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Hom-Leibniz superalgebras and hom-Leibniz poisson superalgebras

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Abstract

This paper aims to characterize Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras, presents the methods to construct these superalgebras. Moreover, derivations and representations of Hom-Leibniz Poisson superalgebras are also investigated.

Keywords: Hom-Leibniz superalgebras, Hom-superdialgebras, Hom-Leibniz Poisson superalgebras, endomorphism.

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1. Introduction

Leibniz algebras are introduced by Cuvier and Loday [11,17], motivated by the study of algebraic K-theory. Such algebras are a non-antisymmetric version of Lie algebras. Active investigations on Leibniz algebras show that many results of Lie algebras can be extended to Leibniz algebras [1,5-7,18-19]. Leibniz superalgebras, originally were introduced by Dzhumadil'daev in [12], can be seen as a direct generalization of Leibniz algebras. Some theories of superdialgebras and (co)homology of Leibniz superalgebras are investigated [14-16].

During the past decades, there is an increasing interest in exploring some exotic algebraic structures [9-10]. In particular, Casas and Datuashoili considered algebras with brackets [8]. Such algebras are called noncommutative Leibniz Poisson algebras. On the other hand the dual algebraic operads of the classical operads provide some kinds of algebraic structures: Dialgebras, Dendriform algebras and Trialgebras [20].

Recently, Leibniz algebras are generalized to Hom-Leibniz algebras by Makhlouf and Silvestrov in [21]. Some structure theories of Hom-Leibniz algebras are developed [22].

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Moreover, the dialgebras are also generalized to Hom-dialgebras by Yau in [26], which give rise to Hom-Leibniz algebras. Hom-Lie algebras, Hom-Lie superalgebras and Hom-Lie color algebras have been widely investigated [13,25,3,4,2,27,23-24]. The purpose of this paper is to introduce and study Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras.

The paper is organized as follows. In section 2, we give the definition and some important constructions of Hom-Leibniz superalgebras. In section 3, the notion of Hom-superdialgebras is proposed, the construction of Hom-Leibniz superalgebras is provided. Moreover, we give the definition of representation of Hom-superdialgebras and show that the representation of Hom-superdialgebras gives rise to the representation of Hom-Leibniz superalgebras via a special bracket. In section 4, we introduce the notions of Hom-Leibniz Poisson superalgebras, Hom-associative supertrialgebras and Hom-dendriform superalgebras, furthermore, construct several classes of Hom-Leibniz Poisson superalgebras. Section 5 and Section 6 are devoted to dealing with the derivations and representations of Hom-Leibniz Poisson superalgebras.

Throughout this paper, $\mathbb{K}$ denotes a field of characteristic zero. All vector spaces and algebras are $\mathbb{Z}_2$-graded over $\mathbb{K}$.

2. Hom-Leibniz Superalgebras

In this section, we introduce the notion of Hom-Leibniz superalgebras, and then give the construction of Hom-Leibniz superalgebras.

2.1. Definition. ([3]) *A Hom-associative superalgebra is a triple $(V, \circ, \alpha)$ consisting of a superspace V, an even bilinear map $\circ : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y), \tag{0.1}$$

$$\alpha(x) \circ (y \circ z) = (x \circ y) \circ \alpha(z), \tag{0.2}$$

for all homogeneous elements $x, y, z \in V$.

2.2. Definition. ([3]) *A Hom-Lie superalgebra is a triple $(V, [.,], \alpha)$ consisting of a superspace V, an even bilinear map $[., .] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)], \tag{0.3}$$

$$[x, y] = -(-1)^{|x||y|}[y, x], \tag{0.4}$$

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|x||y|}[\alpha(y), [z, x]] = 0, \tag{0.5}$$

for all homogeneous elements $x, y, z \in V$.

2.3. Definition. *A Hom-Leibniz superalgebra is a triple $(V, [.,], \alpha)$ consisting of a superspace V, an even bilinear map $[., .] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)], \tag{0.6}$$

$$[[x, y], \alpha(z)] = [\alpha(x), [y, z]] + (-1)^{|y||z|}[[x, z], \alpha(y)], \tag{0.7}$$

for all homogeneous elements $x, y, z \in V$.

Let $(V, [., .], \alpha)$ and $(V', [., .]', \alpha')$ be two Hom-Leibniz superalgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-Leibniz superalgebras if

$$f \circ \alpha = \alpha' \circ f, \quad [f(x), f(y)]' = f([x, y]), \forall x, y \in V. \tag{0.8}$$

2.4. Remark. We recover the classical Leibniz superalgebra when α is an identity map and reduces to a Hom-Leibniz algebra when the part of parity one is trivial. Obviously, a Hom-Lie superalgebra is a Hom-Leibniz superalgebra. While a Hom-Leibniz superalgebra is a Hom-Lie superalgebra if and only if $[x,x]=0$, for all homogeneous element $x \in V$.

Suppose that $(V,[.,.],\alpha)$ is a Hom-Leibniz superalgebra. For any $x \in V$, define $Ad_y \in \mathrm{End}(V)$ by

$$Ad_y(x) = (-1)^{|x||y|}[x,y]. \tag{0.9}$$

Then the Hom-Leibniz superalgebra identity (0.7) is written into

$$Ad_{\alpha(z)}([x,y]) = (-1)^{|x||z|}[\alpha(x), Ad_z(y)] + [Ad_z(x), \alpha(y)], \tag{0.10}$$

or into pure operation form

$$Ad_{\alpha(z)}Ad_y = Ad_{Ad_z(y)} \circ \alpha + (-1)^{|y||z|}Ad_{\alpha(y)} \circ Ad_z. \tag{0.11}$$

The following proposition provides a method to construct a Hom-Leibniz superalgebra by a Leibniz superalgebra and an even endomorphism.

2.5. Proposition. *Let $(V,[.,.])$ be a Leibniz superalgebra and $\alpha : V \to V$ be an even Leibniz superalgebra endomorphism. Then $(V,[.,.]_\alpha,\alpha)$ is a Hom-Leibniz superalgebra, where $[x,y]_\alpha = \alpha([x,y])$.*

Moreover, suppose that $(V',[.,.]')$ is another Leibniz superalgebra and $\alpha' : V' \to V'$ is a Leibniz superalgebra endomorphism. If $f : V \to V'$ is a Leibniz superalgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then

$$f : (V,[.,.]_\alpha,\alpha) \to (V',[.,.]'_{\alpha'},\alpha') \tag{0.12}$$

is a morphism of Hom-Leibniz superalgebras.

Proof. We show that $(V,[.,.]_\alpha,\alpha]$ satisfies the Hom-Leibniz superalgebra identity (0.7). In fact,

$$\begin{aligned}
&[\alpha(x),[y,z]_\alpha]_\alpha + (-1)^{|y||z|}[[x,z]_\alpha,\alpha(y)]_\alpha \\
&= \alpha([\alpha(x),\alpha([y,z])]) + (-1)^{|y||z|}\alpha([\alpha([x,z]),\alpha(y)]) \\
&= \alpha^2([x,[y,z]] + (-1)^{|y||z|}[[x,z],y]) \\
&= \alpha^2([[x,y],z]) \\
&= [[x,y]_\alpha,\alpha(z)]_\alpha
\end{aligned}$$

The second assertion follows from

$$\begin{aligned}
f([x,y]_\alpha) &= f([\alpha(x),\alpha(y)]) \\
&= [f\circ\alpha(x), f\circ\alpha(y)]' \\
&= [\alpha'\circ f(x), \alpha'\circ f(y)]' \\
&= [f(x),f(y)]'_{\alpha'}.
\end{aligned}$$

□

2.6. Example. (3-dimensional Hom-Leibniz superalgebras) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 3-dimensional superspace, where $A_{\bar{0}}$ is generated by e_1 and $A_{\bar{1}}$ is generated by e_2, e_3 and the nonzero product is given by $[e_2,e_1] = e_2$. For any $a,b \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by $\alpha(e_1) = ae_1, \alpha(e_3) = be_2$. By Proposition 2.5, for any $a \in \mathbb{K}$, there is the corresponding Hom-Leibniz superalgebra $A_\alpha = (A,[.,.]_\alpha,\alpha)$ with the nonzero product $[e_2, e_1]_\alpha = ae_2$. It is not a Leibniz superalgebra when $a \neq 0, 1$.

2.7. Lemma. *Let V be a Hom-Lie superalgebra, then the bracket*

$$[x \otimes y, a \otimes b] = [[x, y], \alpha(a)] \otimes b + (-1)^{|a||x|+|a||y|} a \otimes [[x, y], \alpha(b)]$$

definies a Hom-Leibniz superalgebra structure on the vector superspace $V \otimes V$.

2.8. Definition. *A representation (module) of the Hom-Leibniz superalgebra $(V, [.,.], \alpha)$ is a Hom-supermodule (U, α_U) equipped with two even V-actions (left and right)*

$$[.,.] : U \times V \to U \quad ((u, x) \mapsto [u, x]) \quad \text{and} \quad [.,.] : V \times U \to U \quad ((x, u) \mapsto [x, u])$$

satisfying the following axioms,

(0.13) $[U_\alpha, V_\beta] \subseteq U_{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{Z}_2,$

(0.14) $[V_\alpha, U_\beta] \subseteq U_{\alpha+\beta}, \forall \alpha, \beta \in \mathbb{Z}_2,$

(0.15) $\alpha_U([u, x]) = [\alpha_U(u), \alpha(x)],$

(0.16) $\alpha_U([x, u]) = [\alpha(x), \alpha_U(u)],$

(0.17) $[[u, x], \alpha(y)] = [\alpha_U(u), [x, y]] + (-1)^{|x||y|}[[u, y], \alpha(x)],$

(0.18) $[[x, u], \alpha(y)] = [\alpha(x), [u, y]] + (-1)^{|u||y|}[[x, y], \alpha_U(u)],$

(0.19) $[[x, y], \alpha_U(u)] = [\alpha(x), [y, u]] + (-1)^{|u||y|}[[x, u], \alpha(y)],$

for all homogeneous elements $x, y \in V$ and $u \in U$.

Note that the last two relations imply the following identity

$$[\alpha(x), [u, y]] + (-1)^{|u||y|}[\alpha(x), [y, u]] = 0.$$

3. Hom-Superdialgebras

In this section, we extend in one hand superdialgebras and the Hom-dialgebras introduced in [14] and [26] to Hom-superdialgebras. In the other hand we describe some constructions of Hom-Leibniz superalgebras.

3.1. Definition. ([14]) *A superdialgebra is a triple $(V, \dashv, \vdash)$ consisting of a superspace V, two even bilinear maps $\dashv, \vdash : V \times V \to V$ satisfying*

(0.20) $x \vdash (y \dashv z) = (x \vdash y) \dashv z,$

(0.21) $x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z),$

(0.22) $x \vdash (y \vdash z) = (x \vdash y) \vdash z = (x \dashv y) \vdash z,$

for all homogeneous elements $x, y, z \in V$.

3.2. Definition. *A Hom-superdialgebra is a tuple $(V, \dashv, \vdash, \alpha)$ consisting of a superspace V, two even bilinear maps $\dashv, \vdash : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying*

(0.23) $\alpha(x \dashv y) = \alpha(x) \dashv \alpha(y), \quad \alpha(x \vdash y) = \alpha(x) \vdash \alpha(y),$

(0.24) $\alpha(x) \dashv (y \dashv z) = (x \dashv y) \dashv \alpha(z) = \alpha(x) \dashv (y \vdash z),$

(0.25) $\alpha(x) \vdash (y \vdash z) = (x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z),$

(0.26) $\alpha(x) \vdash (y \dashv z) = (x \vdash y) \dashv \alpha(z),$

for all homogeneous elements $x, y, z \in V$.

3.3. Remark. We recover the classical superdialgebra [14] when α is an identity map and reduces to a Hom-dialgebra [26] when the part of parity one is trivial. Any Hom-associative superalgebra is a Hom-superdialgebra if $a \vdash b = a \dashv b = ab$.

3.4. Proposition. *If $(V_1, \circ_1, \alpha_1)$ and $(V_2, \circ_2, \alpha_2)$ are two Hom-superdialgebras, then the tensor product $V_1 \otimes V_2$ is a Hom-superdialgebra with*

$$\alpha = \alpha_1 \otimes \alpha_2,$$

and

$$(v_1 \otimes v_2) \star (u_1 \otimes u_2) = (-1)^{|v_2||u_1|}(v_1 \star u_1) \otimes (v_2 \star u_2)$$

for all homogeneous elements $v_1, v_2 \in V_1, u_1, u_2 \in V_2$ and $\star = \dashv, \ \vdash$.

3.5. Definition. *Let $(V, \dashv, \vdash, \alpha)$ and $(V', \dashv', \vdash', \alpha')$ be two Hom-superdialgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-superdialgebras if $f \circ \alpha = \alpha' \circ f$, and $f(x) \dashv' f(y) = f(x \dashv y)$, and $f(x) \vdash' f(y) = f(x \vdash y)$ for any $x, y \in V$.*

3.6. Proposition. *Let $(V, \dashv, \vdash)$ be a superdialgebra and $\alpha : V \to V$ be an even superdialgebra endomorphism. Then $(V, \dashv_\alpha, \vdash_\alpha, \alpha)$ is a Hom-superdialgebra, where $x \dashv_\alpha y = \alpha(x \dashv y)$ and $x \vdash_\alpha y = \alpha(x \vdash y)$.*

Moreover, suppose that $(V', \dashv', \vdash')$ is another superdialgebra and $\alpha' : V' \to V'$ is a superdialgebra endomorphism. If $f : V \to V'$ is a superdialgebra morphism that satisfies $f \circ \alpha = \alpha' \circ f$, then

(0.27) $\quad f : (V, \dashv_\alpha, \vdash_\alpha, \alpha) \to (V', \dashv'_{\alpha'}, \vdash'_{\alpha'}, \alpha')$

is a morphism of Hom-superdialgebras.

Proof. We only need to show that $(V, \dashv_\alpha, \vdash_\alpha, \alpha)$ satisfies the Hom-superdialgebra identity (0.24)-(0.26). Direct calculations show that

$$\begin{aligned} \alpha(x) \dashv_\alpha (y \vdash_\alpha z) &= \alpha(\alpha(x) \dashv \alpha(y \dashv z)) \\ &= \alpha^2(x \dashv (y \dashv z)) \\ &= \alpha^2((x \dashv y) \dashv z) \\ &= (x \dashv_\alpha y) \dashv_\alpha \alpha(z), \end{aligned}$$

and

$$\begin{aligned} \alpha(x) \dashv_\alpha (y \vdash_\alpha z) &= \alpha^2(x \dashv (y \dashv z)) \\ &= \alpha^2(x \dashv (y \vdash z)) \\ &= \alpha(\alpha(x) \dashv (y \vdash_\alpha z)) \\ &= \alpha(x) \dashv_\alpha (y \vdash_\alpha z), \end{aligned}$$

thus (0.24) holds. Similarly, we can prove (0.25) and (0.26).

Setting $\star_\alpha = \dashv_\alpha$ and $\star_\alpha = \vdash_\alpha$. The second assertion follows from

$$f \circ \star_\alpha = f \circ \alpha \circ \star = \alpha' \circ f \circ \star = \alpha' \circ \star' \circ f = \star_{\alpha'} \circ f.$$

□

3.7. Proposition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra. Define an even bilinear map $[.,.] : V \times V \to V$ by*

(0.28) $\quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$

Then $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra.

Proof. We only need to show that $(V, [.,.], \alpha)$ satisfies the Hom-Leibniz superalgebra identity (0.7). Direct calculations show that

$$\begin{aligned}
&[\alpha(x), [y, z]] + (-1)^{|y||z|}[[x, z], \alpha(y)] \\
&= \alpha(x) \dashv (y \dashv z) - (-1)^{|x||y|+|x||z|}(y \dashv z) \vdash \alpha(x) \\
&- (-1)^{|y||z|}\alpha(x) \dashv (z \vdash y) + (-1)^{|x||y|+|x||z|+|y||z|}(z \vdash y) \vdash \alpha(x) \\
&+ (-1)^{|y||z|}(x \dashv z) \dashv \alpha(y) - (-1)^{|x||y|}\alpha(y) \vdash (x \dashv z) \\
&- (-1)^{|x||z|+|y||z|}(z \vdash x) \dashv \alpha(y) + (-1)^{|x||y|+|x||z|}\alpha(y) \vdash (z \vdash x) \\
&= (x \dashv y) \dashv \alpha(z) - (-1)^{|x||z|+|y||z}\alpha(z) \vdash (x \dashv y) \\
&- (-1)^{|x||y|}(y \vdash x) \dashv \alpha(z) + (-1)^{|x||y|+|x|||z|+|y||z|}\alpha(z) \vdash (y \vdash x) \\
&+ (-1)^{|y||z|}\{(x \dashv z) \dashv \alpha(y) - \alpha(x) \dashv (z \vdash y)\} \\
&+ (-1)^{|x||y|+|x||z|}\{\alpha(y) \vdash (z \vdash x) - (y \dashv z) \vdash \alpha(x)\} \\
&= [x \dashv y, \alpha(z)] - (-1)^{|x||y|}[y \vdash x, \alpha(z)] \\
&= [[x, y], \alpha(z)].
\end{aligned}$$

□

3.8. Proposition. *Let $(V, [.,.], \alpha_1)$ be a Hom-Leibniz superalgebra , $(U, \dashv, \vdash, \alpha_2)$ be a super commutative Hom-superdialgebra and let $g = V \otimes U$. Define the operations $\alpha : g \to g$ and $[.,.] : g^{\otimes 2} \to g$ by*

(0.29) $\alpha = \alpha_1 \otimes \alpha_2,$

(0.30) $[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x, y] \otimes (a \vdash b).$

Then $(g, [.,.], \alpha)$ is a Hom-Leibniz superalgebra.

Proof. We only need to show that $(g, [.,.], \alpha)$ satisfies the Hom-Leibniz superalgebra identity (0.7). Direct calculations show that

$$\begin{aligned}
&[\alpha(x \otimes a), [y \otimes b, z \otimes c]] + (-1)^{|y||z|+|y||c|+|b||z|+|b||c|}[[x \otimes a, z \otimes c], \alpha(y \otimes b)] \\
&= [\alpha_1(x) \otimes \alpha_2(a), (-1)^{|b||z|}[y, z] \otimes (b \vdash c)] \\
&+ (-1)^{|y||z|+|y||c|+|b||z|+|b||c|+|a||z|}[[x, z] \otimes (a \vdash c), \alpha_1(y) \otimes \alpha_2(b)] \\
&= (-1)^{|a||y|+|a||z|+|b||z|}[\alpha_1(x), [y, z]] \otimes (\alpha_2(a) \vdash (b \vdash c)) \\
&+ (-1)^{|a||y|+|a||z|+|b||z|+|b||c|+|y||z|}[[x, z], \alpha_1(y)] \otimes ((a \vdash c) \vdash \alpha_2(b)) \\
&= (-1)^{|a||y|+|a||z|+|b||z|}[\alpha_1(x), [y, z]] \otimes (\alpha_2(a) \vdash (b \vdash c)) \\
&+ (-1)^{|a||y|+|a||z|+|b||z|+|y||z|}[[x, z], \alpha_1(y)] \otimes (\alpha_2(a) \vdash (b \vdash c)) \\
&= (-1)^{|a||y|+|a||z|+|b||z|}\{[\alpha_1(x), [y, z]] \\
&+ (-1)^{|y||z|}[[x, z], \alpha_1(y)]\} \otimes (\alpha_2(a) \vdash (b \vdash c)) \\
&= (-1)^{|a||y|+|a||z|+|b||z|}[[x, y], \alpha_1(z)] \otimes (\alpha_2(a) \vdash (b \vdash c)).
\end{aligned}$$

and

$$\begin{aligned}
&[[x \otimes a, y \otimes b], \alpha(z \otimes c)] = [(-1)^{|a||y|}[x, y] \otimes (a \vdash b), \alpha_1(z) \otimes \alpha_2(c)] \\
&= (-1)^{|a||y|+|a||z|+|b||z|}[[x, y], \alpha_1(z)] \otimes ((a \vdash b) \vdash \alpha_2(c)) \\
&= (-1)^{|a||y|+|a||z|+|b||z|}[[x, y], \alpha_1(z)] \otimes (\alpha_2(a) \vdash (b \vdash c)).
\end{aligned}$$

This shows that $(g,[.,.],\alpha)$ is a Hom-Leibniz superalgebra. $\square$

3.9. Definition. *Let $(V,\dashv,\vdash,\alpha)$ be a Hom-superdialgebra and (U,α_U) be a Hom-superspace. The pair (U,α_U) is said to be a V-supermodule if (U,α_U) is a Hom-supermodule equipped with four actions (left and right) of V*

$$V\otimes U\to U\quad (x\otimes u\mapsto x\dashv u\quad or\quad x\vdash u),$$
$$U\otimes V\to U\quad (u\otimes x\mapsto u\dashv x\quad or\quad u\vdash x)$$

satisfying the following axioms

$$\begin{aligned}
\alpha_U(x\dashv u) &= \alpha(x)\dashv\alpha_U(u),\\
\alpha_U(x\vdash u) &= \alpha(x)\vdash\alpha_U(u),\\
\alpha_U(u\dashv x) &= \alpha_U(u)\dashv\alpha(x),\\
\alpha_U(u\vdash x) &= \alpha_U(u)\vdash\alpha(x),\\
\alpha(x)\dashv(y\dashv u) &= (x\dashv y)\dashv\alpha_U(u) = \alpha(x)\dashv(y\vdash u),\\
(x\vdash y)\dashv\alpha_U(u) &= \alpha(x)\vdash(y\dashv u),\\
(x\dashv y)\vdash\alpha_U(u) &= \alpha(x)\vdash(y\vdash u) = (x\vdash y)\vdash\alpha_U(u),\\
\alpha(x)\dashv(u\dashv y) &= (x\dashv u)\dashv\alpha(y) = \alpha(x)\dashv(u\vdash y),\\
(x\vdash u)\dashv\alpha(y) &= \alpha(x)\vdash(u\dashv y),\\
(x\dashv u)\vdash\alpha(y) &= \alpha(x)\vdash(u\vdash y) = (x\vdash u)\vdash\alpha(y),\\
\alpha_U(u)\dashv(x\dashv y) &= (u\dashv x)\dashv\alpha(y) = \alpha_U(u)\dashv(x\vdash y),\\
(u\vdash x)\dashv\alpha(y) &= \alpha_U(u)\vdash(x\dashv y),\\
(u\dashv x)\vdash\alpha(y) &= \alpha_U(u)\vdash(x\vdash y) = (u\vdash x)\vdash\alpha(y),
\end{aligned}$$

for all $x,y\in V$ and $u\in U$.

3.10. Proposition. *Let $(V,\dashv,\vdash,\alpha)$ be a Hom-superdialgebra, $(V,[.,.],\alpha)$ be a Hom-Leibniz superalgebra, where $[x,y]=x\dashv y-(-1)^{|x||y|}y\vdash x$ for any $x,y\in V$, and (U,α_U) be a representation of $(V,\dashv,\vdash,\alpha)$. Then (U,α_U) is also a representation of $(V,[.,.],\alpha)$.*

Proof. We just check

$$[[x,y],\alpha_U(u)]=[\alpha(x),[y,u]]+(-1)^{|y||u|}[[x,u],\alpha(y)].$$

Using the axioms of the supermodule of Hom-superdialgebra, we have

$$\begin{aligned}
&[\alpha(x),[y,u]]+(-1)^{|y||u|}[[x,u],\alpha(y)]\\
&=\alpha(x)\dashv(y\dashv u)-(-1)^{|x||y|+|x||u|}((y\dashv u)\vdash\alpha(x))\\
&-(-1)^{|y||u|}\alpha(x)\dashv(u\vdash y)+(-1)^{|x||y|+|x||u|+|y||u|}(u\vdash y)\vdash\alpha(x)\\
&+(-1)^{|y||u|}(x\dashv u)\dashv\alpha(y)-(-1)^{|x||y|}\alpha(y)\vdash(x\dashv u)\\
&-(-1)^{|x||u|+|y||u|}(u\vdash x)\dashv\alpha(y)+(-1)^{|x||y|+|x||u|}\alpha(y)\vdash(u\vdash x)\\
&=(x\dashv y)\dashv\alpha_U(u)-(-1)^{|x||u|+|y||u|}\alpha_U(u)\vdash(x\dashv y)\\
&-(-1)^{|x||y|}(y\vdash x)\vdash\alpha_U(u)+(-1)^{|x||y|+|y||u|+|x||u|}\alpha_U(u)\vdash(y\vdash x)\\
&=[[x,y],\alpha_U(u)].
\end{aligned}$$

$\square$

4. Hom-Leibniz Poisson Superalgebras

In this section, we introduce the notions of Hom-Leibniz Poisson superalgebras, Hom-associative supertrialgebras and Hom-dendriform superalgebras. Moreover, we construct several classes of Hom-Leibniz Poisson superalgebras.

4.1. Definition. *A Hom-Poisson superalgebra is a tuple $(A, \circ, [.,.], \alpha)$ consisting of a superspace V, two even bilinear maps $\circ, [.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying the following axioms*

(1) $(A, \circ, \alpha)$ *is a Hom-associative superalgebra ,*
(2) $(A, [.,.], \alpha)$ *is a Hom-Lie superalgebra,*
(3) *the Hom-Leibniz superidentity*

$$[x \circ y, \alpha(z)] = \alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y)$$

holds, for all homogeneous elements $x, y, z \in A$.

4.2. Theorem. *Let $(A, \cdot, [.,.])$ be a Poisson superalgebra and $\alpha : A \to A$ be an even Poisson superalgebra endomorphism. Then $(A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ is a Hom-Poisson superalgebra, where $x \cdot_\alpha y = \alpha(x \cdot y)$ and $[x, y]_\alpha = \alpha([x, y])$.*

Proof. It is straightforward. □

This theorem provides a method to construct Hom-Poisson superalgebra by a Poisson superalgebra and an even Poisson superalgebra endomorphism.

4.3. Example. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 2-dimensional superspace, where $A_{\bar{0}}$ is generated by e_1 and $A_{\bar{1}}$ is generated by e_2 and nonzero products are given by

$$e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_2 = e_1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, \quad [e_2, e_2] = 2e_1.$$

For any $a \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2.$$

By Theorem 4.2, for any $a \in \mathbb{K}$, there is the corresponding Hom-Poisson superalgebra $A_a = (A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ with the nonzero products

$$e_1 \cdot_\alpha e_1 = ae_1, \quad e_2 \cdot_\alpha e_2 = ae_1, \quad e_1 \cdot_\alpha e_\alpha = ae_2, \quad [e_2, e_2]_\alpha = 2ae_1.$$

It is not a Poisson superalgebra when $a \neq 0, 1$.

4.4. Example. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a 3-dimensional superspace, where $A_{\bar{0}}$ is generated by e_1, e_2 and $A_{\bar{1}}$ is generated by e_3 and the nonzero products are given by

$$e_1 \cdot e_2 = e_1, \quad e_2 \cdot e_2 = e_2, \quad e_3 \cdot e_2 = e_3, \quad [e_1, e_2] = ae_1.$$

For any $a \in \mathbb{K}$, we consider the homomorphism $\alpha : A \to A$ defined by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_1 + e_2.$$

By Theorem 4.2, for any $a \in \mathbb{K}$, there is the corresponding Hom-Poisson superalgebra $A_\alpha = (A, \cdot_\alpha, [.,.]_\alpha, \alpha)$ with the nonzero products

$$e_1 \cdot_\alpha e_2 = ae_1, \quad e_2 \cdot_\alpha e_2 = e_1 + e_2, \quad [e_1, e_2]_\alpha = ae_1.$$

It is not a Poisson superalgebra when $a \neq 0, 1$.

4.5. Definition. *A Hom-Leibniz Poisson superalgebra is a tuple $(V, \circ, [.,.], \alpha)$ consisting of a superspace V, two even bilinear maps $\circ, [.,.] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying the following axioms*

(1) $(V, \circ, \alpha)$ *is a Hom-associative superalgebra,*
(2) $(V, [.,.], \alpha)$ *is a Hom-Leibniz superalgebra,*

(3) the Hom-Leibniz superidentity

$$[x \circ y, \alpha(z)] = \alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y)$$

holds, for all homogeneous elements $x, y, z \in V$.

4.6. Definition. *Let $(V, \circ, [.,.], \alpha)$ and $(V', \circ', [.,.]', \alpha')$ be two Hom-Leibniz Poisson superalgebras. An even homomorphism $f : V \to V'$ is said to be a morphism of Hom-Leibniz Poisson superalgebras if*

(0.31) $f \circ \alpha = \alpha' \circ f$

(0.32) $f(x) \circ' f(y) = f(x \circ y), \quad [f(x), f(y)]' = f([x, y]), \quad \forall x, y \in V.$

4.7. Remark. Any Hom-Poisson superalgebra is a Hom-Leibniz Poisson superalgebra. Any Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$ is a Hom-Poisson superalgebra if and only if $[x, y] + (-1)^{|x||y|}[y, x] = 0$ holds, for all homogeneous elements $x, y \in V$. If $\alpha = Id$, then a Hom-Leibniz Poisson superalgebra becomes a Leibniz-Poisson superalgebra. On the other hand, any Hom-associative superalgebra is a Hom-Leibniz Poisson superalgebra with usual bracket $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$.

4.8. Proposition. *Let $(V, \dashv, \vdash, \alpha)$ be a Hom-superdialgebra and $\circ, [.,.] : V \times V \to V$ be two binary operations on V defined by*

$$x \circ y = x \vdash y, \quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.], \alpha)$ is a Hom-Leibniz Poisson superalgebra.

Proof. It is obvious that $(V, \circ, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 3.7, it follows that $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra. Next we show the remaining Hom-Leibniz superidentity. In fact

$$\begin{aligned}
&\alpha(x) \circ [y, z] + (-1)^{|y||z|}[x, z] \circ \alpha(y) \\
&= \alpha(x) \vdash (y \dashv z) - (-1)^{|y||z|}\alpha(x) \vdash (z \vdash y) \\
&+ (-1)^{|y||z|}(x \dashv z) \vdash \alpha(y) - (-1)^{|x||z|+|y||z|}(z \vdash x) \vdash \alpha(y) \\
&= (x \vdash y) \dashv \alpha(z) - (-1)^{|x||z|+|y||z|}\alpha(z) \vdash (x \vdash y) \\
&= [x \circ y, \alpha(z)].
\end{aligned}$$

□

Taking $\alpha = Id$ in Proposition 4.8, we obtain the following result about Leibniz-Poisson superalgebras.

4.9. Corollary. *Let $(V, \dashv, \vdash)$ be a superdialgebra and $\circ, [.,.] : V \times V \to V$ be two binary operations on V defined by*

$$x \circ y = x \vdash y, \quad [x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.])$ is a Leibniz-Poisson superalgebra.

4.10. Proposition. *Let $(V, \circ, [.,.])$ be a Leibniz-Poisson superalgebra and $\alpha : V \to V$ be an even Leibniz-Poisson superalgebras endomorphism. Then $(V, \circ_\alpha, [.,.]_\alpha, \alpha)$ is a Hom-Leibniz Poisson superalgebra, where $x \circ_\alpha y = \alpha(x \circ y)$ and $[x, y]_\alpha = \alpha([x, y])$.*

Moreover, suppose that $(V', \circ', [.,.]')$ is another Leibniz superalgebra and $\alpha' : V' \to V'$ is a Leibniz superalgebras endomorphism. If $f : V \to V'$ is a Leibniz superalgebra *morphism that satisfies* $f \circ \alpha = \alpha' \circ f$*, then*

$$f : (V, \circ_\alpha, [.,.]_\alpha, \alpha) \to (V', \circ_{\alpha'}, [.,.]_{\alpha'}, \alpha')$$

is a morphism of Hom-Leibniz superalgebras.

Proof. It is obvious that $(V, \circ_\alpha, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 2.5, we have $(V, [.,.]_\alpha, \alpha)$ is a Hom-Leibniz superalgebra. Next we will show that the Hom-Leibniz superidentity holds. In fact

$$\begin{aligned}
&\alpha(x) \circ_\alpha [y,z]_\alpha + (-1)^{|y||z|}[x,z]_\alpha \circ_\alpha \alpha(y) \\
&= \alpha(\alpha(x) \circ \alpha([y,z])) + (-1)^{|y||z|}\alpha(\alpha([x,z]) \circ \alpha(y)) \\
&= \alpha^2(x \circ [y,z] + (-1)^{|y||z|}[x,z] \circ y) \\
&= \alpha^2[x \circ y, z] \\
&= \alpha([x \circ_\alpha y, \alpha(z)]) \\
&= [x \circ_\alpha y, \alpha(z)]_\alpha.
\end{aligned}$$

By Proposition 2.5, the second assertion is straightforward. □

4.11. Definition. *An Hom-associative supertrialgebra is a quintuple* $(V, \dashv, \vdash, \perp, \alpha)$ *consisting of a superspace* V*, three even bilinear maps* $\dashv, \vdash, \perp : V \times V \to V$ *and an even superspace homomorphism* $\alpha : V \to V$ *satisfying the following axioms*

$$\begin{array}{rclrcl}
\alpha(x \dashv y) &=& \alpha(x) \dashv \alpha(y), & \alpha(x \vdash y) &=& \alpha(x) \vdash \alpha(y), \\
\alpha(x \perp y) &=& \alpha(x) \perp \alpha(y), & (x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \dashv z), \\
(x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \vdash z), & (x \vdash y) \dashv \alpha(z) &=& \alpha(x) \vdash (y \dashv z), \\
(x \dashv y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), & (x \vdash y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), \\
(x \dashv y) \dashv \alpha(z) &=& \alpha(x) \dashv (y \perp z), & (x \perp y) \dashv \alpha(z) &=& \alpha(x) \perp (y \dashv z), \\
(x \dashv y) \perp \alpha(z) &=& \alpha(x) \perp (y \vdash z), & (x \vdash y) \perp \alpha(z) &=& \alpha(x) \vdash (y \perp z), \\
(x \perp y) \vdash \alpha(z) &=& \alpha(x) \vdash (y \vdash z), & (x \perp y) \perp \alpha(z) &=& \alpha(x) \perp (y \perp z).
\end{array}$$

4.12. Remark. We recover the classical associative trialgebra when $\alpha = Id$ and the part of parity one is trivial in [14,20]. The associative supertrialgebra is obtained when $\alpha = Id$. Any Hom-associative supertrialgebra gives rise to a Hom-associative superdialgebra by forgetting the operation $\perp$.

4.13. Proposition. *Let* $(V, \dashv, \vdash, \perp, \alpha)$ *be a Hom-associative supertrialgebra and* $\circ, [.,.] : V \times V \to V$ *be two binary operations on* V *defined by*

$$x \circ y = x \perp y, \qquad [x,y] = x \dashv y - (-1)^{|x||y|} y \vdash x, \forall x, y \in V.$$

Then $(V, \circ, [.,.], \alpha)$ *is a Hom-Leibniz Poisson superalgebra.*

Proof. It is obvious that $(V, \circ, \alpha)$ is a Hom-associative superalgebra. Moreover, from Proposition 3.7, we have $(V, [.,.], \alpha)$ is a Hom-Leibniz superalgebra. Next we will show that the remaining Hom-Leibniz superidentity holds. In fact,

$$\begin{aligned}
[\alpha(x), y \circ z] + (-1)^{|y||z|}[x \circ z, \alpha(y)] &= \alpha(x) \perp (y \dashv z) \\
&- (-1)^{|y||z|}\alpha(x) \perp (z \vdash y) \\
&+ (-1)^{|y||z|}(x \dashv z) \perp \alpha(y) \\
&- (-1)^{|x||z|+|y||z|}(z \vdash x) \perp \alpha(y) \\
&= (x \perp y) \dashv \alpha(z) - (-1)^{|x||z|+|y||z|}\alpha(z) \vdash (x \perp y) \\
&= [x \circ y, \alpha(z)].
\end{aligned}$$

□

4.14. Definition. [24] *A Hom-dendriform superalgebra is a tuple $(V, \prec, \succ, \alpha)$ consisting of a superspace V, two even bilinear maps $\prec, \succ: V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying the following axioms*

$$\begin{aligned}
\alpha(x < y) &= \alpha(x) < \alpha(y), \\
\alpha(x > y) &= \alpha(x) > \alpha(y), \\
(x \prec y) \prec \alpha(z) &= \alpha(x) \prec (y \prec z) + \alpha(x) \prec (y \succ z), \\
(x \succ y) \prec \alpha(z) &= \alpha(x) \succ (y \prec z), \\
(x \prec y) \succ \alpha(z) + (x \succ y) \succ \alpha(z) &= \alpha(x) \succ (y \succ z),
\end{aligned}$$

for all homogeneous elements $x, y, z \in V$.

4.15. Lemma. *Let $(V, \prec, \succ, \alpha)$ be a Hom-dendriform superalgebra, define the product on homogeneous elements by*

$$x * y = x \prec y + x \succ y.$$

Then $(V, *, \alpha)$ is a Hom-associative superalgebra.

4.16. Proposition. *Let $(V, \prec, \succ, \alpha)$ be a Hom-dendriform superalgebra. Define the products on homogeneous elements by*

$$x * y = x \prec y + x \succ y, \qquad [x, y] = x * y - (-1)^{|x||y|} y * x.$$

Then $(V, *, [.,.], \alpha)$ is a Hom-Leibniz Poisson superalgebra.

Proof. It is straightforward. □

5. Derivation of Hom-Leibniz Poisson Superalgebras

In this section, we extend the α-derivations of Hom-Lie algebras introduced in [25] to Hom-Leibniz Poisson superalgebras.

Let $(V, \circ, [.,.], \alpha)$ be a Hom-Leibniz Poisson superalgebra, denote by α^k the k-times composition of α, i.e., $\alpha^k = \alpha \circ \alpha \cdot \cdots \cdot \circ \alpha$ (k-times). In particular, $\alpha^{-1} = 0, \alpha^0 = Id$, and $\alpha^1 = \alpha$.

5.1. Definition. *For any $k \geq -1$, we call $D \in (\mathrm{EndV})_i$, where $i \in \mathbb{Z}_2$, an α^k-derivation of the Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$ if*

(0.33) $\alpha \circ D = D \circ \alpha,$

(0.34) $D([x, y]) = [D(x), \alpha^k(y)] + (-1)^{|x||D|}[\alpha^k(x), D(y)],$

(0.35) $D(x \circ y) = D(x) \circ \alpha^k(y) + (-1)^{|x||D|}\alpha^k(x) \circ D(y),$

for all homogeneous elements $x, y \in V$.

We denote by $\mathrm{Der}_{\alpha^k}(V) = \mathrm{Der}_{\alpha^k}(V)_{\bar{0}} \oplus \mathrm{Der}_{\alpha^k}(V)_{\bar{1}}$ the set of α^k-derivations of the Hom-Leibniz Poisson superalgebra $(V, \circ, [.,.], \alpha)$, and $\mathrm{Der}(V) = \oplus_{k \geq -1} \mathrm{Der}_{\alpha^k}(V)$.

For any homogeneous elements $a \in V$, satisfying $\alpha(a) = a$, define $ad_k(a) \in \mathrm{End}(V)$ by

$$ad_k(a)(x) = -(-1)^{|a||x|}[\alpha^k(x), a], \forall x \in V.$$

Notice that $|ad_k(a)| = |a|$.

5.2. Proposition. *Let $(V, \circ, [.,.], \alpha)$ be a Hom-Leibniz Poisson superalgebra. Then $ad_k(a)$ is an α^{k+1}-derivation, which is said to be an inner α^{k+1}-derivation.*

Proof. Direct calculations show that

$$\begin{aligned} ad_k(a) \circ \alpha(x) &= -(-1)^{|a||x|}[\alpha^{k+1}(x), a] \\ &= -(-1)^{|a||x|}[\alpha^{k+1}(x), \alpha(a)] \\ &= -(-1)^{|a||x|}\alpha([\alpha^k(x), a] \\ &= \alpha \circ ad_k(a)(x), \end{aligned}$$

and

$$\begin{aligned} ad_k(a)([x, y]) &= -(-1)^{|a||x|+|a||y|}[\alpha^k([x, y]), a] \\ &= -(-1)^{|a||x|+|a||y|}[[\alpha^k(x), \alpha^k(y)], \alpha(a)] \\ &= -(-1)^{|a||x|+|a||y|}[\alpha^{k+1}(x), [\alpha^k(y), a]] - (-1)^{|a||x|}[[\alpha^k(x), a], \alpha^{k+1}(y)] \\ &= (-1)^{|a||x|}[\alpha^{k+1}(x), ad_k(a)(y)] + [ad_k(a)(x), \alpha^{k+1}(y)], \end{aligned}$$

and

$$\begin{aligned} ad_k(a)(x \circ y) &= -(-1)^{|a||x|+|a||y|}[\alpha^k(x \circ y), a] \\ &= -(-1)^{|a||x|+|a||y|}[\alpha^k(x) \circ \alpha^k(y), \alpha(a)] \\ &= -(-1)^{|a||x|+|a||y|}\alpha^{k+1}(x) \circ [\alpha^k(y), a]] - (-1)^{|a||x|}[\alpha^k(x), a] \circ \alpha^{k+1}(y) \\ &= (-1)^{|a||x|}\alpha^{k+1}(x) \circ ad_k(a)(y) + ad_k(a)(x) \circ \alpha^{k+1}(y). \end{aligned}$$

Therefore, $ad_k(a)$ is an α^{k+1}-derivation. □

We denote by $\mathrm{Inn}_{\alpha^k}(V)$ the set of inner α^k-derivations, i.e.,

$$\mathrm{Inn}_{\alpha^k}(V) = \{ad_k(a) | a \in V_{\bar{0}} \cup V_{\bar{1}}, \alpha(a) = a\}.$$

For any $D \in \mathrm{Der}(V)$ and $D' \in \mathrm{Der}(V)$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D.$$

5.3. Lemma. *For any $D \in (Der_{\alpha^k}(V))_i$ and $D' \in (Der_{\alpha^k}(V))_j$, then $[D, D'] \in Der_{\alpha^{k+s}}(V)_{|D|+|D'|}$, where $k + s \geq -1$ and $(i, j) \in \mathbb{Z}_2^2$.*

Proof. For any $x, y \in V$, we have

$$\begin{aligned}
[D,D']([x,y]) &= D\circ D'([x,y]) - (-1)^{|D||D'|}D'\circ D([x,y])\\
&= D([D'(x),\alpha^s(y)] + (-1)^{|D'||x|}[\alpha^s(x),D'(y)])\\
&- (-1)^{|D||D'|}D'([D(x),\alpha^k(y)] + (-1)^{|D||x|}[\alpha^k(x),D(y)])\\
&= [DD'(x),\alpha^{k+s}(y)] + (-1)^{|D||D'|+|D||x|}[\alpha^k(D'(x)),D\alpha^s(y)]\\
&+ (-1)^{|D'||x|}[D\alpha^s(x),\alpha^k D'(y)] + (-1)^{|D'||x|+|D||x|}[\alpha^{k+s}(x),DD'(y)]\\
&- (-1)^{|D||D'|}[D'D(x),\alpha^{k+s}(y)] - (-1)^{|D'||x|}[\alpha^s D(x),D'\alpha^k(y)]\\
&- (-1)^{|D||D'|+|D||x|}[D'\alpha^k(x),\alpha^s D(y)]\\
&- (-1)^{|D||D'|+|D||x|+|D'||x|}[\alpha^{k+s}(x),D'D(y)].
\end{aligned}$$

Since D and D' satisfy $D\circ\alpha = \alpha\circ D$ and $D'\circ\alpha = \alpha\circ D'$, we obtain

$$\begin{aligned}
[D,D']([x,y]) &= [DD'(x) - (-1)^{|D||D'|}D'D(x),\alpha^{k+s}(y)]\\
&+ (-1)^{|D||x|+|D'||x|}[\alpha^{k+s}(x),DD'(y) - (-1)^{|D||D'|}D'D(y)]\\
&= [[D,D'](x),\alpha^{k+s}(y)] + (-1)^{|[D,D']||x|}[\alpha^{k+s}(x),[D,D'](y)].
\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}
[D,D'](x\circ y) &= D\circ D'(x\circ y) - (-1)^{|D||D'|}D'\circ D(x\circ y)\\
&= D(D'(x)\circ\alpha^s(y) + (-1)^{|D'||x|}\alpha^s(x)\circ D'(y))\\
&- (-1)^{|D||D'|}D'(D(x)\circ\alpha^k(y) + (-1)^{|D||x|}\alpha^k(x)\circ D(y))\\
&= DD'(x)\circ\alpha^{k+s}(y) + (-1)^{|D||D'|+|D||x|}\alpha^k D'(x)\circ D\alpha^s(y)\\
&+ (-1)^{|D'||x|}D\alpha^s(x)\circ\alpha^k D'(y) + (-1)^{|D'||x|+|D||x|}\alpha^{k+s}(x)\circ DD'(y)\\
&- (-1)^{|D||D'|}D'D(x)\circ\alpha^{k+s}(y) - (-1)^{|D'||x|}\alpha^s D(x)\circ D'\alpha^k(y)\\
&- (-1)^{|D||D'|+|D||x|}D'\alpha^k(x)\circ\alpha^s D(y)\\
&- (-1)^{|D||D'|+|D||x|+|D'||x|}\alpha^{k+s}(x)\circ D'D(y)\\
&= (DD' - (-1)^{|D||D'|}D'D)(x)\circ\alpha^{k+s}(y)\\
&+ (-1)^{|[D,D']||x|}(DD' - (-1)^{|D||D'|}D'D)(y)\\
&= [D,D'](x)\circ\alpha^{k+s}(y) + (-1)^{|[D,D']||x|}\alpha^{k+s}(x)\circ[D,D'](y).
\end{aligned}$$

It is easy to verify that $\alpha\circ[D,D'] = [D,D']\circ\alpha$, which leads to $[D,D'] \in Der_{\alpha^{k+s}}(V)_{|D|+|D'|}$.

5.4. Remark. Obviously, we have

$$\mathrm{Der}_{\alpha^{-1}}(V) = \{D \in \mathrm{End}(V) | D\circ\alpha = \alpha\circ D, D([x,y]) = 0, D(x\circ y) = 0, \forall x,y \in V\}.$$

Thus for any $D, D' \in \mathrm{Der}_{\alpha^{-1}}(V)$, we have $[D,D'] \in \mathrm{Der}_{\alpha^{-1}}(V)$.

5.5. Proposition. *With the above notations,* $\mathrm{Der}(V)$ *is a Hom-Leibniz Poisson superalgebra, in which the bracket is given by* $[D,D'] = DD' - (-1)^{|D||D'|}D'D$ *and an even endomorphism* α' *is defined by* $\alpha'(D) = \alpha\circ D$.

6. Representations of Hom-Leibniz Poisson Superalgebras

Let $(V,\circ,[.,.],\alpha)$ be a Hom-Leibniz Poisson superalgebra, then $(V,\circ,\alpha)$ is a Hom-associative superalgebra and $(V,[.,.],\alpha)$ a Hom-Leibniz superalgebra, so we can study $V-V$-bimodules, and the representation of Hom-Leibniz superalgebras over V.

6.1. Definition. *Let $(V,\circ,[.,.],\alpha)$ be a Hom-Leibniz Poisson superalgebra. A $V-V$-bimodule (M,α_M) is two $\mathbb{K}$-module homomorphisms*

$$[.,.]: V\otimes M\to M, [.,.]: M\otimes V\to M$$

such that the following axioms hold:

$$[V_\alpha, M_\beta]\subseteq M_{\alpha+\beta},\qquad [M_\alpha, V_\beta]\subseteq M_{\alpha+\beta},\qquad \forall\alpha,\beta\in\mathbb{Z}_2,$$

$$\begin{aligned}
\alpha_M([v,m]) &= [\alpha(v),\alpha_M(m)],\\
\alpha_M([m,v]) &= [\alpha_M(m),\alpha(v)],\\
[[v_1,v_2],\alpha_M(m)] &= [\alpha(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha(v_2)],\\
[[v_1,m],\alpha(v_2)] &= [\alpha(v_1),[m,v_2]]+(-1)^{|v_2||m|}[[v_1,v_2],\alpha_M(m)],\\
[[m,v_1],\alpha(v_2)] &= [\alpha_M(m),[v_1,v_2]]+(-1)^{|v_1||v_2|}[[m,v_2],\alpha(v_1)],\\
[v_1\circ m,\alpha(v_2)] &= \alpha(v_1)\circ[m,v_2]+(-1)^{|m||v_2|}[v_1,v_2]\circ\alpha_M(m),\\
[m\circ v_1,\alpha(v_2)] &= \alpha_M(m)\circ[v_1,v_2]+(-1)^{|v_1||v_2|}[m,v_2]\circ\alpha(v_1),\\
[v_1\circ v_2,\alpha_M(m)] &= \alpha(v_1)\circ[v_2,m]+(-1)^{|v_2||m|}[v_1,m]\circ\alpha(v_2),
\end{aligned}$$

for all homogeneous elements $m\in M, v_1,v_2\in V$.

A representation over V is defined by a $V-V$-bimodule (M,α_M).

6.2. Proposition. *Let $(V_1,\circ_1,[.,.]_1,\alpha_1)$ and $(V_2,\circ_2,[.,.]_2,\alpha_2)$ be Hom-Leibniz Poisson superalgebras and $\varphi: V_1\to V_2$ be a morphism of Hom-Leibniz Poisson superalgebras, then V_2 is a representation over V_1 with respect to the operations*

$$v_1\cdot m=\varphi(v_1)\cdot m,\qquad m\cdot v_1=m\cdot\varphi(v_1),$$

$$[v_1,m]=[\varphi(v_1),m],\quad [m,v_1]=[m,\varphi(v_1)],\quad \forall v_1\in V_1,\quad m\in V_2.$$

Proof. For any $v_1,v_2\in V_1, m\in V_2$, We just check

$$[[v_1,v_2],\alpha_2(m)]=[\alpha_1(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha_1(v_2)]$$

and

$$[v_1\cdot v_2,\alpha_2(m)]=\alpha_1(v_1)\cdot[v_2,m]+(-1)^{|v_2||m|}[v_1,m]\cdot\alpha_1(v_2).$$

By the definition of the operations, we have

$$\begin{aligned}
[[v_1,v_2],\alpha_2(m)] &= [\varphi([v_1,v_2]),\alpha_2(m)]\\
&= [[\varphi(v_1),\varphi(v_2)],\alpha_2(m)]\\
&= [\alpha_2\varphi(v_1),[\varphi(v_2),m]]+(-1)^{|v_2||m|}[[\varphi(v_1),m],\alpha_2\varphi(v_2)]\\
&= [\varphi\alpha_1(v_1),[\varphi(v_2),m]]+(-1)^{|v_2||m|}[[\varphi(v_1),m],\varphi\alpha_1(v_2)]\\
&= [\alpha_1(v_1),[v_2,m]]+(-1)^{|v_2||m|}[[v_1,m],\alpha_1(v_2)].
\end{aligned}$$

and

$$\begin{aligned}
[v_1 \cdot v_2, \alpha_2(m)] &= [\varphi(v_1 \cdot v_2), \alpha_2(m)] \\
&= [\varphi(v_1) \cdot \varphi(v_2), \alpha_2(m)] \\
&= \alpha_2\varphi(v_1) \cdot [\varphi(v_2), m] + (-1)^{|v_2||m|}[\varphi(v_1), m] \cdot \alpha_2\varphi(v_2) \\
&= \varphi\alpha_1(v_1) \cdot [v_2, m] + (-1)^{|v_2||m|}[v_1, m] \cdot \varphi\alpha_1(v_2) \\
&= \alpha_1(v_1) \cdot [v_2, m] + (-1)^{|v_2||m|}[v_1, m] \cdot \alpha_1(v_2).
\end{aligned}$$

□

6.3. Proposition. *Let $(V, \circ, [.,.], \alpha)$ be a Hom-Leibniz Poisson superalgebra, then $(\mathrm{End}(V), \alpha')$ can be endowed with a representation over V by means of the operations*

$$\alpha'(f) = \alpha \circ f, \quad (v \cdot f)(a) = v \cdot f(a), \quad (f \cdot v)(a) = (-1)^{|f||a|} f(a) \cdot v,$$

$$[v, f](a) = [v, f(a)], \quad [f, v](a) = (-1)^{|v||a|}[f(a), v],$$

for any $a, v \in V$, $f \in \mathrm{End}(V)$.

Proof. For any $v_1, v_2 \in V, f \in \mathrm{End}(V)$, We just check

$$[[v_1, v_2], \alpha'(f)] = [\alpha(v_1), [v_2, f]] + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)]$$

and

$$[v_1 \cdot v_2, \alpha'(f)] = \alpha(v_1) \cdot [v_2, f] + (-1)^{|v_2||f|}[v_1, f] \cdot \alpha(v_2).$$

By the definition of the operations, we have

$$\begin{aligned}
[[v_1, v_2], \alpha'(f)](a) &= [[v_1, v_2], \alpha'(f)(a)] \\
&= [[v_1, v_2], \alpha \circ f(a)] \\
&= [\alpha(v_1), [v_2, f(a)]] + (-1)^{|v_2||f|+|v_2||a|}[[v_1, f(a)], \alpha(v_2)] \\
&= [\alpha(v_1), [v_2, f](a)] + (-1)^{|v_2||f|+|v_2||a|}[[v_1, f](a), \alpha(v_2)] \\
&= [\alpha(v_1), [v_2, f]](a) + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)](a).
\end{aligned}$$

Then $[[v_1, v_2], \alpha'(f)] = [\alpha(v_1), [v_2, f]] + (-1)^{|v_2||f|}[[v_1, f], \alpha(v_2)]$. Since

$$\begin{aligned}
[v_1 \cdot v_2, \alpha'(f)](a) &= [v_1 \cdot v_2, \alpha(f(a))] \\
&= \alpha(v_1) \cdot [v_2, f(a)] + (-1)^{|v_2||f|+|v_2||a|}[v_1, f(a)] \cdot \alpha(v_2) \\
&= \alpha(v_1) \cdot [v_2, f](a) + (-1)^{|v_2||f|+|v_2||a|}[v_1, f](a) \cdot \alpha(v_2) \\
&= (\alpha(v_1) \cdot [v_2, f])(a) + (-1)^{|v_2||f|}([v_1, f] \cdot \alpha(v_2))(a).
\end{aligned}$$

We obtain $[v_1 \cdot v_2, \alpha'(f)] = \alpha(v_1) \cdot [v_2, f] + (-1)^{|v_2||f|}[v_1, f] \cdot \alpha(v_2)$. □

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