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Numerical solution of fourth order parabolic partial differential equation using parametric septic splines

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Abstract

In this paper, we report three level implicit method of high accuracy schemes for the numerical solution of fourth order nonhomogeneous parabolic partial differential equation, that governs the behavior of a vibrating beam. Parametric septic spline is used in space and finite difference discretization in time. The linear stability of the presented method is investigated. The computed results for three examples are compared wherever possible with those already available in literature which shows the superiority of the proposed method.

Keywords: Parametric septic splines; Fourth order parabolic equation; Stability analysis; Vibrating beam; Finite difference scheme.

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1. Introduction

In this paper, we consider the problem of undamped transverse vibration of a flexible straight beam in such a way that its support do not contribute to the strain energy of the system and is represented by the fourth order parabolic partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = f(x,t), \quad a \le x \le b, \quad t > 0, \tag{1.1}$$

subject to the initial conditions

$$\left.\begin{aligned} u(x,0) &= g_0(x), \quad a \le x \le b, \\ u_t(x,0) &= g_1(x), \quad a \le x \le b \end{aligned}\right\} \tag{1.2}$$

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and the boundary conditions

$$\left.\begin{aligned} u(a,t) = f_0(t), \quad u(b,t) = f_1(t), \quad t \geq 0, \\ u_{xx}(a,t) = q_0(t), \quad u_{xx}(b,t) = q_1(t), \quad t \geq 0, \end{aligned}\right\} \tag{1.3}$$

where u is the transverse displacement of the beam, $g_0(x),\ g_1(x),\ f_0(t),\ f_1(t),\ q_0(t),\ q_1(t)$ are continuous functions, t and x are time and distance variables respectively and $f(x,t)$ is dynamic driving force per unit mass [10,21,22,35].

Numerical methods for the solution of equation (1.1) have been carried out by many authors. Jain et al. [24], Danaee and Evans [1], Evans [8], Collatz [20], Andrade and Mckee [7] and Evans and Yousif [9] used finite difference methods for the numerical solution of transverse vibrations. Fairweather and Gourlay [13] derived explicit and implicit finite difference methods based on the semi explicit method. Parametric quintic spline methods are given by Rashidinia and Aziz [16] using nodal points. Collatz [20], Crandall [33], Jain [23], Conte [32], Jain et al. [24] and Todd [18] have proposed both explicit and implicit methods successfully. Five level, unconditionally stable, explicit method with truncation error of $O(k^2 + h^2 + (\frac{k}{h})^2)$ has been given by Albrecht [15]. All the above authors considered the homogeneous case of equation (1.1) with a constant coefficients. The analytical solution of homogeneous case of equation (1.1) has been obtained by using Adomain decomposition method by Wazwaz [3,4]. The nonhomogeneous problem with constant coefficients has been studied by Aziz et al. [34] based on parametric quintic spline and by Khan et al. [2] based on sextic spline by using nodal points. Khaliq and Twizell [6] and Twizell and Khaliq [11] developed a family of numerical methods, which are second order accurate in space and time, based on exact recurrence relation for a homogeneous case of equation (1.1) with a variable coefficient. Rashidinia and Mohammadi [17] developed three level implicit methods of $O(k^2+h^4)$ and $O(k^4+h^4)$ for the numerical solution of equation (1.1) with variable coefficients by using sextic spline. Wazwaz [5] has developed analytical solution of variable coefficient fourth order parabolic partial differential equation in two and three space dimensions. Khan et al. [25] have introduced a new algorithm, namely Laplace Decomposition Algorithm for fourth order parabolic partial differential equations with variable coefficients. In [26], the homotopy analysis method (HAM) is applied to solve such problems. Khan et al. [27] have studied numerical solution of time fractional fourth order partial differential equations with variable coefficients. They have implemented reliable series solution techniques namely, Adomian Decomposition Method (ADM) and He's Variational Iteration Method (HVIM). A family of B-spline methods have been considered by Caglar [14]. In [28], Mittal and Jain discussed two methods. In Method-I, they decomposed equation (1.1) in a system of second order equations and have solved them by using cubic B-spline and in Method-II, they have solved equation (1.1) directly by using quintic B-spline method. Talwar et al. [19] and Mohanty et al. [29-31] have used high accuracy spline scheme for solving one dimensional partial differential equations.

In this paper, parametric septic spline relations have been derived using nodal points. We have used parametric septic spline functions to develop a new numerical method for obtaining smooth approximations to the solution of nonhomogeneous parabolic partial differential equations dealing with vibrations of beams. In section 2, parametric septic spline and spline relations are developed. In section 3, we have presented the formulation of our method. Development of boundary equations are given in section 4. In section 5, truncation error and class of methods are given. Stability analysis is discussed in section 6. Finally in section 7, three examples are given to demonstrate the practical usefulness and superiority of our method.

2. Parametric septic spline

Let a set of grid points in the interval $[a,b]$ such that

$$x_j = a + jh,\ j = 0(1)N,\ h = \frac{(b-a)}{N}. \tag{2.1}$$

A function $S_\Delta(x,\tau)$ of class $C^6[a,b]$ which interpolates $u(x)$ at the mesh point x_j depends on a parameter τ, and as $\tau \to 0$ it reduces to septic spline $S_\Delta(x)$ in $[a,b]$ is termed as parametric septic spline function. Since the parameter τ can occur in $S_\Delta(x)$ in many ways such a spline is not unique.

If $S_\Delta(x,\tau) = S_\Delta(x)$ is a piecewise function satisfying the following differential equation in the interval $[x_{j-1}, x_j]$

$$\begin{aligned} S_\Delta^{(6)}(x) - \tau^2 S_\Delta''(x) &= (Q_j - \tau^2 M_j)\frac{x - x_{j-1}}{h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{x_j - x}{h} \\ &= A_j z + A_{j-1}\bar{z}, \end{aligned} \tag{2.2}$$

where

$$z = \frac{x - x_{j-1}}{h},\ \bar{z} = 1 - z,\ A_i = Q_i - \tau^2 M_i,$$

$$S_\Delta''(x_i,\tau) = M_i,\quad S_\Delta^{(6)}(x_i,\tau) = Q_i,\quad i = j-1, j;\quad \tau > 0,$$

then it is termed as parametric septic spline II.
Solving equation (2.2), we get

$$\begin{aligned} S_\triangle(x) &= A_1 + A_2 x + A_3 \cosh\sqrt{\tau}x + A_4 \sinh\sqrt{\tau}x + A_5 \cos\sqrt{\tau}x + A_6 \sin\sqrt{\tau}x \\ &\quad -\frac{1}{\tau^2}\left\{(Q_j - \tau^2 M_j)\frac{(x - x_{j-1})^3}{6h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{(x_j - x)^3}{6h}\right\} \end{aligned} \tag{2.3}$$

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$\left.\begin{aligned} &S_\Delta(x_j) = u_j,\ S_\Delta(x_{j+1}) = u_{j+1}, \\ &S_\Delta''(x_j) = M_j,\ S_\Delta''(x_{j+1}) = M_{j+1}, \\ &S_\Delta^{(4)}(x_j) = F_j,\ S_\Delta^{(4)}(x_{j+1}) = F_{j+1}. \end{aligned}\right\} \tag{2.4}$$

To define spline in terms of u_j's, M_j's and F_j's, the coefficients introduced in Eq.(2.3) are calculated as

$$\begin{aligned}
A_1 &= u_{j-1} + \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) - \frac{F_{j-1}}{\tau^2} \\
&\quad - \frac{x_{j-1}}{h}\Bigg[(u_j - u_{j-1}) - \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) + \frac{h^2}{6\tau^2}(Q_j - \tau^2 M_j) + \frac{1}{\tau^2}(F_{j-1} - F_j)\Bigg], \\
A_2 &= \frac{1}{h}(u_j - u_{j-1}) + \frac{h}{6\tau^2}\Bigg[-(Q_{j-1} - \tau^2 M_{j-1}) + (Q_j - \tau^2 M_j)\Bigg] + \frac{1}{\tau^2 h}(F_{j-1} - F_j), \\
A_3 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Bigg[\frac{1}{2}\sinh\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) - \frac{1}{2}\sinh\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right) \\
&\quad - \frac{1}{\tau}\sinh\sqrt{\tau}x_{j-1}Q_j + \frac{1}{\tau}\sinh\sqrt{\tau}x_j Q_{j-1}\Bigg], \\
A_4 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Bigg[-\frac{1}{2}\cosh\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) + \frac{1}{2}\cosh\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right) \\
&\quad + \frac{1}{\tau}\cosh\sqrt{\tau}x_{j-1}Q_j - \frac{1}{\tau}\cosh\sqrt{\tau}x_j Q_{j-1}\Bigg], \\
A_5 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Bigg[\sin\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) - \sin\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\Bigg], \\
A_6 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Bigg[-\cos\sqrt{\tau}x_j\left(F_{j-1} - \frac{Q_{j-1}}{\tau}\right) + \cos\sqrt{\tau}x_{j-1}\left(F_j - \frac{Q_j}{\tau}\right)\Bigg].
\end{aligned} \tag{2.5}$$

Substituting these values in (2.3), we get

$$\begin{aligned}
S_\Delta(x) &= zu_j + \bar{z}u_{j-1} + \frac{h^2}{6}\Bigg[p(z)M_j + p(\bar{z})M_{j-1}\Bigg] + \frac{h^4}{2}\Bigg[r(z)F_j + r(\bar{z})F_{j-1}\Bigg] \\
&\quad + \frac{h^6}{6}\Bigg[q(z)Q_j + q(\bar{z})Q_{j-1}\Bigg],
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
p_1(z) &= z^3 - z, \quad q_1(z) = \frac{z}{\omega^4} - \frac{z^3}{\omega^4} + \frac{3\sinh\omega z}{\omega^6 \sinh\omega} - \frac{3\sin\omega z}{\omega^6 \sin\omega}, \\
r_1(z) &= \frac{-2z}{\omega^4} + \frac{\sinh\omega z}{\omega^4 \sinh\omega} + \frac{\sin\omega z}{\omega^4 \sin\omega} \quad \mathit{and} \quad \omega = \sqrt{\tau}h.
\end{aligned} \tag{2.7}$$

Applying the first, third and fifth derivative continuities at the knots, i.e. $S_\Delta^{(\mu)}(x_j^-) = S_\Delta^{(\mu)}(x_j^+)$, $\mu = 1, 3$ and 5, the following consistency relations are derived:

$$\begin{aligned}
M_{j+1} + 4M_j + M_{j-1} &= \frac{6}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + 3h^2(\alpha_2 F_{j+1} + 2\beta_2 F_j + \alpha_2 F_{j-1}) \\
&\quad + h^4(\alpha_1 Q_{j+1} + 2\beta_1 Q_j + \alpha_1 Q_{j-1}), \quad j = 1(1)N-1.
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
M_{j+1} - 2M_j + M_{j-1} &= \frac{h^2}{6}[(1 - \omega^4\alpha_1)F_{j+1} + 2(2 - \omega^4\beta_1)F_j + (1 - \omega^4\alpha_1)F_{j-1}] \\
&\quad - \frac{h^4}{2}(\alpha_2 Q_{j+1} + 2\beta_2 Q_j + \alpha_2 Q_{j-1}), \quad j = 1(1)N-1.
\end{aligned} \tag{2.9}$$

$$h^2[(1-\omega^4\alpha_1)Q_{j+1}+2(2-\omega^4\beta_1)Q_j+(1-\omega^4\alpha_1)Q_{j-1}] =$$
$$3[(\omega^4\alpha_2+2)F_{j+1}+2(\omega^4\beta_2-2)F_j+(\omega^4\alpha_2+2)F_{j-1}], \quad j=1(1)N-1, \tag{2.10}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{\omega^4}+\frac{3}{\omega^5\sinh\omega}-\frac{3}{\omega^5\sin\omega},\\
\beta_1 &= \frac{2}{\omega^4}-\frac{3}{\omega^5}\coth\omega+\frac{3}{\omega^5}\cot\omega,\\
\alpha_2 &= \frac{-2}{\omega^4}+\frac{1}{\omega^3\sinh\omega}+\frac{1}{\omega^3\sin\omega},\\
\beta_2 &= \frac{2}{\omega^4}-\frac{1}{\omega^3}\coth\omega-\frac{1}{\omega^3}\cot\omega.
\end{aligned} \tag{2.11}$$

As $\tau \to 0$ that is $\omega \to 0$ then $(\alpha_1,\beta_1,\alpha_2,\beta_2) \to (\frac{-31}{2520},\frac{-4}{315},\frac{7}{180},\frac{2}{45})$.

Using equations (2.8)-(2.10), we obtain the following scheme

$$\begin{aligned}
&(e_1u_{j-3}+e_2u_{j-2}+e_3u_{j-1}+e_4u_j+e_3u_{j+1}+e_2u_{j+2}+e_1u_{j+3})\\
&=\frac{h^4}{6}(p_1F_{j-3}+p_2F_{j-2}+p_3F_{j-1}+p_4F_j+p_3F_{j+1}+p_2F_{j+2}+p_1F_{j+3}),\ j=3(1)N-3,
\end{aligned} \tag{2.12}$$

where the coefficients (e_1,e_2,e_3,e_4) and (p_1,p_2,p_3,p_4) of the developed scheme are given by

$$\begin{aligned}
e_1 &= 1-3\omega^4\alpha_1+3\omega^8\alpha_1^2-\omega^{12}\alpha_1^3,\\
e_2 &= 4\omega^4\alpha_1-2\omega^4\beta_1-8\omega^8\alpha_1^2+4\omega^8\alpha_1\beta_1-2\omega^{12}\alpha_1^2\beta_1,\\
e_3 &= 7(1-\omega^4\alpha_1)^3-8(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1),\\
e_4 &= 12(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1)-8(1-\omega^4\alpha_1)^3,\\
p_1 &= c_1(1-\omega^4\alpha_1)^2,\\
p_2 &= 2c_1(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+c_2(1-\omega^4\alpha_1)^2-3d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_3 &= (c_1+c_3)(1-\omega^4\alpha_1)^2+6d_1(1-\omega^4\alpha_1)(2-\omega^4\beta_2)+2c_2(1-\omega^4\alpha_1)(2-\omega^4\beta_1)\\
&\quad -3d_2(1-\omega^4\alpha_1)(2+\omega^4\alpha_2),\\
p_4 &= 2c_2(1-\omega^4\alpha_1)^2-6d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2)-6d_1(2-\omega^4\beta_1)(2-\omega^4\beta_2)\\
&\quad +2c_3(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+6d_2(1-\omega^4\alpha_1)(2-\omega^4\beta_2).
\end{aligned} \tag{2.13}$$

Also

$$\begin{aligned}
c_1 &= \frac{1}{6}\omega^8\alpha_1^2-\frac{3}{2}\omega^4\alpha_2^2-\frac{1}{3}\omega^4\alpha_1-6\alpha_1-6\alpha_2+\frac{1}{6},\\
c_2 &= \frac{2}{3}\omega^8\alpha_1^2+\frac{1}{3}\omega^8\alpha_1\beta_1-18\omega^4\alpha_1\alpha_2-3\omega^4\alpha_2\beta_2-6\omega^4\alpha_2^2-2\omega^4\alpha_1-\frac{1}{3}\omega^4\beta_1\\
&\quad -12\alpha_1-6\beta_2+\frac{4}{3},\\
c_3 &= \frac{1}{3}\omega^8\alpha_1^2+\frac{4}{3}\omega^8\alpha_1\beta_1-36\omega^4\alpha_1\beta_2-12\omega^4\alpha_2\beta_2-3\omega^4\alpha_2^2-\frac{10}{3}\omega^4\alpha_1-\frac{4}{3}\omega^4\beta_1\\
&\quad +36\alpha_1+12\alpha_2+12\beta_2+3,\\
d_1 &= \omega^4\alpha_2\beta_1-\omega^4\alpha_1\beta_2+6\omega^4\alpha_1^2-10\alpha_1-2\alpha_2+2\beta_1+\beta_2,\\
d_2 &= 4\omega^4\alpha_2\beta_1-4\omega^4\alpha_1\beta_2+12\omega^4\alpha_1\beta_1-16\alpha_1-18\alpha_2-4\beta_1+4\beta_2.
\end{aligned} \tag{2.14}$$

As $\tau \to 0$ that is $\omega \to 0$, we have

(i)$(e_1, e_2, e_3, e_4) \longrightarrow (1, 0, -9, 16)$,
(ii)$(c_1, c_2, c_3, d_1, d_2) \longrightarrow \left(\frac{1}{140}, \frac{17}{14}, \frac{249}{70}, \frac{9}{140}, \frac{4}{35}\right)$,
(iii)$(p_1, p_2, p_3, p_4) \longrightarrow \left(\frac{1}{140}, \frac{6}{7}, \frac{1191}{140}, \frac{604}{35}\right)$.

[Remarks:] For these values our scheme reduces to the polynomial septic spline for fourth order boundary value problem which is given as equation (7) in G. Akram and S. S. Siddiqi [12].
Here, we have taken $(e_1, e_2, e_3, e_4) = (1, 0, -9, 16)$, therefore scheme (2.12) becomes

$$\begin{aligned} &p_1(F_{j-3} + F_{j+3}) + p_2(F_{j-2} + F_{j+2}) + p_3(F_{j-1} + F_{j+1}) + p_4 F_j \\ &= \frac{6}{h^4}\Big[(u_{j-3} + u_{j+3}) - 9(u_{j-1} + u_{j+1}) + 16u_j\Big], \ j = 3(1)N-3. \end{aligned} \tag{2.15}$$

We can also write (2.15) as

$$\Lambda_x F_j = \frac{6}{h^4}(6\delta_x^4 + \delta_x^6)u_j, \tag{2.16}$$

where δ is the central difference operator and operator Λ_x for any function W is defined by

$$\Lambda_x W_j = p_1(W_{j-3} + W_{j+3}) + p_2(W_{j-2} + W_{j+2}) + p_3(W_{j-1} + W_{j+1}) + p_4 W_j. \tag{2.17}$$

3. Derivation of the method

Let the region $R = [a, b] \times [0, \infty)$ be discretized by a set of points $R_{h,k}$ which are the vertices of a grid points (x_j, t_m), where $x_j = jh, j = 0(1)N, Nh = b - a$ and $t_m = mk,\ m = 0, 1, 2, 3, ...$ The quantities h and k are mesh sizes in the space and time directions respectively.

We have developed an approximation for (1.1) in which the time derivative is replaced by a finite difference approximation and space derivative is replaced by the parametric septic spline function approximation. We need the following finite difference approximation for the time partial derivative of u :

$$\overline{u}_{tt_j}^m = k^{-2}\delta_t^2(1 + \sigma\delta_t^2)^{-1}u_j^m, \tag{3.1}$$

where σ is a parameter such that the finite difference approximation to the time derivative is $O(k^2)$ for arbitrary σ and $O(k^4)$ for $\sigma = 1/12$. u_j^m is the approximate solution of (1.1) at (x_j, t_m) and δ_t is the central difference operator with respect to t so that

$$\delta_t^2 u_j^m = u_j^{m+1} - 2u_j^m + u_j^{m-1}.$$

At the grid point (j, m) the differential equation may be discretized by

$$\overline{u}_{tt_j}^m + \overline{u}_{xxxx_j}^m = f_j^m, \tag{3.2}$$

where $\overline{u}_{xxxx_j}^m$ is the fourth order spline derivative at (x_j, t_m) denoted by $F_j^m = S_\Delta^{(4)}(x_j, t_m)$ with respect to the space variable $f_j^m = f(x_j, t_m)$. Using (3.1) and replacing fourth order spline derivative by F_j^m, we have

$$k^{-2}\delta_t^2(1 + \sigma\delta_t^2)^{-1}u_j^m + F_j^m = f_j^m. \tag{3.3}$$

Operating Λ_x on both sides of (3.3) and using (2.16), we obtain

$$\begin{aligned} &\delta_t^2[p_1(u_{j-3}^m + u_{j+3}^m) + p_2(u_{j-2}^m + u_{j+2}^m) + p_3(u_{j-1}^m + u_{j+1}^m) + p_4 u_j^m] + 6r^2(1 + \sigma\delta_t^2)(6\delta_x^4 + \delta_x^6)u_j^m \\ &= k^2(1 + \sigma\delta_t^2)[p_1(f_{j-3}^m + f_{j+3}^m) + p_2(f_{j-2}^m + f_{j+2}^m) + p_3(f_{j-1}^m + f_{j+1}^m) + p_4 f_j^m],\ j = 3(1)N-3, \end{aligned} \tag{3.4}$$

where $r = \frac{k}{h^2}$ is the mesh ratio and p_1, p_2, p_3, p_4 are parameters. After simplifying the above equation, we obtain

$$\delta_t^2[(2p_1+2p_2+2p_3+p_4)+(9p_1+4p_2+p_3)\delta_x^2+(6p_1+p_2+36\sigma r^2)\delta_x^4+(p_1+6\sigma r^2)\delta_x^6]u_j^m$$
$$+6r^2(6\delta_x^4+\delta_x^6)u_j^m = k^2(1+\sigma\delta_t^2)[p_1(f_{j-3}^m+f_{j+3}^m)+p_2(f_{j-2}^m+f_{j+2}^m)+p_3(f_{j-1}^m+f_{j+1}^m)+p_4 f_j^m],$$
$$j = 3(1)N-3. \tag{3.5}$$

This scheme (3.5) is finite difference in time and spline scheme in space variable, which on simplification can be written as

$$\begin{aligned}
&[P_1(u_{j-3}^{m+1}+u_{j+3}^{m+1})+P_2(u_{j-2}^{m+1}+u_{j+2}^{m+1})+P_3(u_{j-1}^{m+1}+u_{j+1}^{m+1})+P_4u_j^{m+1}]\\
&+[S_1(u_{j-3}^{m}+u_{j+3}^{m})+S_2(u_{j-2}^{m}+u_{j+2}^{m})+S_3(u_{j-1}^{m}+u_{j+1}^{m})+S_4u_j^{m}]\\
&+[P_1(u_{j-3}^{m-1}+u_{j+3}^{m-1})+P_2(u_{j-2}^{m-1}+u_{j+2}^{m-1})+P_3(u_{j-1}^{m-1}+u_{j+1}^{m-1})+P_4u_j^{m-1}]\\
&=K_1[p_1(f_{j-3}^{m+1}+f_{j+3}^{m+1})+p_2(f_{j-2}^{m+1}+f_{j+2}^{m+1})+p_3(f_{j-1}^{m+1}+f_{j+1}^{m+1})+p_4f_j^{m+1}]\\
&+K_2[p_1(f_{j-3}^{m}+f_{j+3}^{m})+p_2(f_{j-2}^{m}+f_{j+2}^{m})+p_3(f_{j-1}^{m}+f_{j+1}^{m})+p_4f_j^{m}]\\
&+K_1[p_1(f_{j-3}^{m-1}+f_{j+3}^{m-1})+p_2(f_{j-2}^{m-1}+f_{j+2}^{m-1})+p_3(f_{j-1}^{m-1}+f_{j+1}^{m-1})+p_4f_j^{m-1}],\ j=3(1)N-3.
\end{aligned} \tag{3.6}$$

The final scheme (3.6) may be written in the schematic form as

$$\left.\begin{matrix} P_1 & P_2 & P_3 & P_4 & P_3 & P_2 & P_1 \\ S_1 & S_2 & S_3 & S_4 & S_3 & S_2 & S_1 \\ P_1 & P_2 & P_3 & P_4 & P_3 & P_2 & P_1 \end{matrix}\right\} u_j^m = \left.\begin{matrix} K_1p_1 & K_1p_2 & K_1p_3 & K_1p_4 & K_1p_3 & K_1p_2 & K_1p_1 \\ K_2p_1 & K_2p_2 & K_2p_3 & K_2p_4 & K_2p_3 & K_2p_2 & K_2p_1 \\ K_1p_1 & K_1p_2 & K_1p_3 & K_1p_4 & K_1p_3 & K_1p_2 & K_1p_1 \end{matrix}\right\} f_j^m,$$

where

$$\left.\begin{matrix} P_1 = p_1+6\sigma r^2, P_2 = p_2, P_3 = p_3-54\sigma r^2, P_4 = p_4+96\sigma r^2, \\ S_1 = -2P_1+6\sigma r^2, S_2 = -2P_2, S_3 = -2P_3-54\sigma r^2, S_4 = -2P_4+96\sigma r^2, \\ K_1 = \sigma k^2, K_2 = k^2(1-2\sigma). \end{matrix}\right\}$$

4. Development of boundary equations

The relation (3.6) gives $N-5$ linear algebraic equations in $N-1$ unknowns u_j, $j = 3(1)N-3$. We need four more equations, two at each end of the range of integration, for the direct computation of $u_j, j = 1(1)N-1$.

(i)
$$\frac{1392}{7}u_1^m - \frac{2340}{7}u_2^m + \frac{20320}{63}u_3^m - \frac{1320}{7}u_4^m + \frac{432}{7}u_5^m - \frac{548}{63}u_6^m = \frac{464}{9}u_0^m - \frac{80}{7}h^2(u_0^m)'', j = 1,$$

(ii)
$$-\frac{10280}{469}u_1^m + \frac{8746}{87}u_2^m - \frac{19309}{97}u_3^m + \frac{11287}{52}u_4^m - \frac{960}{7}u_5^m + \frac{9793}{207}u_6^m - \frac{464}{67}u_7^m$$
$$= -\frac{720}{469}h^2(u_0^m)'', j = 2,$$

(iii)
$$-\frac{464}{67}u_{N-7}^m + \frac{9793}{207}u_{N-6}^m - \frac{960}{7}u_{N-5}^m + \frac{11287}{52}u_{N-4}^m - \frac{19309}{97}u_{N-3}^m + \frac{8746}{87}u_{N-2}^m$$
$$-\frac{10280}{469}u_{N-1}^m = -\frac{720}{469}h^2(u_N^m)'', j = N-2,$$

(iv)
$$-\frac{548}{63}u_{N-6}^m + \frac{432}{7}u_{N-5}^m - \frac{1320}{7}u_{N-4}^m + \frac{20320}{63}u_{N-3}^m - \frac{2340}{7}u_{N-2}^m + \frac{1392}{7}u_{N-1}^m$$
$$= \frac{464}{9}u_N^m - \frac{80}{7}h^2(u_N^m)'', j = N-1.$$

For high accuracy formula of $O(k^6+h^{10})$, we use the following equations for approximating the boundary equations:
(i)

$$\frac{15928}{35}u_1^m-\frac{5141}{5}u_2^m+\frac{32427}{22}u_3^m-\frac{11441}{8}u_4^m+\frac{15875}{17}u_5^m-\frac{17741}{45}u_6^m+\frac{6919}{71}u_7^m-\frac{3853}{359}u_8^m$$

$$=\frac{6985}{72}u_0^m-\frac{12600}{761}h^2(u_0^m)'', j=1,$$

(ii) $$-\frac{4883}{125}u_1^m+\frac{18192}{79}u_2^m-\frac{38050}{61}u_3^m+\frac{21181}{21}u_4^m-\frac{275354}{261}u_5^m+\frac{138411}{191}u_6^m-\frac{25063}{79}u_7^m$$

$$+\frac{3707}{46}u_8^m-\frac{4288}{473}u_9^m=-\frac{967}{625}h^2(u_0^m)'', j=2,$$

(iii) $$-\frac{4288}{473}u_{N-9}^m+\frac{3707}{46}u_{N-8}^m-\frac{25063}{79}u_{N-7}^m+\frac{138411}{191}u_{N-6}^m-\frac{275354}{261}u_{N-5}^m+\frac{21181}{21}u_{N-4}^m$$

$$-\frac{38050}{61}u_{N-3}^m+\frac{18192}{79}u_{N-2}^m-\frac{4883}{125}u_{N-1}^m=-\frac{967}{625}h^2(u_N^m)'', j=N-2,$$

(iv) $$-\frac{3853}{359}u_{N-8}^m+\frac{6919}{71}u_{N-7}^m-\frac{17741}{45}u_{N-6}^m+\frac{15875}{17}u_{N-5}^m-\frac{11441}{8}u_{N-4}^m+\frac{32427}{22}u_{N-3}^m$$

$$-\frac{5141}{5}u_{N-2}^m+\frac{15928}{35}u_{N-1}^m=\frac{6985}{72}u_N^m-\frac{12600}{761}h^2(u_N^m)'', j=N-1.$$

5. Truncation error and class of methods

Expanding (3.5) in Taylor series in terms of $u(x_j,t_m)$ and its derivatives, we obtain the following relations

$$\begin{aligned}
\delta_x^6 u(x_j,t_m) &= \Big[h^6D_x^6+\frac{1008}{8!}h^8D_x^8+\frac{105840}{10!}h^{10}D_x^{10}+\frac{1013760}{12!}h^{12}D_x^{12}\\
&\quad+\frac{9369360}{14!}h^{14}D_x^{14}+\ldots\Big]u(x_j,t_m)\\
\delta_x^4 u(x_j,t_m) &= \Big[h^4D_x^4+\frac{120}{6!}h^6D_x^6+\frac{505}{8!}h^8D_x^8+\frac{1016}{10!}h^{10}D_x^{10}+\frac{2040}{12!}h^{12}D_x^{12}+\ldots\Big]u(x_j,t_m)\\
\delta_t^2 u(x_j,t_m) &= \Big[-r^2h^4D_x^4+\frac{1}{12}r^4h^8D_x^8-\frac{1}{360}r^6h^{12}D_x^{12}+\frac{1}{20160}r^8h^{16}D_x^{16}+\ldots\Big]u(x_j,t_m),
\end{aligned} \tag{5.1}$$

where $(D_t^2 + D_x^4)u(x_j, t_m) = f(x_j, t_m)$. Using (3.5) and (5.1), we obtain the truncation error

$$
\begin{aligned}
T_j^m \;=\;& \Big[(2p_1 + 2p_2 + 2p_3 + p_4) + (9p_1 + 4p_2 + p_3)\delta_x^2 + (6p_1 + p_2 + 36\sigma r^2)\delta_x^4 \\
&+ (p_1 + 6\sigma r^2)\delta_x^6\Big]\delta_t^2 u_j^m + 6r^2(6\delta_x^4 + \delta_x^6)u_j^m \\
&- k^2(1 + \sigma\delta_t^2)\Big[p_1(f_{j-3}^m + f_{j+3}^m) + p_2(f_{j-2}^m + f_{j+2}^m) + p_3(f_{j-1}^m + f_{j+1}^m) + p_4 f_j^m\Big] \\
=\;& \Big[o_1\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big) \\
&+ (o_2h^2D_x^2)\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big) \\
&+ (2o_2 + 24o_3)\frac{h^4D_x^4}{4!}\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big) \\
&+ (2o_2 + 120o_3 + 720o_4)\frac{h^6D_x^6}{6!}\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big) \\
&+ (2o_2 + 504o_3 + 10080o_4)\frac{h^8D_x^8}{8!}\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big) \\
&+ (2o_2 + 1016o_3 + 105840o_4)\frac{h^{10}D_x^{10}}{10!}\Big(k^2D_t^2 + \frac{2k^4D_t^4}{4!} + \frac{2k^6D_t^6}{6!} + \frac{2k^8D_t^8}{8!} + \ldots\Big)\Big]u_j^m \\
&+ 6r^2\Big(\frac{144}{4!}h^4D_x^4 + \frac{1440}{6!}h^6D_x^6 + \frac{4032}{8!}h^8D_x^8 + \frac{111936}{10!}h^{10}D_x^{10} + \frac{10266000}{12!}h^{12}D_x^{12} \\
&+ \ldots\Big)u_j^m - \Big[o_1k^2 + o_1\sigma k^4D_t^2 + \frac{2o_1}{4!}\sigma k^6D_t^4 + \frac{2o_1}{6!}\sigma k^8D_t^6 + \frac{2o_1}{8!}\sigma k^{10}D_t^{10} + \ldots \\
&\Big(o_2h^2D_x^2 + 2(81p_1 + 16p_2 + p_3)\frac{h^4D_x^4}{4!} + 2(729p_1 + 64p_2 + p_3)\frac{h^6D_x^6}{6!} \\
&+ 2(6561p_1 + 256p_2 + p_3)\frac{h^8D_x^8}{8!} + \ldots\Big) \times \\
&\Big(k^2 + \sigma k^4D_t^2 + \frac{2\sigma k^6D_t^4}{4!} + \frac{2\sigma k^8D_t^6}{6!} + \frac{2\sigma k^{10}D_t^8}{8!} + \ldots\Big)\Big](D_t^2 + D_x^4)u_j^m
\end{aligned}
$$

which may be written as

$$
\begin{aligned}
T_j^m \;=\;& \Big[(36 - o_1)r^2h^4D_x^4 + (12 - o_2)r^2h^6D_x^6 + \Big(\frac{3}{5} - \frac{1}{12}(81p_1 + 16p_2 + p_3)\Big)r^2h^8D_x^8 \\
&+ \Big(\frac{583}{3150} - \frac{1}{360}(729p_1 + 64p_2 + p_3)\Big)r^2h^{10}D_x^{10} \\
&+ \Big(\frac{2565}{199584} - \frac{1}{20160}(6561p_1 + 256p_2 + p_3)\Big)r^2h^{12}D_x^{12} + \ldots \\
&+ \Big(\frac{1}{12} - \sigma\Big)o_1k^4D_t^4 + \Big(\frac{1}{12} - \sigma\Big)o_2h^2k^4D_x^2D_t^4 + \Big(\frac{1}{360} - \frac{\sigma}{12}\Big)o_2h^2k^6D_x^2D_t^6 \\
&+ \Big(\frac{h^4}{12}(o_2 + 12o_3) - o_1\sigma k^2 - \frac{h^4}{12}(81p_1 + 16p_2 + p_3)\Big)k^4D_x^4D_t^2 \\
&+ \Big(\frac{h^4}{360}(o_2 + 60o_3 + 360o_4) - o_2\sigma k^2 - \frac{\sigma h^4}{360}(729p_1 + 64p_2 + p_3)\Big)h^2k^4D_x^6D_t^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{h^4}{144}(o_2+12o_3)-\frac{1}{12}o_1\sigma k^2-\frac{\sigma h^4}{12}(81p_1+16p_2+p3)\right)k^4D_x^4D_t^4\\
&+\left(\frac{1}{20160}-\frac{\sigma}{360}\right)o_1k^8D_t^8+\left(\frac{1}{360}-\frac{\sigma}{12}\right)o_1k^6D_t^6+\left(\frac{2}{8!}-\frac{2\sigma}{6!}\right)o_2h^2k^8D_x^2D_t^8\\
&+\left(\frac{h^4}{8!}(2o_2+504o_3+10080o_4)-\frac{\sigma k^2}{12}(81p_1+16p_2+p_3)\right.\\
&\left.-\frac{2\sigma h^4}{8!}(6561p_1+256p_2+p_3)\right)h^4k^2D_x^8D_t^2\\
&\left.+\left(\frac{h^4}{4320}(o_2+12o_3)-\frac{1}{360}o_1\sigma k^2-\frac{\sigma h^4}{144}(81p_1+16p_2+p_3)\right)k^6D_x^4D_t^6+...\right]u_j^m\\
&+...,
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{gathered}
o_1=2p_1+2p_2+2p_3+p_4,\quad o_2=9p_1+4p_2+p_3,\\
o_3=6p_1+p_2+36\sigma r^2,\quad o_4=p_1+6\sigma r^2,\ D_x=\frac{\partial}{\partial x},\quad D_t=\frac{\partial}{\partial t}.
\end{gathered}
\tag{5.3}
$$

For various values of parameters p_1,p_2,p_3,p_4 and σ, we obtain the following class of methods:

Case 1: If $36-o_1=0$, we obtain various schemes of $O(k^2+h^2)$ for arbitrary values of σ.

Case 2: If $36-o_1=0$ and $12-o_2=0$, we obtain various schemes of $O(k^2+h^4)$ for arbitrary values of σ.

Case 3: If $36-o_1=0,\ 12-o_2=0$, and $\frac{3}{5}-\frac{1}{12}(81p_1+16p_2+p_3)=0$, we obtain various schemes of $O(k^4+h^8)$ for $\sigma\neq\frac{1}{12}$ and $O(k^6+h^8)$ for $\sigma=\frac{1}{12}$.

Case 4: For $(p_1,p_2,p_3,p_4,\sigma)=\left(\frac{16}{75},\frac{-42}{25},\frac{84}{5},\frac{16}{3},\frac{1}{12}\right)$, we obtain a scheme of $O(k^6+h^{10})$.

6. Stability analysis

To investigate the stability analysis of the scheme (3.6), we use the Von Neumann method. We have assumed that the solution of (3.6) at the grid point (x_j,t_m) is of the form

$$u_j^m=\xi^m e^{ji\theta}, \tag{6.1}$$

where $i=\sqrt{-1},\ \theta$ is real and ξ in general is complex.

Substituting (6.1) in homogeneous part of (3.6), we obtain a characteristic equation

$$X\xi^2+Y\xi+Z=0, \tag{6.2}$$

where

$$
\begin{gathered}
X=Z=P_1\cos 3\theta+P_2\cos 2\theta+P_3\cos\theta+2P_4,\\
Y=S_1\cos 3\theta+S_2\cos 2\theta+S_3\cos\theta+2S_4.
\end{gathered}
$$

Under the transformation $\xi=\frac{1+\eta}{1-\eta}$, equation (6.2) becomes

$$(X-Y+Z)\eta^2+2(X-Z)\eta+(X+Y+Z)=0. \tag{6.3}$$

The necessary and sufficient condition for $|\xi|\leq 1$ is that $X-Y+Z>0,\ X-Z>0$ and $X+Y+Z>0$.

The conditions $X-Z>0$ and $X+Y+Z>0$ are always satisfied for all real values of θ. From the condition $X-Y+Z>0$, we get that the scheme (3.6) is unconditionally stable if $\sigma\geq\frac{1}{4}$ and conditionally stable if $\sigma<\frac{1}{4}$ for all real values of p_1,p_2,p_3,p_4 and θ.

We summarized the above results in the following theorem:

Theorem: The scheme (3.6) for solving (1.1) is unconditionally stable if $\sigma \geq \frac{1}{4}$ and conditionally stable if $\sigma < \frac{1}{4}$. By using the Lax theorem, we can conclude that the present method is converge as long as stability criterion is satisfied.

7. Numerical results and discussions

We have applied the presented method on the fourth order parabolic partial differential equation and have considered one homogeneous and two nonhomogeneous examples. The proposed method (3.6) is implicit three level method based on parametric septic spline function.

Example 1: Consider a nonhomogeneous fourth order parabolic partial differential equation [2,9,17,34]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = (\pi^4 - 1)\sin \pi x \cos t, \quad 0 \leq x \leq 1,\ t \geq 0,$$

subject to the initial conditions

$$u(x,0) = \sin \pi x, \quad u_t(x,0) = 0, \quad 0 \leq x \leq 1$$

and the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \geq 0.$$

The analytical solution for this example is

$$u(x,t) = \sin \pi x \cos t.$$

We have solved the above example with $h = 0.05$ and $k = 0.005$ giving $r = 2$ and by choosing $\sigma = \frac{1}{4}, \frac{1}{12}$ with $O(k^4 + h^8), O(k^6 + h^8)$ and $O(k^6 + h^{10})$ for arbitrary choices of parameters p_1, p_2, p_3 and p_4. All computations have been done over ten time steps. The absolute errors at particular points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ and comparison with other existing methods [2,8,15,26] are tabulated in table 1. We repeat the computations for 16 time steps with $r = 0.5$.

Table 1. Absolute errors for example 1

Methods $(p_1, p_2, p_3, p_4, \sigma)$	r	Time steps	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$O(k^6+h^{10})$							
$\left(\frac{16}{75}, \frac{-42}{25}, \frac{84}{5}, \frac{16}{3}, \frac{1}{12}\right)$	2	10	2.05(−6)	4.37(−7)	1.04(−7)	1.19(−8)	5.61(−8)
	0.5	16	4.04(−7)	7.81(−9)	8.34(−9)	4.56(−8)	5.10(−8)
$O(k^6+h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	2	10	2.63(−6)	2.86(−8)	4.34(−8)	7.02(−8)	6.31(−8)
	0.5	16	6.80(−7)	3.52(−7)	9.65(−7)	1.39(−6)	1.53(−6)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	2	10	2.76(−6)	4.65(−8)	3.55(−8)	6.71(−8)	6.06(−8)
	0.5	16	1.08(−6)	2.14(−7)	1.02(−6)	1.59(−6)	1.78(−6)
$O(k^4+h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	2	10	9.02(−6)	8.37(−7)	3.92(−7)	1.60(−8)	9.53(−8)
	0.5	16	3.84(−6)	1.60(−6)	3.04(−7)	1.98(−6)	2.66(−6)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	2	10	9.10(−6)	8.43(−7)	3.96(−7)	1.67(−8)	9.57(−8)
	0.5	16	2.14(−6)	1.10(−6)	1.20(−7)	1.07(−6)	1.51(−6)
[17] $O(k^2+h^4), \sigma=\frac{1}{4}$,	2	10	3.09(−6)	4.04(−6)	1.65(−6)	2.44(−6)	2.73(−7)
	0.5	16	5.25(−7)	2.87(−7)	1.54(−7)	1.64(−7)	1.76(−7)
$O(k^4+h^4), \sigma=\frac{1}{12}$,	2	10	2.91(−6)	1.73(−6)	1.60(−6)	2.33(−6)	2.60(−7)
	0.5	16	4.47(−7)	2.66(−7)	1.39(−7)	1.55(−7)	1.57(−7)
[2]	2	10	1.87(−6)	2.13(−5)	1.49(−5)	8.60(−6)	5.96(−6)
	0.5	16	9.07(−6)	7.79(−6)	2.75(−6)	1.01(−6)	2.59(−6)
[34]	2	10	1.80(−5)	2.00(−5)	1.40(−5)	8.30(−6)	5.70(−6)
	0.5	16	9.20(−6)	7.90(−6)	2.80(−6)	9.80(−7)	2.50(−6)
[9]	2	10	2.20(−4)	4.10(−4)	5.40(−4)	6.20(−4)	6.50(−4)
	0.5	16	2.50(−5)	4.70(−5)	6.60(−5)	7.80(−5)	8.20(−5)

Example 2: Consider a homogeneous fourth order parabolic partial differential equation [18,33,34]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 \le x \le 1, \; t \ge 0,$$

subject to the initial conditions

$$u(x,0) = \frac{x}{12}(2x^2 - x^3 - 1), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$$

and the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \ge 0.$$

The analytical solution for this example is

$$u(x,t) = \sum_{s=0}^{\infty} d_s \sin(2s+1)\pi x \cos(2s+1)^2 \pi^2 t,$$

where

$$d_s = \frac{-8}{[(2s+1)^5 \pi^5]}.$$

We have solved this example with $h = 0.1$ for $\sigma = \frac{1}{4}, \frac{1}{12}$. The absolute errors at particular points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ with $r = 2$ and 50 time steps of $O(k^4 + h^8), O(k^6 + h^8)$ and $O(k^6 + h^{10})$ for arbitrary choices of parameters p_1, p_2, p_3 and p_4 and comparison with other existing methods are tabulated in table 2. We repeat the computations for 100 time steps with $r = \sqrt{\frac{1}{6}}, \sqrt{\frac{7}{60}}$ and $r = \sqrt{\frac{1}{84}}$. We have also included results given by unconditionally stable method.

Table 2. Absolute errors for example 2

Methods $(p_1, p_2, p_3, p_4, \sigma)$	r^2	Time steps	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
$O(k^6 + h^{10})$							
$\left(\frac{16}{75}, \frac{-42}{25}, \frac{84}{5}, \frac{16}{3}, \frac{1}{12}\right)$	4	50	2.66(−12)	5.48(−12)	1.12(−12)	6.07(−13)	7.42(−13)
	$\frac{1}{6}$	100	1.29(−12)	3.24(−13)	1.79(−12)	3.63(−12)	5.06(−12)
	$\frac{7}{60}$	100	1.79(−12)	4.02(−12)	1.38(−11)	2.40(−11)	3.11(−11)
	$\frac{1}{84}$	100	2.17(−13)	1.33(−13)	7.43(−13)	1.35(−12)	1.50(−12)
$O(k^6 + h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	4	50	1.69(−12)	1.11(−12)	6.43(−13)	9.50(−13)	1.54(−12)
	$\frac{1}{6}$	100	2.44(−12)	3.47(−13)	3.06(−12)	3.98(−12)	3.71(−12)
	$\frac{7}{60}$	100	3.00(−12)	4.44(−12)	1.55(−11)	2.45(−11)	3.24(−11)
	$\frac{1}{84}$	100	2.74(−13)	1.83(−13)	9.23(−13)	1.39(−12)	1.71(−12)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	4	50	1.54(−12)	1.00(−12)	6.33(−13)	9.57(−13)	1.52(−12)
	$\frac{1}{6}$	100	3.06(−12)	6.12(−13)	3.26(−12)	4.06(−12)	3.46(−12)
	$\frac{7}{60}$	100	3.73(−12)	4.14(−12)	1.57(−11)	2.46(−11)	3.21(−11)
	$\frac{1}{84}$	100	3.39(−13)	1.56(−13)	9.46(−13)	1.40(−12)	1.68(−12)
$O(k^4 + h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	4	50	4.77(−13)	1.85(−12)	4.04(−12)	6.80(−12)	9.90(−12)
	$\frac{1}{6}$	100	4.64(−12)	1.43(−12)	3.58(−12)	4.21(−12)	4.01(−12)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	4	50	5.59(−13)	3.20(−13)	5.36(−13)	1.24(−12)	2.71(−12)
	$\frac{1}{6}$	100	8.74(−12)	3.30(−12)	4.79(−12)	4.75(−12)	4.50(−12)
[18], $\sigma = \frac{1}{4}$	4	50	3.19(−4)	6.19(−4)	8.81(−4)	1.87(−3)	1.15(−3)
	$\frac{1}{6}$	100	2.61(−4)	4.43(−4)	5.47(−4)	6.08(−4)	6.33(−4)
[34], $\sigma = \frac{1}{4}$	4	50	3.21(−4)	5.77(−4)	7.24(−4)	7.89(−4)	8.10(−4)
	$\frac{1}{6}$	100	3.81(−4)	3.33(−4)	7.74(−4)	7.81(−4)	7.66(−4)
[33], $\sigma = \frac{1}{12}$	4	50	4.32(−4)	8.34(−4)	1.18(−4)	1.42(−3)	1.52(−3)
	$\frac{1}{6}$	100	2.30(−4)	4.08(−4)	5.40(−4)	6.56(−4)	7.02(−4)
[34], $\sigma = \frac{1}{12}$	4	50	1.00(−5)	5.00(−5)	1.73(−4)	3.33(−4)	4.10(−4)
	$\frac{1}{6}$	100	3.52(−4)	6.30(−4)	7.77(−4)	7.72(−4)	7.38(−4)
	$\frac{7}{60}$	100	1.38(−4)	1.74(−4)	9.05(−5)	3.40(−4)	9.60(−4)
	$\frac{1}{84}$	100	3.53(−5)	6.22(−5)	7.11(−5)	6.11(−5)	5.53(−5)

Example 3: Consider a nonhomogeneous fourth order parabolic partial differential

equation [32]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = [24 - x^2(1-x)^2]\cos t, \quad 0 \le x \le 1, \ t > 0,$$

subject to the initial conditions

$$u(x,0) = x^2(1-x)^2, \quad u_t(x,0) = 0, \quad 0 \le x \le 1$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, u_{xx}(0,t) = u_{xx}(1,t) = 2\cos t, \quad t \ge 0.$$

The analytical solution for this example is

$$u(x,t) = x^2(1-x)^2 \cos t.$$

We have solved this example with $h = 0.05$ for $\sigma = \frac{1}{4}, \frac{1}{12}$. The absolute errors at particular points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ with $r = 2$ and 10 time steps for $O(k^4+h^8), O(k^6+h^8)$ and $O(k^6+h^{10})$ using arbitrary choices of parameters p_1, p_2, p_3 and p_4 are tabulated in table 3. We repeat the computations for 16 time steps with $r = 0.5$.

Table 3. Absolute errors for example 3

Methods (p_1,p_2,p_3,p_4,σ)	r	Time steps	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$O(k^6+h^{10})$							
$\left(\frac{16}{75}, \frac{-42}{25}, \frac{84}{5}, \frac{16}{3}, \frac{1}{12}\right)$	2	10	3.16(−4)	2.74(−5)	4.18(−6)	8.92(−7)	1.17(−8)
	0.5	16	1.35(−5)	6.11(−6)	2.99(−6)	1.15(−5)	1.48(−5)
$O(k^6+h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	2	10	3.97(−4)	2.82(−5)	4.92(−6)	1.20(−6)	1.77(−7)
	0.5	16	1.12(−4)	2.17(−5)	6.58(−5)	9.42(−5)	1.04(−4)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	2	10	3.98(−4)	2.90(−5)	5.00(−6)	1.16(−6)	1.26(−7)
	0.5	16	1.36(−4)	1.28(−5)	7.08(−5)	1.10(−4)	1.24(−4)
$O(k^4+h^8)$							
$\left(\frac{12}{25}, \frac{-82}{25}, \frac{104}{5}, 0, \frac{1}{4}\right)$	2	10	1.05(−3)	7.46(−5)	5.42(−5)	7.48(−6)	4.98(−6)
	0.5	16	1.50(−4)	9.27(−5)	1.83(−5)	1.20(−4)	1.61(−4)
$\left(\frac{43}{100}, \frac{-149}{150}, \frac{401}{20}, 1, \frac{1}{4}\right)$	2	10	1.04(−3)	7.43(−5)	5.43(−5)	7.46(−6)	5.00(−6)
	0.5	16	5.31(−5)	6.32(−5)	6.29(−7)	6.61(−5)	9.31(−5)

Conclusion

The parametric septic spline function have been developed to obtain three level implicit methods for solving fourth order parabolic partial differential equations. The developed methods are tested on three examples. The performance of these methods have been examined by comparing solution of homogeneous and nonhomogeneous fourth order parabolic partial differential equations with available results. In examples 1, 2 and 3, we have computed absolute errors at the points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ for the sake of comparison with our references and results are tabulated in tables 1-3. Tables show that our results are more accurate than the results obtained by previous methods.

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