when $M = 1$. Not looking that the notion of END random variables seems to be a straightforward generalization of the notion of ND, the END structure is substantially more comprehensive. As it is mentioned in Liu[3], the END structure can reflect not only a negative dependent structure but also a positive one, to some extent. Joag-Dev and Proschan[5] also pointed out that negatively associated (NA) random variables must be ND, therefore NA random variables are also END. Some applications for sequences of END random variables have been found. We refer to Shen[6] for the probability inequalities, Liu[3] for the precise large deviations, Chen[7] for the strong law of large numbers and applications to risk theory and renewal theory.

Recently, Baek et al.[8] discussed the complete convergence of weighted sums for arrays of row-wise NA random variables and obtained the following result:

1.3. Theorem. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise NA random variables with $EX_{ni} = 0$ and for some random variable X and constant $C > 0$,

$P(|X_{ni}| > x) \leq CP(|X| > x)$ for all $i \geq 1, n \geq 1$ and $x \geq 0$. Suppose that $\beta \geq -1$, and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants such that

$$(1.3) \quad \sup_{i \geq 1} |a_{ni}| = O(n^{-r}) \text{ for some } r > 0$$

and

$$(1.4) \quad \sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \text{ for some } \alpha \in [0, r).$$

(i) If $\alpha + \beta + 1 > 0$ and there exists some $\delta > 0$ such that $\frac{\alpha}{r} + 1 < \delta \leq 2$, and $s = \max(1 + \frac{\alpha + \beta + 1}{r}, \delta)$, then, under $E|X|^s < \infty$, we have

$$(1.5) \quad \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

(ii) If $\alpha + \beta + 1 = 0$, then, under $E|X| \log(1 + |X|) < \infty$, (1.5) remains true.

If $\beta < -1$, then (1.5) is immediate. Hence Theorem 1.3 is of interest only for $\beta \geq -1$. Baek and Park [9] extended Theorem 1.3 to the case of arrays of row-wise pairwise negatively quadrant dependent (NQD) random variables. However, there is a question in the proofs of Theorem 1.3(i) in Baek and Park [9]. The Rosenthal type inequality plays a key role in this proof, but it is still an open problem to obtain Rosenthal type inequality for pairwise NQD random variables.

When $\beta > -1$, Wu [10] dealt with more general weight and proved the following complete convergence for weighted sums of arrays of row-wise ND random variables. But, the proof of Wu[10] does not work for the case of $\beta = -1$.

1.4. Theorem. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise ND random variables and for some random variable X and constant $C > 0$, $P(|X_{ni}| > x) \leq CP(|X| > x)$ for all $i \geq 1, n \geq 1$ and $x \geq 0$. Let $\beta > -1$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and

$$(1.6) \quad \sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(n^{\alpha}) \text{ for some } 0 < \theta < 2 \text{ and some } \alpha \text{ such that } \theta + \alpha/r < 2.$$

Denote $s = \theta + (\alpha + \beta + 1)/r$. When $s \geq 1$, further assume that $EX_{ni} = 0$ for any $i \geq 1, n \geq 1$.

(i) If $\alpha + \beta + 1 > 0$ and $E|X|^s < \infty$, then (1.5) holds.

(ii) If $\alpha + \beta + 1 = 0$ and $E|X|^{\theta} \log(1 + |X|) < \infty$, then (1.5) holds.

The concept of complete moment convergence was introduced firstly by Chow [11]. As we know, the complete moment convergence implies complete convergence. Moreover, the complete moment convergence can more exactly describe the convergence rate of a sequence of random variables than the complete convergence. So, a study on complete moment convergence is of interest. Liang et al. [12] obtained the complete q th moment convergence theorems of sequences of identically distributed NA random variables. Sung [13] proposed sets of sufficient conditions for complete q th moment convergence of arrays of random variables satisfying Marcinkiewicz-Zygmund and Rosenthal type inequalities. Guo [14] provided some

sufficient conditions for complete moment convergence of row-wise NA arrays of random variables. Li and Zhang [15] established the complete moment convergence of moving average processes based on a sequence of identically distributed NA random variables as follows.

1.5. Theorem. *Suppose that $Y_n = \sum_{i=-\infty}^{\infty} a_{i+n}X_i$, $n \geq 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{-\infty}^{\infty} |a_i| < \infty$ and $\{X_i, -\infty < i < \infty\}$ is a sequence of identically distributed and negatively associated random variables with $EX_1 = 0, EX_1^2 < \infty$. Let $1/2 < r \leq 1, p \geq 1 + 1/(2r)$. Then $E|X_1|^p < \infty$ implies that*

$$\sum_{n=1}^{\infty} n^{rp-2-r}l(n)E\left(\left|\sum_{i=1}^n Y_i\right| - \epsilon n^r\right)^+ < \infty \text{ for all } \epsilon > 0.$$

The aim of this paper is to give a sufficient condition concerning complete q th moment convergence for arrays of row-wise END random variables. As an application, we not only generalize and extend the corresponding results of Baek et al. [8] and Wu [10] under some weaker conditions, but also greatly simplify their proof. Moreover, the complete q th moment convergence of moving average processes based on a sequence of END random variables is also obtained, which improves the result of Li and Zhang [15]. The Baum-Katz type result for arrays of row-wise END random variables is also established.

Before we start our main results, we firstly state some definitions and lemmas which will be useful in the proofs of our main results. Throughout this paper, the symbol C stands for a generic positive constant which may differ from one place to another. The symbol $I(A)$ denotes the indicator function of A . Let $a_n \ll b_n$ denote that there exists a constant $C > 0$ such that $a_n \leq Cb_n$ for all $n \geq 1$. Denote $(x)_+^q = (\max(x, 0))^q$, $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $\log x = \ln \max(e, x)$.

1.6. Definition. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C , such that $P(|X_n| > x) \leq CP(|X| > x)$ for all $x \geq 0$ and $n \geq 1$.

The following lemma establish the fundamental inequalities for stochastic domination, the proof is due to Wu [16].

1.7. Lemma. *Let the sequence $\{X_n, n \geq 1\}$ of random variables be stochastically dominated by a random variable X . Then for any $n \geq 1, p > 0, x > 0$, the following two statements hold:*

$$\begin{aligned} E|X_n|^p I(|X_n| \leq x) &\leq C(E|X|^p I(|X| \leq x) + x^p P(|X| > x)), \\ E|X_n|^p I(|X_n| > x) &\leq CE|X|^p I(|X| > x). \end{aligned}$$

The following lemma is the Hoffmann-Jørgensen type inequality for sequences of END random variables and is obtained by Shen [6].

1.8. Lemma. *Let $\{X_i, i \geq 1\}$ be a sequence of END random variables with $EX_i = 0$ and $EX_i^2 < \infty$ for every $i \geq 1$ and set $B_n = \sum_{i=1}^n EX_i^2$ for any $n \geq 1$. Then for all $y > 0, t > 0, n \geq 1$,*

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq y\right) \leq P\left(\max_{1 \leq k \leq n} |X_k| > t\right) + 2M \cdot \exp\left\{\frac{y}{t} - \frac{y}{t} \log\left(1 + \frac{yt}{B_n}\right)\right\}.$$

1.9. Definition. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if $\lim_{x \rightarrow \infty} \frac{l(x\lambda)}{l(x)} = 1$ for each $\lambda > 0$.

1.10. Lemma. Let X be a random variable and $l(x) > 0$ be a slowly varying function. Then

- (i) $\sum_{n=1}^{\infty} n^{-1} E|X|^\alpha I(|X| > n^\gamma) \leq CE|X|^\alpha \log(1 + |X|)$ for any $\alpha \geq 0, \gamma > 0$,
- (ii) $\sum_{n=1}^{\infty} n^\beta l(n) E|X|^\alpha I(|X| > n^\gamma) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma})$ for any $\beta > -1, \alpha \geq 0, \gamma > 0$,
- (iii) $\sum_{n=1}^{\infty} n^\beta l(n) E|X|^\alpha I(|X| \leq n^\gamma) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma})$ for any $\beta < -1, \alpha \geq 0, \gamma > 0$.

Proof. We only prove (ii). Noting that $\beta > -1$, we have by Lemma 1.5 in Guo[14] that

$$\begin{aligned} \sum_{n=1}^{\infty} n^\beta l(n) E|X|^\alpha I(|X| > n^\gamma) &= \sum_{n=1}^{\infty} n^\beta l(n) \sum_{k=n}^{\infty} E|X|^\alpha I(k^\gamma < |X| \leq (k+1)^\gamma) \\ &= \sum_{k=1}^{\infty} E|X|^\alpha I(k^\gamma < |X| \leq (k+1)^\gamma) \sum_{n=1}^k n^\beta l(n) \\ &\leq C \sum_{k=1}^{\infty} k^{\beta+1} l(k) E|X|^\alpha I(k^\gamma < |X| \leq (k+1)^\gamma) \leq CE|X|^{\alpha+(\beta+1)/\gamma} l(|X|^{1/\gamma}). \end{aligned}$$

□

2. Main Results and the Proofs

In this section, let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of row-wise END random variables with the same M in each row. Let $\{k_n, n \geq 1\}$ be a sequence of positive integers and $\{a_n, n \geq 1\}$ be a sequence of positive constants. If $k_n = \infty$ we will assume that the series $\sum_{i=1}^{\infty} X_{ni}$ converges a.s. For any $x \geq 1, q > 0$, set

$$X'_{ni}(x) = x^{1/q} I(X_{ni} > x^{1/q}) + X_{ni} I(|X_{ni}| \leq x^{1/q}) - x^{1/q} I(X_{ni} < -x^{1/q}),$$

$1 \leq i \leq k_n, n \geq 1$. For any $x \geq 1, q > 0$, it is clear that $\{X'_{ni}(x), 1 \leq i \leq k_n, n \geq 1\}$ is an array of row-wise END random variables, since it is a sequence of monotone transformations of $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$.

2.1. Theorem. Suppose that $q > 0$ and the following three conditions hold:

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$,
- (ii) there exist $0 < r \leq 2$ and $s > q/r$ such that

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r \right)^s < \infty,$$

(iii) $\sup_{x \geq 1} x^{-1/q} \sum_{i=1}^{k_n} |EX'_{ni}(x)| \rightarrow 0$, as $n \rightarrow \infty$. Then for all $\epsilon > 0$,

$$(2.1) \quad \sum_{n=1}^{\infty} a_n E \left(\left| \sum_{i=1}^{k_n} X_{ni} \right| - \epsilon \right)_+^q < \infty.$$

Proof. By Fubini's theorem, we get that

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n E \left(\left| \sum_{i=1}^{k_n} X_{ni} \right| - \epsilon \right)_+^q = \sum_{n=1}^{\infty} a_n \int_0^{\infty} P \left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \epsilon + x^{1/q} \right) dx \\ & \leq \sum_{n=1}^{\infty} a_n P \left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > \epsilon \right) + \sum_{n=1}^{\infty} a_n \int_1^{\infty} P \left(\left| \sum_{i=1}^{k_n} X_{ni} \right| > x^{1/q} \right) dx =: I_1 + I_2. \end{aligned}$$

We prove only $I_2 < \infty$, the proof of $I_1 < \infty$ is analogous. Using a simple integral and Fubini's theorem, we obtain that for any $q > 0$ and a random variable X ,

$$(2.2) \quad \int_1^{\infty} P(|X| > x^{1/q}) dx \leq E|X|^q I(|X| > 1).$$

Then by (2.2) and the subadditivity of probability measure we obtain the estimate

$$\begin{aligned} I_2 & \leq \sum_{n=1}^{\infty} a_n \int_1^{\infty} P \left(\left| \sum_{i=1}^{k_n} X'_{ni}(x) \right| > x^{1/q} \right) dx + \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P(|X_{ni}| > x^{1/q}) dx \\ & \leq \sum_{n=1}^{\infty} a_n \int_1^{\infty} P \left(\left| \sum_{i=1}^{k_n} X'_{ni}(x) \right| > x^{1/q} \right) dx + \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} E|X_{ni}|^q I(|X_{ni}| > 1) \\ & =: I_3 + I_4. \end{aligned}$$

By assumption (i), we have $I_4 < \infty$. By assumption (iii), we deduce that

$$(2.3) \quad I_3 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} P \left(\left| \sum_{i=1}^{k_n} (X'_{ni}(x) - EX'_{ni}(x)) \right| > x^{1/q}/2 \right) dx.$$

Set $B_n = \sum_{i=1}^{k_n} E(X'_{ni}(x) - EX'_{ni}(x))^2$, $y = x^{1/q}/2$, $t = x^{1/q}/(2s)$, we have by assumption (iii) and Lemma 1.8 that

$$\begin{aligned} & P \left(\left| \sum_{i=1}^{k_n} (X'_{ni}(x) - EX'_{ni}(x)) \right| > x^{1/q}/2 \right) \\ & \leq P \left(\max_{1 \leq i \leq k_n} |X'_{ni}(x) - EX'_{ni}(x)| > x^{1/q}/(2s) \right) + 2Me^s \cdot \left(1 + \frac{x^{2/q}}{4sB_n} \right)^{-s} \\ (2.4) \quad & \leq P \left(\max_{1 \leq i \leq k_n} |X'_{ni}(x)| > x^{1/q}/(4s) \right) + 2Me^s (4s)^s x^{-2s/q} B_n^s \\ & \leq \sum_{i=1}^{k_n} P \left(|X'_{ni}(x)| > x^{1/q}/(4s) \right) + 2Me^s (4s)^s x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2 \right)^s \\ & \ll \sum_{i=1}^{k_n} P \left(|X'_{ni}(x)| > x^{1/q}/(4s) \right) + x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2 \right)^s. \end{aligned}$$

By (2.3) and (2.4), we obtain that

$$(2.1) \quad I_3 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) dx \\ + \sum_{n=1}^{\infty} a_n \int_1^{\infty} x^{-2s/q} \left(\sum_{i=1}^{k_n} E(X'_{ni}(x))^2\right)^s dx \\ = I_4 + I_5.$$

Since $|X'_{ni}(x)| \leq |X_{ni}|$, we have $P\left(|X'_{ni}(x)| > x^{1/q}/(4s)\right) \leq P\left(|X_{ni}| > x^{1/q}/(4s)\right)$. By (2.2) and assumption (i), we conclude that

$$I_4 \leq \sum_{n=1}^{\infty} a_n \int_1^{\infty} \sum_{i=1}^{k_n} P\left(|X_{ni}| > x^{1/q}/(4s)\right) dx \\ \leq \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} (4s)^q E|X_{ni}|^q I(|X_{ni}| > 1/(4s)) < \infty.$$

Hence, to complete the proof, it suffices to show that $I_5 < \infty$. From the definition of $X'_{ni}(x)$, since $0 < r \leq 2$, we have by C_r -inequality that

$$(2.5) \quad E(X'_{ni}(x))^2 \ll EX_{ni}^2 I(|X_{ni}| \leq x^{1/q}) + x^{2/q} P(|X_{ni}| > x^{1/q}) \leq 2x^{(2-r)/q} E|X_{ni}|^r.$$

Noting that $s > q/r$, it is clear that $\int_1^{\infty} x^{-sr/q} dx < \infty$. Then we have by (2.5) and assumption (ii) that

$$I_5 \ll \sum_{n=1}^{\infty} a_n \int_1^{\infty} x^{-2s/q} \left(\sum_{i=1}^{k_n} x^{(2-r)/q} E|X_{ni}|^r\right)^s dx \\ \leq \sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r\right)^s \int_1^{\infty} x^{-sr/q} dx \ll \sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^r\right)^s < \infty.$$

Therefore, (2.1) holds. \square

2.2. Remark. Note that

$$\sum_{n=1}^{\infty} a_n E\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| - \epsilon\right)_+^q = \int_0^{\infty} \sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon + x^{1/q}\right) dx.$$

Thus, we obtain that the complete q th moment convergence implies the complete convergence, i.e., (2.1) implies

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \epsilon\right) < \infty \text{ for all } \epsilon > 0.$$

2.3. Theorem. Suppose that $\beta > -1$, $p > 0$, $q > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by

a random variable X . Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and

$$(2.6) \quad \sum_{i=1}^{\infty} |a_{ni}|^t \ll n^{-1-\beta+r(p-t)} \text{ for some } 0 < t < p.$$

Furthermore, assume that

$$(2.7) \quad \sum_{i=1}^{\infty} a_{ni}^2 \ll n^{-\mu} \text{ for some } \mu > 0$$

if $p \geq 2$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Then

$$(2.8) \quad \begin{cases} E|X|^q < \infty, & \text{if } q > p, \\ E|X|^p \log(1 + |X|) < \infty, & \text{if } q = p, \\ E|X|^p < \infty, & \text{if } q < p, \end{cases}$$

implies

$$(2.9) \quad \sum_{n=1}^{\infty} n^{\beta} E \left(\left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| - \epsilon \right)_+^q < \infty \text{ for all } \epsilon > 0.$$

Proof. We will apply Theorem 2.1 with $a_n = n^{\beta}$, $k_n = \infty$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ replaced by $\{a_{ni} X_{ni}, i \geq 1, n \geq 1\}$. Without loss of generality, we can assume that $a_{ni} > 0$ for all $i \geq 1, n \geq 1$ (otherwise, we use a_{ni}^+ and a_{ni}^- instead of a_{ni} , respectively, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). From (1.3) and (2.6), we can assume that

$$(2.10) \quad \sup_{i \geq 1} |a_{ni}| \leq n^{-r}, \quad \sum_{i=1}^{\infty} |a_{ni}|^t \leq n^{-1-\beta+r(p-t)}.$$

Hence for any $q \geq t$, we obtain by (2.10) that

$$(2.11) \quad \sum_{i=1}^{\infty} |a_{ni}|^q = \sum_{i=1}^{\infty} |a_{ni}|^t |a_{ni}|^{q-t} \leq n^{-r(q-t)} \sum_{i=1}^{\infty} |a_{ni}|^t \leq n^{-1-\beta+r(p-q)}.$$

For all $\epsilon > 0$, we have by (1.3), (2.8), (2.11), Lemma 1.7 and Lemma 1.10 that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^q I(|a_{ni}X_{ni}| > \epsilon) \\
& \ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^q E|X|^q I(|X| > \epsilon n^r) \\
& \leq \sum_{n=1}^{\infty} n^{-1+r(p-q)} E|X|^q I(|X| > \epsilon n^r) \\
(2.12) \quad & \leq \begin{cases} \sum_{n=1}^{\infty} n^{-1+r(p-q)} E|X|^q, & \text{if } q > p, \\ \sum_{n=1}^{\infty} n^{-1} E|X|^p I(|X| > \epsilon n^r), & \text{if } q = p, \end{cases} \\
& \ll \begin{cases} \sum_{n=1}^{\infty} n^{-1+r(p-q)}, & \text{if } q > p, \\ E|X|^p \log(1 + |X|), & \text{if } q = p, \end{cases} \\
& < \infty.
\end{aligned}$$

When $q < p$, taking q' such that $\max(q, t) < q' < p$, we have by (1.3), (2.8), (2.11), Lemma 1.7 and Lemma 1.10 that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^q I(|a_{ni}X_{ni}| > \epsilon) \\
& \leq \epsilon^{q-q'} \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^{q'} I(|a_{ni}X_{ni}| > \epsilon) \\
(2.13) \quad & \ll \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} |a_{ni}|^{q'} E|X|^{q'} I(|X| > \epsilon n^r) \\
& \leq \sum_{n=1}^{\infty} n^{-1+r(p-q')} E|X|^{q'} I(|X| > \epsilon n^r) \\
& \ll E|X|^p < \infty.
\end{aligned}$$

It is obvious that (2.8) implies $E|X|^p < \infty$. When $p \geq 2$, It is clear that $EX^2 < \infty$. Noting that $\mu > 0$, we can choose sufficiently large s such that $\beta - \mu s < -1$ and $s > q/2$. Then, by Lemma 1.7, (2.7) and $EX^2 < \infty$ we get that

$$(2.14) \quad \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} E a_{ni}^2 X_{ni}^2 \right)^s \ll \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} a_{ni}^2 \right)^s \ll \sum_{n=1}^{\infty} n^{\beta - \mu s} < \infty.$$

When $p < 2$, since $\beta > -1$, we can choose sufficiently large s such that $\beta + s(-1 - \beta) < -1$ and $s > q/p$, we have by (2.11), $E|X|^p < \infty$ and Lemma 1.7 that

$$(2.15) \quad \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^p \right)^s \ll \sum_{n=1}^{\infty} n^{\beta} \left(\sum_{i=1}^{\infty} |a_{ni}|^p \right)^s \leq \sum_{n=1}^{\infty} n^{\beta+s(-1-\beta)} < \infty.$$

When $p < 1$, combining (2.11), $E|X|^p < \infty$, $\beta > -1$, C_r -inequality and Lemma 1.7, we obtain that

$$(2.16) \quad \begin{aligned} & \sup_{x \geq 1} x^{-1/q} \sum_{i=1}^{\infty} |EX'_{ni}(x)| \leq \\ & \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x \geq 1} x^{-1/q} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| \leq x^{1/q}) \\ & \leq \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x \geq 1} x^{-p/q} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^p I(|a_{ni}X_{ni}| \leq x^{1/q}) \\ & \leq 2 \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|^p \ll \sum_{i=1}^{\infty} |a_{ni}|^p \\ & \leq n^{-1-\beta} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

When $p \geq 1$, since $EX_{ni} = 0$, we get that

$$Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| \leq x^{1/q}) = -Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| > x^{1/q}).$$

Thus, we have by $E|X|^p < \infty$, $\beta > -1$, C_r -inequality and Lemma 1.7 that

$$(2.17) \quad \begin{aligned} & \sup_{x \geq 1} x^{-1/q} \sum_{i=1}^{\infty} |EX'_{ni}(x)| \\ & \leq \sum_{i=1}^{\infty} P(|a_{ni}X_{ni}| > 1) + \sup_{x \geq 1} x^{-1/q} \sum_{i=1}^{\infty} \left| Ea_{ni}X_{ni}I(|a_{ni}X_{ni}| > x^{1/q}) \right| \\ & \leq 2 \sum_{i=1}^{\infty} E|a_{ni}X_{ni}| I(|a_{ni}X_{ni}| > 1) \ll \sum_{i=1}^{\infty} |a_{ni}|^p E|X|^p I(|X| > n^r) \ll \sum_{i=1}^{\infty} |a_{ni}|^p \\ & \leq n^{-1-\beta} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus, by (2.12)–(2.17), we see that assumptions (i), (ii) and (iii) in Theorem 2.1 are fulfilled. Therefore (2.9) holds by Theorem 2.1. \square

2.4. Remark. When $1 + \alpha + \beta > 0$, the conditions (1.3), (2.6) and (2.7) are weaker than the conditions (1.3) and (1.6). In fact, taking $t = \theta$, $p = \theta + (1 + \alpha + \beta)/r$, we immediately get (2.6) by (1.6). Noting that $\theta < 2$, we obtain by (1.3) and (1.6) that

$$\sum_{i=1}^{\infty} a_{ni}^2 \leq \sup_{i \geq 1} |a_{ni}|^{2-\theta} \sum_{i=1}^{\infty} |a_{ni}|^{\theta} \ll n^{-(r(2-\theta)-\alpha)}.$$

Since $\theta < 2 - \alpha/r$, we have $\mu =: r(2 - \theta) - \alpha > 0$. Therefore (2.7) holds. So, Theorem 2.3 not only extends the result of Wu [10] for ND random variables to

END case, but also obtains the weaker sufficient condition of complete q th moment convergence of weighted sums for arrays of row-wise END random variables. It is worthy to point out that the method used in this article is novel, which differs from that of Wu [10]. Our method greatly simplify the proof of Wu [10].

Note that conditions (1.3) and (2.6) together imply

$$(2.18) \quad \sum_{i=1}^{\infty} |a_{ni}|^p \ll n^{-1-\beta}.$$

From the proof of Theorem 2.3, we can easily see that if $q > 0$ of Theorem 2.3 is replaced by $q \geq p$, then condition (2.6) can be replaced by the weaker condition (2.18).

2.5. Theorem. *Suppose that $\beta > -1$, $p > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X . Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3) and (2.18). Furthermore, assume that (2.7) holds if $p \geq 2$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Then*

$$(2.19) \quad \begin{cases} E|X|^q < \infty, & \text{if } q > p, \\ E|X|^p \log(1 + |X|) < \infty, & \text{if } q = p, \end{cases}$$

implies that (2.9) holds.

2.6. Remark. As in Remark 2.4, when $1 + \alpha + \beta = 0$, the conditions (1.3), (2.7) and (2.18) are weaker than the conditions (1.3) and (1.6).

Take $q < p$ in Theorem 2.3 and $q = p$ in Theorem 2.5, by Remark 2.2 we can immediately obtain the following corollary:

2.7. Corollary. *Suppose that $\beta > -1$, $p > 0$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X . Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq 1$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants satisfying (1.3), (2.7) and*

$$(2.20) \quad \sum_{i=1}^{\infty} |a_{ni}|^t \ll n^{-1-\beta+r(p-t)} \text{ for some } 0 < t \leq p.$$

- (i) *If $t < p$, then $E|X|^p < \infty$ implies (1.5).*
- (ii) *If $t = p$, then $E|X|^p \log(1 + |X|) < \infty$ implies (1.5).*

The following corollary establish complete q th moment convergence for moving average processes under a sequence of END non-identically distributed random variables, which extends the corresponding results of Li and Zhang [15] to the case of sequences of END non-identically distributed random variables. Moreover, our result covers the case of $r > 1$, which was not considered by Li and Zhang [15].

2.8. Corollary. *Suppose that $Y_n = \sum_{i=-\infty}^{\infty} a_{i+n} X_i$, $n \geq 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers with $\sum_{-\infty}^{\infty} |a_i| < \infty$ and $\{X_i, -\infty < i < \infty\}$ is a sequence of END random variables with mean zero which are stochastically*

dominated by a random variable X . Let $r > 1/2$, $p \geq 1 + 1/(2r)$, $q > 0$. Then

$$(2.21) \quad \begin{cases} E|X|^q < \infty, & \text{if } q > p, \\ E|X|^p \log(1 + |X|) < \infty, & \text{if } q = p, \\ E|X|^p < \infty, & \text{if } q < p, \end{cases}$$

implies that

$$(2.22) \quad \sum_{n=1}^{\infty} n^{rp-2} E \left(\left| n^{-r} \sum_{i=1}^n Y_i \right| - \epsilon \right)_+^q < \infty, \text{ for all } \epsilon > 0.$$

Proof. Note that

$$n^{-r} \sum_{i=1}^n Y_i = \sum_{i=-\infty}^{\infty} \left(n^{-r} \sum_{j=1}^n a_{i+j} \right) X_i.$$

We will apply Theorem 2.3 with $\beta = rp - 2$, $t = 1$, $a_{ni} = n^{-r} \sum_{j=1}^n a_{i+j}$ and $\{X_{ni}, i \geq 1, n \geq 1\}$ replaced by $\{X_i, -\infty < i < \infty\}$. Noting that $\sum_{-\infty}^{\infty} |a_i| < \infty$, $r > 1/2$ and $p \geq 1 + 1/(2r)$, we can easily see that the conditions (1.3) and (1.6) hold for $\theta = 1$, $\alpha = 1 - r$. Therefore (2.22) holds by (2.21), Theorem 2.3 and Remark 2.2. \square

Similar to the proof of Corollary 2.8, we can get the following Baum-Katz type result for arrays of row-wise END random variables as follows.

2.9. Corollary. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of row-wise END random variables which are stochastically dominated by a random variable X . Let $r > 1/2$, $p > 1$, $q > 0$. Assume further that $EX_{ni} = 0$ for all $i \geq 1$ and $n \geq 1$ when $p \geq r$. Then

$$\begin{cases} E|X|^q < \infty, & \text{if } q > p/r, \\ E|X|^{p/r} \log(1 + |X|) < \infty, & \text{if } q = p/r, \\ E|X|^{p/r} < \infty, & \text{if } q < p/r, \end{cases}$$

implies that

$$\sum_{n=1}^{\infty} n^{p-2-rq} E \left(\left| \sum_{i=1}^n X_{ni} \right| - \epsilon n^r \right)_+^q < \infty, \text{ for all } \epsilon > 0.$$

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