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On the bivariate and multivariate weighted generalized exponential distributions

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Abstract

This article proposes a particular member of the weighted bivariate distribution, namely, bivariate weighted generalized exponential distribution. This distribution is obtained via conditioning, starting from three independent generalized exponential distributions with different shape but equal scale parameters. Several structural properties of the proposed bivariate weighted generalized exponential distribution including total positivity of order two, marginal moments, reliability parameter and estimation of the model parameters are studied. A multivariate extension of the proposed model is discussed with some properties. Small simulation experiments have been performed to see the behavior of the maximum likelihood estimators, and one data analysis has been presented for illustrative purposes.

Keywords: Weighted distributions, Total positivity of order 2, Reliability parameter, Multivariate weighted distributions.

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1. Introduction

In recent times, there have been numerous studies on the family of weighted distributions that emerges as a center of attraction in the development of application (see Arellano-Valle and Azzalini (2006) and the references therein). The weighted distribution arises when the density $g(x;\theta_1)$ of the potential observation x gets contaminated so that it is multiplied by some non-negative weight function $w(x;\theta_1,\theta_2)$ involving an additional parameter vector θ_2. Then, the observed data is a random realization from a weighted distribution with density

$$f(x;\theta_1,\theta_2)=\frac{w(x;\theta_1,\theta_2)g(x;\theta_1)}{E\left[w(X;\theta_1,\theta_2)\right]}, \tag{1.1}$$

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where the expectation in the denominator is just a normalizing constant. An extensive class of weighted distributions are discussed in Rao (1965,1985), Bayarri and DeGroot (1992), Arnold and Beaver (2002), Branco and Dey (2001), Azzalini (1985) and Kim (2005). As elaborated in the articles by Arnold and Nagaraja (1991) as well as in the book by Genton (2004), the application of the weighted distribution extends to the areas of econometrics, astronomy, engineering, medicine as well as psychology. In particular in scenarios where the observed random phenomena can be described by (1.1). Again, if the potential observation x is obtained only from a selected portion of the population of interest, then (1.1) is called a selection model. Weighted distributions, establishing links with selection models obtained from various forms of selection mechanisms are well addressed in the literature; see Genton (2004), Arellano-Valle et al. (2006) and the references therein. The main objective of this study, described here, is to investigate various properties of a class of weighted distributions arising via conditioning where the underlying distributions are independent generalized exponential. Although the class has some resemblance with the selection distributions developed by Arellano-Valle et al. (2006), we are not aware of any detailed exposition of the distributional properties. This lack of detailed exposition motivates the investigation described in this article. This class apart from a theoretical interest, is worthy of investigation from an applied point of view. In the applied view point, the class produce new models that provide us a means to analyze non-normal data such as interval grouped data, screened data and skewed data. We envision a real life scenario as a genesis of the proposed bivariate weighted distribution in a classical stress-strength model context.
Assume a system has two independent components with strengths W_1 and W_2, and suppose that to run the process each component strength has to overcome an outside stress W_0 which is independent of both (W_1 and W_2). If we define $(X,Y) \stackrel{d}{=} ((W_1,W_2)|(\min(W_1,W_2)) > W_0)$ where the W_i's have absolutely continuous distributions, then the resulting joint distribution of (W_1,W_2) is the type of bivariate weighted distribution to be investigated in this paper.

2. The bivariate weighted generalized exponential distribution

Let W_1, W_2 and W_0 be independent random variables with density functions $f_{W_i}(w_i)$, $i=0,1,2$. Define $(X,Y) \stackrel{d}{=} ((W_1,W_2)|W_0 < \min(W_1,W_2))$, then the density function of the corresponding bivariate weighted distribution is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{f_{W_1}(x)f_{W_2}(y)P(W_0<\min(W_1,W_2)|W_1=x,W_2=y)}{P(W_0<\min(W_1,W_2))} \\ &= \frac{f_{W_1}(x)f_{W_2}(y)F_{W_0}(\min(x,y))}{P(W_0<\min(W_1,W_2))}. \qquad (2.1) \end{aligned}$$

Indeed the density in (2.1) is a bivariate weighted distribution of (X,Y) with the weight $P(W_0<\min(W_1,W_2))$. This method was first proposed by Al-Mutairi at el. (2011).

If W_i's for $i=0,1,2$, are identically distributed with common density function $f_W(w)$, then $P(W_0<\min(W_1,W_2)) = \frac{1}{3}$. Hence, (2.1) reduces to

$$f_{X,Y}(x,y) = 3f_W(x)f_W(y)F_{W_0}(\min(x,y)). \qquad (2.2)$$

Next, we consider a member of the weighted family in (2.1), the bivariate weighted generalized exponential distribution. The exponentiated exponential distribution (Gupta and Kundu, 2001), known in the literature as the generalized exponential distribution (GED), is a two-parameter right skewed unimodal distribution where the behavior of the density and the hazard functions are quite similar to the density and the hazard functions

of the gamma and Weibull distributions. The generalized exponential distribution can also be used effectively to analyze lifetime data.
Next, if W_i's are independent generalized exponential random variables with parameters (α_i, θ) for $i = 0, 1, 2$. Then the normalizing constant is

$$\begin{aligned}
P(W_0 < \min(W_1, W_2)) &= \int_0^\infty \int_0^\infty f_{W_1}(x) f_{W_2}(y) F_{W_0}(\min(x,y)) dx dy \\
&= \int_0^\infty \int_0^\infty \theta^{-2} \alpha_1 e^{-x/\theta} (1 - e^{-x/\theta})^{\alpha_1 - 1} \\
&\quad \times e^{-y/\theta} \alpha_2 (1 - e^{-y/\theta})^{\alpha_2 - 1} (1 - e^{-\min(x,y)/\theta})^{\alpha_0} dx dy \\
&= \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2 + \alpha_0} \left(\frac{1}{\alpha_1 + \alpha_0} + \frac{1}{\alpha_2 + \alpha_0} \right). \qquad (2.3)
\end{aligned}$$

From (2.1), the density function of the proposed bivariate generalized exponential distribution can be written as

$$\begin{aligned}
f_{X,Y}(x,y) &= \theta^{-2} \delta(\alpha_0, \alpha_1, \alpha_2) e^{-(x/\theta + y/\theta)} (1 - e^{-x/\theta})^{\alpha_1 - 1} (1 - e^{-y/\theta})^{\alpha_2 - 1} \\
&\quad \times (1 - e^{-\min(x,y)/\theta})^{\alpha_0} \times I(x > 0, y > 0), \qquad (2.4)
\end{aligned}$$

where $\delta(\alpha_0, \alpha_1, \alpha_2) = \left\{ \frac{1}{\alpha_1 + \alpha_2 + \alpha_0} \left(\frac{1}{\alpha_1 + \alpha_0} + \frac{1}{\alpha_2 + \alpha_0} \right) \right\}^{-1}$, $\alpha_i > 0$ for $i = 0, 1, 2$ and $\theta > 0$.
A bivariate random variable (X, Y) with the joint p.d.f $f(x, y)$ in (2.4) is said to follow the bivariate weighted generalized exponential distribution with parameters α_0, α_1, α_2 and θ and will be denoted by BWGED$(\alpha_0, \alpha_1, \alpha_2, \theta)$. When $\alpha_0 = \alpha_1 = \alpha_2 = 1$, the BWGED reduces to the bivariate weighted exponential distribution (BWED) with parameters $\lambda_0 = \lambda_1 = \lambda_2 = 1/\theta$ [Al-Mutairi et al., 2011]. Also, when $\alpha_0 \longrightarrow 0$ and $\alpha_1 = \alpha_2 = 1$, the BWGED reduces to the bivariate exponential distribution where X and Y are independent and follow $Exp(\theta)$ distribution.
In Figure 1, various density and contour plots of BWGED density are provided. Figure 1 shows that the joint density function is very flexible in terms of shapes, it can assume various shapes such as strictly decreasing and concave down. The shape of the distribution is strictly decreasing whenever $\alpha_i < 1$, $i = 0, 1, 2$. Also, it appears from the plots that the BWGED density is a unimodal distribution.

The remainder of this paper is organized as follows: In section 3, some properties of the bivariate generalized exponential distribution in (2.4) are discussed. In section 4, some discussion on the multivariate extension of the proposed family is provided. Section 5 deals with the estimation of the bivariate generalized exponential distribution parameters. For illustrative purposes, one data set is studied in section 6. In section 7, some concluding remarks are made regarding the BWGED model.

3. Properties of the bivariate generalized exponential distribution

In this section we discuss various structural properties of the BWGED including moment generating functions, marginal distributions and distributions of the minimum and maximum.

3.1. Moment generating function. The moment generating function of BWGED in (2.4) is

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right) = I_1 + I_2, \quad \text{say} \qquad (3.1)$$

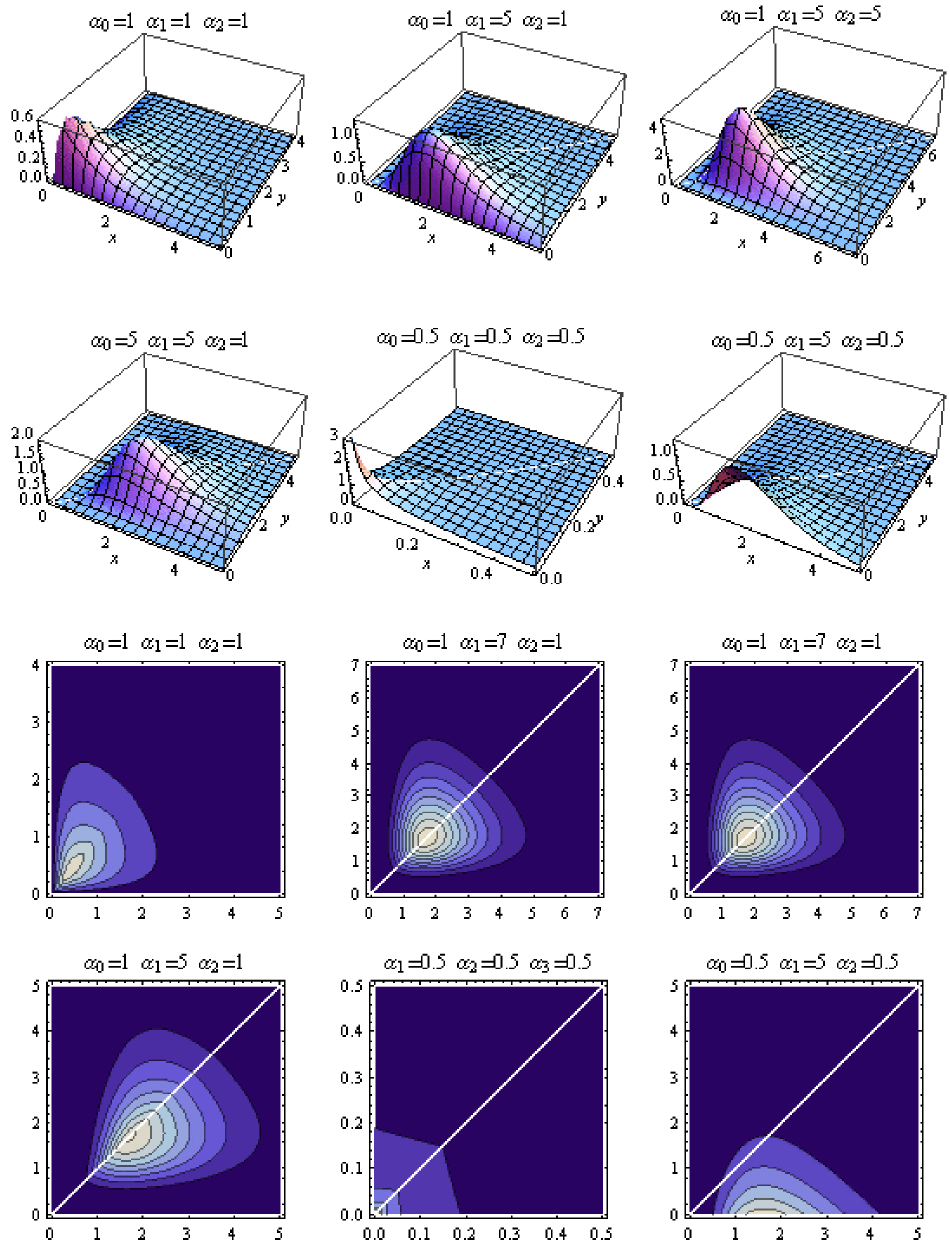

Figure 1. The density and contour plots for various values of α_0, α_1 and α_2.

where

$$I_1 = \int_0^{\infty} \int_0^{x} e^{t_1 x + t_2 y} e^{-(x/\theta + y/\theta)} (1 - e^{-x/\theta})^{\alpha_1 - 1} (1 - e^{-y/\theta})^{\alpha_0 + \alpha_2 - 1} dy dx,$$

and

$$I_2 = \int_0^\infty \int_0^y e^{t_1 x + t_2 y} e^{-(x/\theta + y/\theta)} (1 - e^{-x/\theta})^{\alpha_0+\alpha_1-1} (1 - e^{-y/\theta})^{\alpha_2 - 1} dx dy.$$

For I_1, $\int_0^y e^{t_2\theta} e^{-\lambda(y/\theta)} (1 - e^{-y/\theta})^{\lambda(\alpha_0+\alpha_2-1)} dx = \theta\, B_{1-e^{-x/\theta}}(\alpha_0 + \alpha_2, 1 - \theta\, t_2)$, $|t_2| < \theta^{-1}$ and $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$, is the incomplete beta function. On using the series representation,
$B_x(a,b) = \sum_{k=0}^\infty \frac{(1-b)_k\, x^{a+k}}{k!\,(a+k)}$ where $(a)_k = a(a-1)\cdots(a-k+1)$,
[http://mathworld.wolfram.com/IncompleteBetaFunction.html], one can show

$$I_1 = \theta^2 \sum_{k=0}^\infty \frac{(\theta\, t_2)_k}{k!\,(\alpha_0 + \alpha_2 + k)} B(\alpha_0 + \alpha_1 + \alpha_2 + k, 1 - \theta\, t_1), \quad |t_1|, |t_2| < \theta^{-1}. \tag{3.2}$$

Similarly,

$$I_2 = \theta^2 \sum_{k=0}^\infty \frac{(\theta\, t_1)_k}{k!\,(\alpha_0 + \alpha_1 + k)} B(\alpha_0 + \alpha_1 + \alpha_2 + k, 1 - \theta\, t_2), \quad |t_1|, |t_2| < \theta^{-1}. \tag{3.3}$$

Substituting (3.2) and (3.3) in (3.1), we get an expression for the joint moment generating function of (X, Y).

3.2. Marginal distributions. From (2.4), the marginal density of X is

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{X,Y}(x,y) dy \\ &= \theta^{-1} \delta(\alpha_0, \alpha_1, \alpha_2) e^{-x/\theta} \\ & \quad \left(\left(\frac{1}{\alpha_2 + \alpha_0} - \frac{1}{\alpha_2} \right) (1 - e^{-x/\theta})^{\alpha_1 + \alpha_2 + \alpha_0 - 1} + \frac{1}{\alpha_2} (1 - e^{-x/\theta})^{\alpha_1 + \alpha_0 - 1} \right) \\ & \quad \times I(x > 0). \end{aligned} \tag{3.4}$$

Similarly, the marginal density of Y is

$$\begin{aligned} f_Y(y) &= \theta^{-1} \delta(\alpha_0, \alpha_1, \alpha_2) e^{-y/\theta} \\ & \quad \left(\left(\frac{1}{\alpha_1 + \alpha_0} - \frac{1}{\alpha_1} \right) (1 - e^{-y/\theta})^{\alpha_1 + \alpha_2 + \alpha_0 - 1} + \frac{1}{\alpha_1} (1 - e^{-y/\theta})^{\alpha_2 + \alpha_0 - 1} \right) \\ & \quad \times I(y > 0). \end{aligned} \tag{3.5}$$

Lemma 1. The marginal distributions of X and Y are weighted generalized exponential distributions.

Proof. From (3.4), one can write $f_X(x) = \sum_{i=1}^2 a_i f_{X_i}(x_i)$, where $\sum_{i=1}^2 a_i = 1$, $X_1 \sim GED(\alpha_1 + \alpha_2 + \alpha_0, \theta)$, $X_2 \sim GED(\alpha_1 + \alpha_0, \theta)$, $a_1 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{\alpha_1+\alpha_2+\alpha_0} \left(\frac{1}{\alpha_0+\alpha_2} - \frac{1}{\alpha_2} \right)$ and $a_2 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{(\alpha_0+\alpha_1)\alpha_2}$. Similarly, one can write (3.5) as $f_Y(y) = \sum_{i=1}^2 b_i f_{Y_i}(y_i)$, where $b_1 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{\alpha_1+\alpha_2+\alpha_0} \left(\frac{1}{\alpha_0+\alpha_1} - \frac{1}{\alpha_1} \right)$ and $b_2 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{(\alpha_0+\alpha_1)\alpha_1}$ and $Y_1 \sim GED(\alpha_1 + \alpha_2 + \alpha_0, \theta)$, $Y_2 \sim GED(\alpha_2 + \alpha_0, \theta)$. □

Now, consider the following lemma from Gupta and Kundu (2001).

Lemma 2. If T follows generalized exponential distribution (GED) with parameters (α, λ), then,

(i) $M_T(t) = \alpha B(\alpha, 1 - t/\lambda), \quad |t| < \lambda.$
(ii) $E(T) = (\psi(\alpha+1) - \psi(1))/\lambda$, where $\psi(.)$ is the digamma function.

From Lemma 1, the moment generating function of X and Y, respectively, can be written as

$$M_X(t) = a_1 M_{X_1}(t) + a_2 M_{X_2}(t), \tag{3.6}$$

$$M_Y(t) = b_1 M_{Y_1}(t) + b_2 M_{Y_2}(t), \tag{3.7}$$

where a_1, a_2, b_1 and b_2 are mentioned in the proof of Lemma 1. Here, $X_1, Y_1 \sim GED(\alpha_1 + \alpha_2 + \alpha_0, \theta)$, $X_2 \sim GED(\alpha_1 + \alpha_0, \theta)$ and $Y_2 \sim GED(\alpha_2 + \alpha_0, \theta)$.
Hence, using (3.6), (3.7) and Lemma 2, we get

$$\begin{aligned} M_X(t) &= (\alpha_0 + \alpha_1 + \alpha_2) a_1 B(\alpha_0 + \alpha_1 + \alpha_2, 1 - t/\theta) \\ &\quad + (\alpha_0 + \alpha_1) a_2 B(\alpha_0 + \alpha_1, 1 - t/\theta), |t| < \theta, \end{aligned}$$

$$\begin{aligned} M_Y(t) &= (\alpha_0 + \alpha_1 + \alpha_2) b_1 B(\alpha_0 + \alpha_1 + \alpha_2, 1 - t/\theta) \\ &\quad + (\alpha_0 + \alpha_2) b_2 B(\alpha_0 + \alpha_2, 1 - t/\theta), |t| < \theta, \end{aligned}$$

$$E(X) = a_1\, \theta^{-1}\, \psi(\alpha_0 + \alpha_1 + \alpha_2 + 1) + a_2\, \theta^{-1}\, \psi(\alpha_1 + \alpha_0 + 1) - (a_1 + a_2)\, \theta^{-1}\, \psi(1),$$

and

$$E(Y) = b_1\, \theta^{-1}\, \psi(\alpha_0 + \alpha_1 + \alpha_2 + 1) + b_2\, \theta^{-1}\, \psi(\alpha_2 + \alpha_0 + 1) - (b_1 + b_2)\, \theta^{-1}\, \psi(1).$$

3.3. Distributions of max(X,Y) and min(X,Y). To find the distribution of $Z = \min(X, Y)$, we consider the following: For any $z \in (0, \infty)$

$$\begin{aligned} & P(Z > z) \\ &= \int_z^\infty \int_z^y f(x,y)dxdy + \int_z^\infty \int_z^x f(x,y)dydx \\ &= \frac{\delta(\alpha_0, \alpha_1, \alpha_2)}{(\alpha_1 + \alpha_0)} \left(\frac{1}{\alpha_0 + \alpha_1 + \alpha_2} - \frac{(1 - e^{-z/\theta})^{\alpha_1 + \alpha_0}}{\alpha_2} + \frac{(\alpha_1 + \alpha_0)(1 - e^{-z/\theta})^{\alpha_0 + \alpha_1 + \alpha_2}}{\alpha_2(\alpha_0 + \alpha_1 + \alpha_2)} \right) \\ &+ \frac{\delta(\alpha_0, \alpha_1, \alpha_2)}{(\alpha_2 + \alpha_0)} \left(\frac{1}{\alpha_0 + \alpha_1 + \alpha_2} - \frac{1}{\alpha_1}(1 - e^{-z/\theta})^{\alpha_2 + \alpha_0} + \frac{\alpha_2 + \alpha_0}{\alpha_1(\alpha_0 + \alpha_1 + \alpha_2)}(1 - e^{-z/\theta})^{\alpha_0 + \alpha_1 + \alpha_2} \right). \end{aligned} \tag{3.8}$$

On differentiation (3.8), we get

$$\begin{aligned} f(z) &= \theta^{-1} \delta(\alpha_0, \alpha_1, \alpha_2) e^{-z/\theta} \\ &\times \left(\frac{1}{\alpha_1}(1 - e^{-z/\theta})^{\alpha_2 + \alpha_0 - 1} + \frac{1}{\alpha_2}(1 - e^{-z/\theta})^{\alpha_1 + \alpha_0 - 1} \right. \\ &\left. - \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) (1 - e^{-z/\theta})^{\alpha_0 + \alpha_1 + \alpha_2 - 1} \right) \times I(z > 0). \end{aligned} \tag{3.9}$$

Lemma 3. The distribution of $\min(X, Y)$ is a weighted generalized exponential distribution.

Proof. From (3.9), $f_Z(z) = \sum_{i=1}^{3} c_i f_{Z_i}(z_i)$, where $\sum_{i=1}^{3} c_i = 1$, $Z_1 \sim GED(\alpha_2+\alpha_0,\theta)$, $Z_2 \sim GED(\alpha_1+\alpha_0,\theta)$, $Z_3 \sim GED(\alpha_0+\alpha_1+\alpha_2,\theta)$ and $c_1 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{\alpha_1(\alpha_2+\alpha_0)}$, $c_2 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{(\alpha_0+\alpha_1)\alpha_2}$ and $c_3 = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{\alpha_2+\alpha_1+\alpha_0}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)$. □

For the distribution of $W = \max(X,Y)$, note that for any $w \in (0,\infty)$,

$$
\begin{aligned}
\overline{F}_W(w) &= P(W > w) \\
&= P(X > w \quad or \quad Y > w) \\
&= P(X > w) + P(Y > w) - P(X > w \quad and \quad Y > w) \\
&= P(X > w) + P(Y > w) - P(Z > w) \\
&= \overline{F}_X(w) + \overline{F}_Y(w) - \overline{F}_Z(w). \qquad (3.10)
\end{aligned}
$$

Differentiating (3.10) with respect to w and using (3.4), (3.5) and (3.9) we get:

$$
\begin{aligned}
f_W(w) &= f_X(w) + f_Y(w) - f_Z(w) \\
&= \theta^{-1}(\alpha_0+\alpha_1+\alpha_2)e^{-w/\theta}(1-e^{-w/\theta})^{\alpha_0+\alpha_1+\alpha_2-1} \times I(w>0). \qquad (3.11)
\end{aligned}
$$

From (3.11), $W = \max(X,Y)$, follows the generalized exponential distribution with parameters $\alpha_0+\alpha_1+\alpha_2$ and θ. Using equations (3.9), (3.11) and Lemma 2, the moment generating functions and the means of Z and W are:

(i) $M_Z(t) = c_1(\alpha_2+\alpha_0)B(\alpha_2+\alpha_0, 1-t/\theta) + c_2(\alpha_1+\alpha_0)B(\alpha_1+\alpha_0, 1-t/\theta) + c_3(\alpha_0+\alpha_1+\alpha_2)B(\alpha_0+\alpha_1+\alpha_2, 1-t/\theta)$, $|t| < \theta$.
$E(Z) = c_1\theta^{-1}\psi(\alpha_2+\alpha_0+1) + c_2\theta^{-1}\psi(\alpha_2+\alpha_0+1) + c_3\theta^{-1}\psi(\alpha_2+\alpha_1+\alpha_0+1) - (c_1+c_2+c_3)\theta^{-1}\psi(1)$.

(ii) $M_W(t) = (\alpha_2+\alpha_1+\alpha_0)B(\alpha_2+\alpha_1+\alpha_0, 1-t/\theta)$, $|t| < \theta$.
$E(W) = \theta^{-1}\left(\psi(\alpha_2+\alpha_1+\alpha_0+1) - \psi(1)\right)$.

3.4. Renyi Entropy. Shannon's (1948), pioneering work, entropy has been used as a major tool in information theory and in almost every branch of science and engineering. One of the main extensions of Shannon entropy was defined by Renyi (1961). This generalized entropy measure is given by

$$
I_R(\lambda) = \frac{\log(G(\lambda))}{1-\lambda}, \quad \lambda > 0, \lambda \neq 1. \qquad (3.12)
$$

Where $G(\lambda) = \int_{\mathbb{X}} f^{\lambda} d\mu$, and μ is a σ-finite measure on $\mathbb{X}$. One can get an expression for the Shannon entropy from (3.12) by taking limit for $\lambda \to 1$.

3.1. Theorem. *The Renyi entropy for the bivariate generalized exponential distribution in (2.4) is* $I_R(\lambda) = (1-\lambda)^{-1}\log(G(\lambda))$, *where*

(3.13)

$$
G(\lambda) = \theta^{2-2\lambda}\delta^{\lambda}(\alpha_0,\alpha_1,\alpha_2)\sum_{k=0}^{\infty}\left(T_{\alpha_0+\alpha_1}^{\lambda,k} + T_{\alpha_0+\alpha_2}^{\lambda,k}\right)B(\lambda, \lambda(\alpha_2+\alpha_1+\alpha_0-2)+k+2),
$$

and $T_x^{\lambda,k} = \frac{(1-\lambda)_k}{k!\,[\lambda(x-1)+k+1]}$.

Proof. From (2.4), we can write

$$G(\lambda) = \theta^{-2\lambda}\delta^{\lambda}(\alpha_0,\alpha_1,\alpha_2) \times (I_1 + I_2), \tag{3.14}$$

where $I_1 = \int_0^\infty \int_0^y e^{-\lambda(x/\theta+y/\theta)}(1-e^{-x/\theta})^{\lambda(\alpha_0+\alpha_1-1)}(1-e^{-y/\theta})^{\lambda(\alpha_2-1)}dxdy$ and $I_2 = \int_0^\infty \int_0^x e^{-\lambda(x/\theta+y/\theta)}(1-e^{-x/\theta})^{\lambda(\alpha_1-1)}(1-e^{-y/\theta})^{\lambda(\alpha_0+\alpha_2-1)}dydx$.
The result in (3.13) follows from (3.14) by using similar approach as in equations (3.2) and (3.3). □

3.5. Stochastic properties. Let t_{11}, t_{12}, t_{21} and t_{22} be real numbers with $0 < t_{11} < t_{12}$ and $0 < t_{21} < t_{22}$. Then (X,Y) has the total positivity of order two (TP_2) property iff

$$f_{X,Y}(t_{11},t_{21})f_{X,Y}(t_{12},t_{22}) - f_{X,Y}(t_{12},t_{21})f_{X,Y}(t_{11},t_{22}) \geq 0. \tag{3.15}$$

3.2. Theorem. *The bivariate generalized exponential distribution in (2.4) has the TP_2 property.*

Proof. Let us consider different cases separately. If $0 < t_{11} < t_{21} < t_{12} < t_{22}$, then for the density function in (2.4), one can easily show that the condition in (3.15) is equivalent to $e^{-t_{21}/\theta} - e^{-t_{12}/\theta} \geq 0$. This inequality holds because $t_{21} < t_{12}$. The other cases can be shown similarly. □

The reliability parameter R is defined as $R = P(X > Y)$, where X and Y are independent random variables. Numerous applications of the reliability parameter have appeared in the literature such as the area of classical stress-strength model and the break down of a system having two components. Other applications of the reliability parameter can be found in Hall (1984) and Weerahandi and Johnson (1992).

3.3. Theorem. *The reliability parameter of the bivariate weighted generalized exponential distribution is*

$$R = \frac{\delta(\alpha_0,\alpha_1,\alpha_2)}{\alpha_1\alpha_2}\left\{\frac{\alpha_1}{\alpha_1+\alpha_2} - \frac{\alpha_0}{\alpha_0+\alpha_2} + \frac{\alpha_0\alpha_2}{\alpha_0+\alpha_1+\alpha_2}\right\}.$$

Proof. Note that $(X,Y) \overset{d}{=} [(W_1,W_2)|W_0 < \min(W_1,W_2)]$ where the W_i's are independent and $W_i \sim \mathrm{GED}(\alpha_i,\theta)$ for $i = 0,1,2$. Thus,

$$\begin{aligned} P(X > Y) &= P(W_1 > W_2 | W_0 < \min(W_1,W_2)) \\ &= \frac{P(W_0 < W_2 < W_1)}{P(W_0 < \min(W_1,W_2))} \end{aligned} \tag{3.16}$$

By using straightforward integration one can easily show that

$$P(W_0 < W_2 < W_1) = \frac{\alpha_1}{\alpha_1+\alpha_2} - \frac{\alpha_0}{\alpha_0+\alpha_2} + \frac{\alpha_0\alpha_2}{\alpha_0+\alpha_1+\alpha_2}. \tag{3.17}$$

Substituting (2.3) and (3.17) in (3.16), the result follows immediately. □

4. Multivariate weighted generalized exponential distribution

One can obtain a multivariate version of (2.1) by assuming $W_i \sim f_{W_i}(w_i)$ for $i = 0, 1, \cdots, k$ are independent random variables. The resulting multivariate weighted density function is given by

$$f_{X_1,X_2,\ldots X_k}(x_1, x_2, \cdots, x_k) = \frac{\left[\prod_{i=1}^{k} f_{W_i}(x_i)\right] F_{W_0}(\min(x_1, x_2, \ldots x_k))}{P(W_0 < \min(W_1, W_2, \ldots, W_k))}. \tag{4.1}$$

From (4.1), a multivariate extension of the bivariate weighted generalized exponential model in (2.4) is given by

(4.2)

$$f(x_1, x_2, \ldots, x_k) \propto \left(\prod_{i=1}^{k} \frac{\alpha_i}{\theta}\right) e^{(-\sum_{i=1}^{k} \frac{x_i}{\theta})} \left(\prod_{i=1}^{k} (1 - e^{-\frac{x_i}{\theta}})^{\alpha_i - 1}\right) \left(1 - e^{-\frac{x_{1:k}}{\theta}}\right)^{\alpha_i - 1} \times I(\underline{x} > \underline{0}),$$

where $x_{1:k} = \min\{(x_1, x_2, \ldots, x_k\}$.
As a motivation, one can consider the following scenario: suppose that a system consists of k components whose random strengths are denoted by $W_1, W_2, \ldots W_k$ and the random stress is given by W_0. Next, if the system has a series structure then one would be interested to know the distribution of $W_1, W_2, \ldots W_k | W_0 < \min(W_1, W_2, \ldots, W_k)$. In fact the system reliability in that case would be given by $R = P(W_0 < \min(W_1, W_2, \ldots, W_k))$.
Next, consider the model in which $Y_1, Y_2, \ldots, Y_j$ are i.i.d. random variables with distribution and density functions G_0 and g_0; $X_1, X_2, \ldots, X_k$ are i.i.d. random variables with distribution and density functions F_0 and f_0 and $Z_1, Z_2, \ldots, Z_\ell$ are i.i.d. random variables with distribution and density functions H_0 and h_0. In this case we have

$$f(x_1, x_2, \ldots, x_k) \propto \left[\prod_{i=1}^{k} f_0(x_i)\right] [G_0(x_{1:k})]^j [1 - H_0(x_{k:k})]^\ell, \tag{4.3}$$

where $x_{k:k} = \max\{(x_1, x_2, \ldots, x_k\}$.
In some specific scenarios it will be possible to evaluate the normalizing constant in (4.3). For example, when the three distributions are generalized exponential, (4.3) reduces to

$$\begin{aligned} f(x_1, x_2, \ldots, x_k) &\propto \left[\alpha_1^k e^{-\theta^{-1}(x_1 + x_2 + \cdots + x_k)}\right] \prod_{i=1}^{k} \left(1 - e^{-x_i/\theta}\right)^{\alpha_1} \\ &\times \left[1 - \left(1 - e^{-x_{k:k}/\theta}\right)^{\alpha_0}\right]^\ell \left(1 - e^{-x_{k:k}/\theta}\right)^{j\,\alpha_2}. \end{aligned} \tag{4.4}$$

To identify the required normalizing constant we must evaluate

$$\begin{aligned} &\int_0^\infty \int_0^\infty \ldots \int_0^\infty \alpha_1^k e^{-\theta^{-1}(x_1 + x_2 + \cdots + x_k)} \prod_{i=1}^{k} \left(1 - e^{-x_i/\theta}\right)^{\alpha_1} \\ &\times \left[1 - \left(1 - e^{-x_{k:k}/\theta}\right)^{\alpha_0}\right]^\ell \left(\left(1 - e^{-x_{k:k}/\theta}\right)^{\alpha_2}\right)^j dx_1 dx_2 .. dx_k \\ &= \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \binom{\ell}{k_1} \binom{\alpha_0 k_1}{k_2} \binom{\alpha_2 j}{k_3} (-1)^{k_1 + k_2 + k_3} E\left(e^{-k_1 X_{1:k}/\theta - k_2 X_{k:k}/\theta}\right), \end{aligned} \tag{4.5}$$

where the X_i's have the generalized exponential (α_1, θ) distribution. So we need the joint moment generating function of $(X_{1:k}, X_{k:k})$. Next, the joint distribution of $(X_{1:k}, X_{k:k})$

is

$$f(x_{1:k}, x_{k:k})$$
$$= \frac{k^3(k-1)\alpha_1^2}{\theta^2} e^{-x_{1:k}/\theta - x_{k:k}/\theta} \left(1 - e^{-x_{k:k}/\theta}\right)^{k\alpha_1 - 1} \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1 - 1}$$
$$\left(1 - \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1}\right)^{k-1} \left(\left(1 - e^{-x_{k:k}/\theta}\right)^{\alpha_1} - \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1}\right)^{k-2}$$
$$\times I(0 < x_{1:k} < x_{k:k} < \infty)$$

Now,

$$E\left(e^{-X_{1:k}/\theta - X_{k:k}/\theta}\right)$$
$$= \int_0^\infty \int_0^{x_{k:k}} \frac{k^3(k-1)\alpha_1^2}{\theta^2} e^{-2x_{1:k}/\theta - 2x_{k:k}/\theta} \left(1 - e^{-x_{k:k}/\theta}\right)^{k\alpha_1 - 1} \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1 - 1}$$
$$\times \left(1 - \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1}\right)^{k-1} \left(\left(1 - e^{-x_{k:k}/\theta}\right)^{\alpha_1} - \left(1 - e^{-x_{1:k}/\theta}\right)^{\alpha_1}\right)^{k-2} dx_{1:k} dx_{k:k}, \tag{4.6}$$

which can be written as $\sum_{j=0}^\infty (-1)^j \frac{\theta}{2+j} \left(\binom{(2k+1)\alpha_1}{j} + \binom{(2k+1)\alpha_1 - 1}{j}\right)$, after some algebraic simplification. Hence, using (4.6) in (4.5), the normalizing constant corresponds to the distribution in (4.4) is

$$C = \theta \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k_1+k_2+k_3+j}}{2+j} \binom{\ell}{k_1} \binom{\alpha_0 k_1}{k_2} \binom{\alpha_2 j}{k_3}$$
$$\left(\binom{(2k+1)\alpha_1}{j} + \binom{(2k+1)\alpha_1 - 1}{j}\right),$$

Corollary 1. If $(X_1, X_2, ..., X_k)$ has a multivariate weighted generalized exponential distribution in (4.1) with parameters $(\alpha_i, \theta), i = 0, 1, 2, \cdots k$, then the normalizing constant, $C_1 = P(X_1 < \min(X_2, \cdots, X_k)) = \frac{\sum_{i=0}^k \alpha_i}{\prod_{i=1}^k \alpha_i} \left(\sum_{i=1}^k \frac{1}{\alpha_i + \alpha_0}\right)^{-1}$.

Proof. The result follows immediately by using the same logic as in (2.3). □

Corollary 2. If $(X_1, X_2, ..., X_k)$ has a multivariate weighted generalized exponential distribution with parameters $(\alpha_i, \theta), i = 0, 1, 2, \cdots k$, then the distribution of $Z = \min(X_1, X_2, \cdots, X_k)$ has the density

$$f(z) = \theta^{-1} C_1^{-1} e^{-z/\theta} \left(\sum_{i=1}^k \frac{1}{\alpha_i} (1 - e^{-z/\theta})^{\alpha_i + \alpha_0 - 1} - \frac{1}{\sum_{i=1}^k \alpha_i} (1 - e^{-z/\theta})^{\sum_{i=0}^k \alpha_i - 1}\right)$$
$$\times I(z > 0),$$

where C_1 is the constant in Corollary 1.

5. Estimation

In this section, we consider the maximum likelihood method to estimate the model parameters of the bivariate generalized exponential distribution in (2.4).

5.1. Maximum likelihood estimation. Assume that a random sample of size n observations are taken from the bivariate density in (2.4), then the corresponding log-likelihood function can be written as

$$\begin{aligned}\ell(\alpha_0,\alpha_1,\alpha_2,\theta) \quad = \quad & -2n\log\theta + n\log(\alpha_0+\alpha_1+\alpha_2) + n\log(\alpha_0+\alpha_1) + n\log(\alpha_0+\alpha_2)\\ & -n\log(2\alpha_0+\alpha_1+\alpha_2) - n\theta^{-1}\left(\bar{x}+\bar{y}\right) + (\alpha_1-1)\sum_{i=1}^{n}\log\left(1-e^{-x_i/\theta}\right)\\ & +(\alpha_2-1)\sum_{i=1}^{n}\log\left(1-e^{-y_i/\theta}\right) + \alpha_0\sum_{i=1}^{n}\log\left(1-e^{-\min(x_i,y_i)/\theta}\right). \end{aligned} \tag{5.1}$$

Differentiating (5.1) with respect to α_0, α_1, α_2, and θ we get

$$\begin{aligned}\frac{\partial}{\partial\alpha_0}\ell(\alpha_0,\alpha_1,\alpha_2,\theta) = & \frac{n}{\alpha_0+\alpha_1+\alpha_2} - \frac{2n}{2\alpha_0+\alpha_1+\alpha_2} + \frac{n}{\alpha_0+\alpha_1} + \frac{n}{\alpha_0+\alpha_2}\\ & +\sum_{i=1}^{n}\log\left(1-e^{-\min(x_i,y_i)/\theta}\right). \end{aligned} \tag{5.2}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha_1}\ell(\alpha_0,\alpha_1,\alpha_2,\theta) = & \frac{n}{\alpha_0+\alpha_1+\alpha_2} - \frac{n}{2\alpha_0+\alpha_1+\alpha_2} + \frac{n}{\alpha_0+\alpha_1}\\ & +\sum_{i=1}^{n}\log\left(1-e^{-x_i/\theta}\right). \end{aligned} \tag{5.3}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha_2}\ell(\alpha_0,\alpha_1,\alpha_2,\theta) = & \frac{n}{\alpha_0+\alpha_1+\alpha_2} - \frac{n}{2\alpha_0+\alpha_1+\alpha_2} + \frac{n}{\alpha_0+\alpha_2}\\ & +\sum_{i=1}^{n}\log\left(1-e^{-y_i/\theta}\right). \end{aligned} \tag{5.4}$$

$$\begin{aligned}&\frac{\partial}{\partial\theta}\ell(\alpha_0,\alpha_1,\alpha_2,\theta)\\ &= -2n\theta^{-1} + \theta^{-2}n(\bar{x}+\bar{y}) - (\alpha_1-1)\theta^{-2}\sum_{i=1}^{n}x_i\left(e^{x_i/\theta}-1\right)^{-1}\\ &\quad -(\alpha_2-1)\theta^{-2}\sum_{i=1}^{n}y_i\left(e^{y_i/\theta}-1\right)^{-1} - \alpha_0\theta^{-2}\sum_{i=1}^{n}\min(x_i,y_i)\left(e^{\min(x_i,y_i/\theta)}-1\right)^{-1}. \end{aligned} \tag{5.5}$$

Setting (5.2), (5.3), (5.4) and (5.5) to 0 and solving simultaneously, we get the maximum likelihood estimates for α_0, α_1, α_2 and θ.

If the scale parameter θ is assumed to be known, then setting equations (5.2), (5.3) and (5.4) equal to zero, we get,

$$\frac{1}{\alpha} - \frac{2}{\alpha_0+\alpha} + \frac{1}{\alpha_0+\alpha_1} + \frac{1}{\alpha_0+\alpha_2} = C. \tag{5.6}$$

$$\frac{1}{\alpha} - \frac{1}{\alpha_0+\alpha} + \frac{1}{\alpha_0+\alpha_1} = A. \tag{5.7}$$

$$\frac{1}{\alpha} - \frac{1}{\alpha_0+\alpha} + \frac{1}{\alpha_0+\alpha_2} = B. \tag{5.8}$$

where $A = -\sum_{i=1}^{n}\log\left(1-e^{-X_i/\theta}\right)$, $B = -\sum_{i=1}^{n}\log\left(1-e^{-Y_i/\theta}\right)$,
$C = -\sum_{i=1}^{n}\log\left(1-e^{-\min(X_i,Y_i)/\theta}\right)$ and $\alpha = \alpha_0+\alpha_1+\alpha_2$.

Adding (5.7) and (5.8) and then subtracting from (5.6), we get

$$\alpha = \frac{1}{A + B - C}. \tag{5.9}$$

On using (5.7) and (5.8) and then simplifying, we get

$$\alpha_2 = \alpha - \left(A - B + \frac{1}{\alpha - \alpha_1} \right)^{-1}. \tag{5.10}$$

Therefore, using equations (5.9), (5.10) and the fact that $\alpha_0 = \alpha - \alpha_1 - \alpha_2$, one can easily solve equation (5.6) for α_1. This will increase the calculation efficiency in order to obtain the numerical solution faster. The Fisher information matrix when θ is known, $I(\underline{\delta}) = -E\left(\frac{\partial^2}{\partial \delta_i \partial \delta_j} \log\left(f(X|\underline{\delta})\right)\right) = \{U_{rs}; r, s = \alpha_0, \alpha_1, \alpha_2\}$, can be obtained from equations (30)-(32) as follows:

$$U_{\alpha_0\alpha_0} = n\left(\alpha^{-2} - 4(\alpha_0 + \alpha)^{-2} + (\alpha_0 + \alpha_1)^{-2} + (\alpha_0 + \alpha_2)^{-2}\right).$$

$$U_{\alpha_0\alpha_1} = n\left(\alpha^{-2} - 2(\alpha_0 + \alpha)^{-2} + (\alpha_0 + \alpha_1)^{-2}\right).$$

$$U_{\alpha_0\alpha_2} = n\left(\alpha^{-2} - 2(\alpha_0 + \alpha)^{-2} + (\alpha_0 + \alpha_2)^{-2}\right).$$

$$U_{\alpha_1\alpha_1} = n\left(\alpha^{-2} - (\alpha_0 + \alpha)^{-2} + (\alpha_0 + \alpha_1)^{-2}\right).$$

$$U_{\alpha_1\alpha_2} = n\left(\alpha^{-2} - (\alpha_0 + \alpha)^{-2}\right),$$

$$U_{\alpha_2\alpha_2} = n\left(\alpha^{-2} - (\alpha_0 + \alpha)^{-2} + (\alpha_0 + \alpha_2)^{-2}\right).$$

The Fisher information matrix can be used to obtain interval estimation of the model parameters. Under standard regularity conditions, the multivariate normal $N_3(0, I(\widehat{\underline{\delta}})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. The matrix, $I(\widehat{\underline{\delta}})$ is the Fisher information matrix evaluated at $\widehat{\underline{\delta}}$. Therefore, the $100(1-a)\%$ confidence intervals for α_0, α_1 and α_2 are given by $\hat{\alpha_0} \pm z_{a/2} \times \sqrt{var(\hat{\alpha_0})}$, $\hat{\alpha_1} \pm z_{a/2} \times \sqrt{var(\hat{\alpha_1})}$, and $\hat{\alpha_2} \pm z_{a/2} \times \sqrt{var(\hat{\alpha_2})}$, respectively, where

$$Var(\widehat{\alpha}_0) = \frac{10\alpha_0^4 + 22\alpha_0^3(\alpha_1+\alpha_2) + (\alpha_1+\alpha_2)^4 + \alpha_0^2\left(20\alpha_1^2 + 34\alpha_1\alpha_2 + 20\alpha_2^2\right) + 4\alpha_0\left(2\alpha_1^3 + 5\alpha_1^2\alpha_2 + 5\alpha_1\alpha_2^2 + 2\alpha_2^3\right)}{2n(\alpha_0+\alpha_1)(\alpha_0+\alpha_2)},$$

$$Var(\widehat{\alpha}_1) = \frac{5\alpha_0^3 + 8\alpha_0^2\alpha_1 + 11\alpha_0^2\alpha_2 + 6\alpha_0\alpha_1^2 + 12\alpha_0\alpha_1\alpha_2 + 7\alpha_0\alpha_2^2 + 2\alpha_1^3 + 4\alpha_1^2\alpha_2 + 4\alpha_1\alpha_2^2 + \alpha_2^3}{2n(\alpha_0+\alpha_1)},$$

$$Var(\widehat{\alpha}_2) = \frac{5\alpha_0^3 + 11\alpha_0^2\alpha_1 + 8\alpha_0^2\alpha_2 + 7\alpha_0\alpha_1^2 + 12\alpha_0\alpha_1\alpha_2 + 6\alpha_0\alpha_2^2 + \alpha_1^3 + 4\alpha_1^2\alpha_2 + 4\alpha_1\alpha_2^2 + 2\alpha_2^3}{2n(\alpha_0+\alpha_2)}.$$

5.2. Simulation study. To illustrate the application of the bivariate generalized exponential distribution in (2.4), a small simulation study is conducted. However, in this paper we report only the results for estimation of the model parameters using the maximum likelihood estimation procedure. Bivariate random samples of size 50, 100 and 200 were generated from the density in (2.4) with the following parameter values; Set I: $\alpha_0 = 1$, $\alpha_1 = 5$, $\alpha_2 = 5$ and $\theta = 1$ and Set II: $\alpha_0 = 1$, $\alpha_1 = 4$, $\alpha_2 = 3$ and $\theta = 3$. Since both the conditional distributions of the bivariate density in (2.4), $X|Y$ and $Y|X$, are completely known in closed forms, a Gibbs sampling technique is used to generate bivariate random samples. The simulation is repeated 200 times. The estimated value and the standard deviation of the parameters using the maximum likelihood method are presented in Tables 1 and 2.

Table 1. Parameter estimates and standard deviations for BWGED under set I.

Sample size	$\hat{\alpha_0}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\theta}$
50	1.2372 (0.3220)	5.0955 (0.8543)	5.0902 (0.7065)	0.9903 (0.0379)
100	1.1616 (0.1662)	4.9123 (0.5579)	5.1261 (0.6607)	0.9767 (0.0217)
200	1.1510 (0.1190)	5.0190 (0.4097)	4.9918 (0.2449)	0.9956 (0.0207)

Table 2. Parameter estimates and standard deviations for the BWGED under set II.

Sample size	$\hat{\alpha_0}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\theta}$
50	1.1526 (0.4653)	4.1931 (0.5748)	3.5209 (0.7732)	2.8400 (0.2598)
100	1.2204 (0.2971)	3.9411 (0.4501)	3.5043 (0.6746)	2.9127 (0.2081)
200	1.1374 (0.1573)	4.1051 (0.2225)	3.2157 (0.4294)	2.9677 (0.1140)

From Tables 1 and 2, it appears that the maximum likelihood estimation performs quite effectively to estimate the model parameters.

6. Application

In this section, the BWGED is applied to a data set from Al-Mutairi at el. (2011). The data set represents the scores from twenty five first year graduate students in probability and inference classes of a premier Institute in India. For both the courses, Analysis-I is a prerequisite. It is assumed that the knowledge of Analysis-I affects the scores in both the courses. The data set is

$$X : 53, 55, 85, 87, 22, 23, 25, 93, 51, 62, 53, 32, 43, 47, 30, 88, 59, 49, 42, 71, 41, 82, 75, 93, 37.$$

$$Y : 89, 90, 59, 50, 25, 29, 54, 62, 39, 25, 89, 32, 33, 63, 38, 77, 55, 41, 31, 66, 57, 32, 43, 88, 34.$$

We fit the data set to the BWGED and compared the result with the bivariate weighted exponential distribution (Al- Murairi et al., 2011). The maximum likelihood estimates for both models are reported in Table 3. The Kolmogorov-Smirnov test statistic (K-S) for the distribution functions of the marginal X and Y is used to compare the goodness of fit of the BWGED and the bivariate weighted exponential distribution (BWED). The K-S statistics and the p-value for the K-S statistics for the fitted marginal distributions are reported in Tables 3. From Table 3, the p-values indicate that the marginals of the BWGED gives an adequate fit to the data. Figure (2) displays the empirical and the fitted cumulative distribution functions. This figure supports the results in Table 3.

Table 3. Parameter estimates for the scores data

Distribution	BWED	BWGED
Parameter Estimates	$\hat{\lambda_1} = 0.0263$ $\hat{\lambda_2} = 0.0293$ $\hat{\lambda_3} = 0.0005$	$\hat{\theta} = 20.9321$ $\hat{\alpha_0} = 10.7633$ $\hat{\alpha_1} = 0.9752$ $\hat{\alpha_2} = 1 \times 10^{-6}$
K-S for X	0.3290	0.0790
K-S p-value for X	0.2080	0.9977
K-S for Y	0.2250	0.1300
K-S p-value for Y	0.2860	0.7924

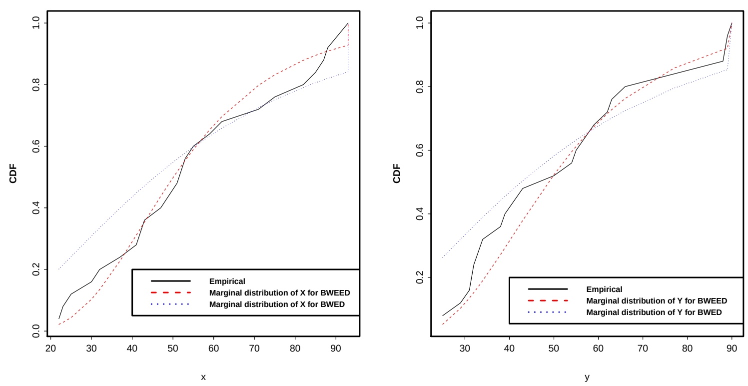

Figure 2. Marginal CDFs for fitted distributions of the scores data

7. Concluding remarks

In this paper, we consider a method for generating bivariate and multivariate generalized exponential distributions. Some structural properties of the bivariate exponentiated-exponential distribution in (2.4) are studied such as marginal distributions, moments, total positivity and parameter estimation. A small simulation study is conducted and the outcome of the simulation study is quite encouraging. Furthermore, one can study general properties for the multivariate generalized exponential distribution in (4.4). Although, in this paper, we focus on the bivariate and multivariate generalized exponential distributions, one can use the techniques in (2.1) and (4.3) to generate different bivariate and multivariate distributions. The analytical tractability of such resulting models is to be investigated before one can explore other properties of the derived model(s).

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