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On some questions regarding projectivity criteria

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Abstract

We investigate questions which are related to projectivity criteria and give some partial answers and related results to them.

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1. Introduction

The purpose of this paper is to investigate questions related to projectivity criteria. It is well-known that if G is a finite group, then a $\mathbb{Z}G$-module M is projective if and only if M is $\mathbb{Z}$-free and $\text{proj.dim}_{\mathbb{Z}G} M < \infty$ (cf. [5]). In [16] we investigated whether or not only finite groups satisfy the criterion above, and showed that this is true in the class of groups $\mathbf{LH}\mathfrak{F}$. For the definitions of $\mathbf{LH}\mathfrak{F}$ and some other terminologies below in this section, see Section 2.

Note that if G is a virtually torsion-free group with $\text{vcd}\, G = n$ and M is a $\mathbb{Z}$-free $\mathbb{Z}G$-module, then $\text{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $\text{proj.dim}_{\mathbb{Z}G} M \le n$ ([5, Theorem X.5.3]). This result was generalized in [15, Theorem 4.7] as follows: if G is a $\mathbf{H}\mathfrak{F}$-group and $\text{spli}\, G < \infty$, then $\text{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $\text{proj.dim}_{\mathbb{Z}G} M \le \text{pccd}\, G$.

It is also known that if H is a subgroup of finite index in G, then a $\mathbb{Z}G$-module M is $\mathbb{Z}G$-projective if and only if M is $\mathbb{Z}H$-projective and $\text{proj.dim}_{\mathbb{Z}G} M < \infty$ ([6, Lemma 4.1 (a)]).

In these viewpoints, we may ask the following questions:

1.1. Question. Let n be a nonnegative integer. Suppose that G satisfies the following property: for any $\mathbb{Z}$-free $\mathbb{Z}G$-module M, $\text{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $\text{proj.dim}_{\mathbb{Z}G} M \le n$. Is it true that $\text{pccd}\, G \le n$?

1.2. Question. Let H be a subgroup of G. Suppose that (G, H) satisfies the property that for any $\mathbb{Z}G$-module M, M is $\mathbb{Z}G$-projective if and only if M is $\mathbb{Z}H$-projective and $\text{proj.dim}_{\mathbb{Z}G} M < \infty$. Is it true that $|G : H| < \infty$?

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It is known in [16, Corollary 2.7] that Question 1.1 has a positive answer for any $\mathbf{LH}\mathfrak{F}$-group and $n = 0$.

On the other hand, recall the following conjecture (a special case of Moore's conjecture ([1, p 64]), which is a far reaching generalization of Serre's theorem [1, p 65]) on cohomological dimension of groups.

1.3. Conjecture. *Let G be a torsion-free group and H a subgroup of finite index in G. Then every $\mathbb{Z}G$-module M which is $\mathbb{Z}H$-projective is also $\mathbb{Z}G$-projective.*

In the same sprit as the questions above, we also naturally ask the following:

1.4. Question. Let G be a torsion-free group and H a subgroup of G. Suppose that (G, H) satisfies the property that every $\mathbb{Z}H$-projective $\mathbb{Z}G$-module is $\mathbb{Z}G$-projective. Is it true that $|G : H| < \infty$?

It can be seen that for a torsion-free group G and its subgroup H, if Question 1.4 has an affirmative answer for (G, H), then Question 1.2 has also an affirmative answer for (G, H).

We give some partial answers and related results to Questions 1.1, 1.2, and 1.4 in Theorems 3.7, 3.8, 3.9, and 3.10 and Corollaries 3.5 and 3.6.

2. Preliminaries

In this section, we briefly introduce some definitions and preliminary results. For more details, we recommend each reference below.

1. ([18, 3]) The class $\mathbf{H}\mathfrak{F}$ is the smallest class of groups containing the class of finite groups and which contains a group G whenever G admits a finite dimensional contractible G-C-complex whose stabilizers are already in $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ is the class of groups such that all of its finitely generated subgroups are in $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ is extension closed, closed under ascending unions, and closed under amalgamated free products and HNN extensions. The class $\mathbf{LH}\mathfrak{F}$ contains, for example, all elementary amenable groups and all linear groups.

2. The cohomological dimension of G, denoted $\operatorname{cd} G$, is the projective dimension of the trivial G-module $\mathbb{Z}$ over $\mathbb{Z}G$. For a virtually torsion-free group G, i.e., G has a torsion-free subgroup of finite index, it was well-known that all torsion-free subgroups of G of finite index have the same cohomological dimension (cf. [5]). The common cohomological dimension of the torsion-free subgroups of finite index is called the virtual cohomological dimension of G and is denoted by $\operatorname{vcd} G$. The finiteness of $\operatorname{vcd} G$ ensures that the Farrell cohomology of a group is well defined. There are other well-known invariants of a group which have been accompanied with the Ikenaga's generalized cohomology ([14]) and the complete cohomology ([4, 12, 19]):

(1) $\underline{\operatorname{cd}}\, G := \sup\{\, n : \operatorname{Ext}^n_{\mathbb{Z}G}(M, F) \neq 0,\ M : \mathbb{Z}\text{-free},\ \ F : \mathbb{Z}G\text{-free}\}$ ([14]).
(2) $\operatorname{spli} G := \sup\{n : \operatorname{Ext}^n_{\mathbb{Z}G}(I, -) \neq 0,\ I : \mathbb{Z}G\text{-injective}\}$ ([11]).
(3) $\operatorname{silp} G := \sup\{n : \operatorname{Ext}^n_{\mathbb{Z}G}(-, P) \neq 0,\ P : \mathbb{Z}G\text{-projective}\}$ ([11]).
(4) $\operatorname{fin.dim} G := \sup\{n : \operatorname{proj.dim}_G M = n < \infty\}$ ([20]).
(5) $\operatorname{pccd} G := \sup\{\, n : H^n(G, P) \neq 0,\ P : \mathbb{Z}G\text{-projective}\}$ ([15]).
(6) $\operatorname{Gcd} G := \operatorname{Gpd}_{\mathbb{Z}G}\mathbb{Z}$, the Gorenstein projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ ([2, 3]).

It is well known from [2, 3, 7, 11, 13, 14, 15, 17, 22] that for any group G,

(a) $\operatorname{pccd} G \leq \underline{\operatorname{cd}}\, G = \operatorname{Gcd} G \leq \operatorname{silp} G = \operatorname{spli} G \leq \underline{\operatorname{cd}}\, G + 1 = \operatorname{Gcd} G + 1$.
(b) $-1 \leq \operatorname{pccd} G \leq \infty$.
(c) If G is the Thompson group T, $\bigoplus_{i=1}^{\infty}\mathbb{Z}$, or $GL_n(K)$, where K is a subfield of the algebraic closure of $\mathbb{Q}$, then $\operatorname{pccd} G = -1$.

(*d*) If $G = *_{n\in\mathbb{N}} G_n$, where $G_n := \bigoplus_{i=1}^{n} \mathbb{Z}$, then $\operatorname{pccd} G = \infty$.
(*e*) If $\operatorname{Gcd} G < \infty$, then $\operatorname{Gcd} G = \operatorname{pccd} G$ and so $-1 < \operatorname{pccd} G < \infty$.
(*f*) $\operatorname{fin.dim} G \leq \operatorname{spli} G$, the equality holds when $G \in \mathbf{LH}\mathfrak{F}$ or $\operatorname{spli} G < \infty$.

3. Main results

In what follows, let G be an arbitrary discrete group and $\mathbb{Z}G$ its group ring. We write "G-module", "G-projective", etc. instead of "$\mathbb{Z}G$-module", "$\mathbb{Z}G$-projective", etc.

3.1. Lemma. *Let G be a group satisfying the following property: for any $\mathbb{Z}$-free G-module M,*

$$\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty \textit{ if and only if } \operatorname{proj.dim}_{\mathbb{Z}G} M \leq n.$$

Then $\operatorname{fin.dim} G \leq n + 1$.

Proof. Let N be a G-module with $\operatorname{proj.dim}_{\mathbb{Z}G} N < \infty$. Consider an exact sequence of G-modules $0 \to K \to P \to N \to 0$, where P is G-projective. It is clear that K is $\mathbb{Z}$-free and $\operatorname{proj.dim}_{\mathbb{Z}G} K < \infty$. Thus $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ by the assumption and so $\operatorname{proj.dim}_{\mathbb{Z}G} N \leq n + 1$. Hence we conclude that $\operatorname{fin.dim} G \leq n + 1$. □

In [8] Dembegioti and Talelli proposed the following conjecture and gave some example of groups satisfying it.

3.2. Conjecture. *For any group G,* $\operatorname{spli} G = \underline{\operatorname{cd}}\, G + 1$.

In [3] Bahlekeh, Dembegioti, and Talelli proposed the following conjecture.

3.3. Conjecture. *For any group G,* $\operatorname{fin.dim} G = \operatorname{Gcd} G + 1$.

Note that $\operatorname{Gcd} G = \underline{\operatorname{cd}}\, G$ for any group G, and $\operatorname{fin.dim} G = \operatorname{spli} G$ when G is an $\mathbf{LH}\mathfrak{F}$-group. Thus Conjecture 3.2 is equivalent to Conjecture 3.3 when G is an $\mathbf{LH}\mathfrak{F}$-group.

3.4. Theorem. *If Conjecture 3.3 is true, then Question 1.1 has an affirmative answer.*

Proof. Assume that G satisfies the property in Question 1.1. Then $\operatorname{fin.dim} G \leq n + 1$ by Lemma 3.1. By the assumption, it follows that $\operatorname{Gcd} G \leq n$. Hence $\operatorname{pccd} G \leq n$. □

3.5. Corollary. *Suppose that G satisfies the one of the following:*

(1) $\underline{\operatorname{cd}}\, G = 0$ *or* 1.
(2) *duality group.*
(3) *fundamental group of graph of finite groups.*
(4) *fundamental group of certain finite graph of group of type FP_∞ in* [8, Theorem 3.5].

Then Question 1.1 has an affirmative answer for G.

Proof. It is known from [8, 10] that if a group G is one of the list above, then G satisfies Conjecture 3.3. Hence the result follows from Theorem 3.4. □

The following corollary shows that the validity of Conjecture 3.3 settles Question A in [16] completely.

3.6. Corollary. *Let G be a group with the property that every $\mathbb{Z}$-free G-module of finite projective dimension is projective. If G satisfies Conjecture 3.3, then G is finite.*

Proof. Note that G is finite if and only if $\operatorname{pccd} G = 0$ ([15, Proposition 3.9]). Hence the result follows immediately from Theorem 3.4. □

3.7. Theorem. *Let G be a virtually torsion-free group. If G is an* $\mathbf{LH}\mathfrak{F}$*-group, then Question 1.1 has an affirmative answer for G.*

Proof. Assume that G satisfies the property in Question 1.1. Then $\mathrm{fin.dim}\, G \le n+1$ by Lemma 3.1. Since G is an $\mathbf{LH}\mathfrak{F}$-group, it follows from [22, Corollary 2] that $\mathrm{spli}\, G \le n+1$. Since $\underline{\mathrm{cd}}\, G \le \mathrm{silp}\, G = \mathrm{spli}\, G$, it follows that $\underline{\mathrm{cd}}\, G \le n+1$. Let H be a torsion-free subgroup of finite index in G. Since G is an $\mathbf{LH}\mathfrak{F}$-group, it follows from [22, Corollary 2] and [11, 5.2] that $\mathrm{fin.dim}\, G = \mathrm{spli}\, G = \mathrm{spli}\, H = \mathrm{fin.dim}\, H < \infty$. By [22, Corollary 1] we have $\mathrm{cd}\, H < \infty$. Then $\underline{\mathrm{cd}}\, G = \underline{\mathrm{cd}}\, H = \mathrm{cd}\, H = \mathrm{vcd}\, G$ by [14, Proposition 3, Proposition 5] and so $\mathrm{vcd}\, G \le n+1$. Suppose that $\mathrm{vcd}\, G = n+1$. Then $\mathrm{cd}\, H = \mathrm{proj.dim}_{\mathbb{Z}H}\mathbb{Z} = n+1$. But this contradicts to the property in Question 1.1. Hence $\mathrm{pccd}\, G = \mathrm{vcd}\, G \le n$ as required. □

3.8. Theorem. *Let H be a normal subgroup of G. Suppose that every H-projective, G-module M with $\mathrm{proj.dim}_{\mathbb{Z}G} M < \infty$ is H-free. Then Question 1.2 has an affirmative answer for (G, H).*

Proof. Assume that a G-module M is H-projective and $\mathrm{proj.dim}_{\mathbb{Z}G} M < \infty$. Let $Q = G/H$. Since M is H-projective, it follows from a spectral sequence argument as in the proof of [6, Lemma 4.1 (a)] that for any G-module N,

$$\mathrm{Ext}^i_{\mathbb{Z}G}(M, N) \cong H^i(Q, \mathrm{Hom}_{\mathbb{Z}H}(M, N)).$$

Suppose that there exists a projective Q-module L and $k > 0$ such that $H^k(Q, L) \ne 0$. We can regard L as a G-module via the quotient map $q : G \twoheadrightarrow Q$. By the assumption, we see that M is G-projective and thereby for any $i > 0$,

$$\mathrm{Ext}^i_{\mathbb{Z}G}(M, L) \cong H^i(Q, \mathrm{Hom}_{\mathbb{Z}H}(M, L)) = 0.$$

Since M is H-free by the assumption, it follows that

$$\mathrm{Hom}_{\mathbb{Z}H}(M, L) \cong \mathrm{Hom}_{\mathbb{Z}H}(\oplus \mathbb{Z}H, L) \cong \prod \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}H, L) \cong \prod L$$

as Q-modules (cf. [21, Thorem 2.31]). Thus we have

$$H^k(Q, \mathrm{Hom}_{\mathbb{Z}H}(M, L)) \cong H^k(Q, \prod L) \cong \prod H^k(Q, L) \ne 0$$

(cf. [21, Proposition 7.22]), which makes a contradiction. Hence $H^i(Q, S) = 0$ for each $i > 0$ and any projective Q-module S, and therefore $\mathrm{pccd}\, Q \le 0$. But since $\mathrm{pccd}\, Q \ne -1$, we see that $\mathrm{pccd}\, Q = 0$ and so Q is finite by [15, Proposition 3.9]. Hence we conclude that $|G : H| < \infty$. □

3.9. Theorem. *Let H be a normal subgroup of G. Suppose that every H-stably free, G-module M with $\mathrm{proj.dim}_{\mathbb{Z}G} M < \infty$ is H-free. Assume further that for any H-stably-free, G-module M with $\mathrm{proj.dim}_{\mathbb{Z}G} M < \infty$, there exist H-free modules M' and F such that $M \oplus M' \cong F$ as H-modules and the H-free rank of M' is different from that of F. Then Question 1.2 has an affirmative answer for (G, H).*

Proof. Let $Q = G/H$. By the proof of Theorem 3.8, we see that for any G-module N,

$$\mathrm{Ext}^i_{\mathbb{Z}G}(M, N) \cong H^i(Q, \mathrm{Hom}_{\mathbb{Z}H}(M, N)).$$

Suppose that there exists a projective Q-module L and $k > 0$ such that $H^k(Q, L) \ne 0$. By the assumption, it follows that M is G-projective and therefore we have that for any $i > 0$,

$$\mathrm{Ext}^i_{\mathbb{Z}G}(M, L) \cong H^i(Q, \mathrm{Hom}_{\mathbb{Z}H}(M, L)) = 0.$$

Note that $\mathrm{Hom}_{\mathbb{Z}H}(F,L) \cong \prod_I L$ and $\mathrm{Hom}_{\mathbb{Z}H}(M',L) \cong \prod_J L$ as Q-modules, where the cardinalities of I and J are the H-free ranks of F and M', respectively. Note also that

$$\mathrm{Hom}_{\mathbb{Z}H}(F,L) \cong \mathrm{Hom}_{\mathbb{Z}H}(M,L) \oplus \mathrm{Hom}_{\mathbb{Z}H}(M',L).$$

Thus we have

$$\begin{aligned}\prod_J H^k(Q,L) &\cong H^k(Q,\mathrm{Hom}_{\mathbb{Z}H}(M,L)) \oplus (\prod_J H^k(Q,L)) \\ &\cong H^k(Q,\mathrm{Hom}_{\mathbb{Z}H}(M,L)) \oplus H^k(Q,\mathrm{Hom}_{\mathbb{Z}H}(M',L)) \\ &\cong H^k(Q,\mathrm{Hom}_{\mathbb{Z}H}(F,L)) \cong H^k(Q,\prod_I L) \cong \prod_I H^k(Q,L).\end{aligned}$$

But this makes a contradiction, since the H-free rank of F is different from that of M'. Hence we can conclude that Q is finite by the same argument of the proof of Theorem 3.8. Therefore $|G:H| < \infty$. □

3.10. Theorem. *Let G be a torsion-free group and H a normal subgroup of G. Suppose that* $\mathrm{pccd}\,(G/H) > -1$ *and (G,H) satisfies one of the following:*

- *(a) Every H-projective, G-module M with* $\mathrm{proj.dim}_{\mathbb{Z}G}\, M < \infty$ *is H-free.*
- *(b) Every H-stably free, G-module M with* $\mathrm{proj.dim}_{\mathbb{Z}G}\, M < \infty$ *is H-free, and for any H-stably-free, G-module M with* $\mathrm{proj.dim}_{\mathbb{Z}G}\, M < \infty$*, there exist H-free modules M' and F such that $M \oplus M' \cong F$ as H-modules and the H-free rank of M' is different from that of F.*

Then Question 1.4 has an affirmative answer for (G,H).

Proof. It can be proved by the same argument of the proof of Theorems 3.8 and 3.9. □

3.11. Remark. Let X be a CW-complex such that the universal cover $\widetilde{X}$ is $(m-1)$-connected. It is known from [9, Proposition 1.4] that if $m \geq 3$, then X has the m-type of a finite m-complex if and only if its Swan-Wall class $SW_m[X] = 0$, where $SW_m[X] := C_m(\widetilde{X})/B_m(\widetilde{X}) \in C(\pi_1(X))$, and where $C(\pi_1(X))$ is the abelian monoid of stable equivalence classes of finitely generated $\pi_1(X)$-modules. Recall that for an abelian group A and positive integer m, a CW-complex Y is called a Moore space of type $M(A,m)$ if $H_0(Y) = \mathbb{Z}$, $H_m(Y)$ is isomorphic to A, and $H_i(Y) = 0$ for $i \neq 0, m$.

Suppose now that G is a finite group. Let X be a finite dimensional, finite type CW-complex X with $\pi_1(X) \cong G$ such that $\widetilde{X}$ is a Moore space of type $M(A,m)$. Then we see that $\mathrm{proj.dim}_{\mathbb{Z}G}\,(C_m(\widetilde{X})/B_m(\widetilde{X})) < \infty$, since

$$0 \to C_{\dim X}(\widetilde{X}) \to \cdots \to C_m(\widetilde{X}) \to C_m(\widetilde{X})/B_m(\widetilde{X}) \to 0$$

is a G-free resolution of $C_m(\widetilde{X})/B_m(\widetilde{X})$. It is clear that the sequence of G-modules

$$0 \to Z_m(\widetilde{X})/B_m(\widetilde{X}) \to C_m(\widetilde{X})/B_m(\widetilde{X}) \to C_m(\widetilde{X})/Z_m(\widetilde{X}) \to 0$$

is exact. Since $C_m(\widetilde{X})/Z_m(\widetilde{X}) \cong B_{m-1}(\widetilde{X}) \subset C_m(\widetilde{X})$ and $C_m(\widetilde{X})$ is $\mathbb{Z}$-free, it follows that $C_m(\widetilde{X})/Z_m(\widetilde{X})$ is $\mathbb{Z}$-free. Thus we see that $C_m(\widetilde{X})/B_m(\widetilde{X})$ is finitely generated G-projective and so $C_m(\widetilde{X})/B_m(\widetilde{X}) = 0 \in \widetilde{K}_0(\mathbb{Z}\pi_1(X))$. By [9, Proposition 1.4], it follows that X has the m-type of a finite CW-complex. Consequently, we can conclude that if G is a finite group, then every finite dimensional, finite type CW-complex X with $\pi_1(X) \cong G$ such that $\widetilde{X}$ is a Moore space of type $M(A,m)$ has the m-type of a finite CW-complex. But we do not yet know whether the converse of this holds.

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