The Rayleigh paired comparison model with Bayesian analysis

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Abstract

A paired comparison (PC) method is more reliable to rank or compare more than two items/ objects at the same time. It is a well-developed method of ordering attributes or characteristics of a given set of items.The PC model is developed using Rayleigh random variables on the basis of Stern's criteria [17]. The Rayleigh PC model is analyzed in Bayesian framework using non-informative (Jeffreys and Uniform) priors. The Bayesian inference of the developed model is compared with existing the Bradley-Terry model. The preference and predictive probabilities for current and future comparisons are calculated. The posterior probabilities of hypotheses for comparing two parameters are evaluated. The Bayesian 95% credible interval are calculated.Appropriateness of the model is also examined. Graphs of marginal posterior distributions of the parameters are drawn. The Bayesian analysis is performed using real life data sets.

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1. Introduction

A pair of objects is presented for comparison and two are placed in the relationship preferred or not preferred. If the differences among the objects are distinguishable and fairly apparent then ranking of all objects will be preferable where the objects will be given ranked values depends on preferences. For a detailed discussion on PC method and its usefulness, one is referred to [4], [7] and [9]

A PC model based on two Cauchy random variables has been developed by [1]. The model has been analyzed in the Bayesian frame work using informative, Conjugate and non-informative (Jeffreys and Uniform) priors. The real data set of top five ranked one day international cricket teams is collected for the Bayesian analysis. By study it is concluded that Australia has been ranked on the top. The technique of collecting preference data from judges through binary digits have been highlighted by [2]. The preferred item is denoted by one and zero to the non-preferred item. The Bradley-Terry PC model is used for analysis considering a real data set on ice-cream brands. [3] has been worked on two types of models to use ordinal scales for PC analysis for several parameters. He shows for binary scale that logit transformation for the models simplifies them to the basic Bradley-Terry model. [4] has given the Bayesian analysis of the Bradley-Terry and the Rao-Kupper model. The posterior means of the parameters, posterior probabilities of the hypotheses and predictive probabilities for both the models are included in this study. These results were using the non-informative (Jeffreys and Uniform) priors. [5] have estimated the parameters of the Thurstone-Mosteller PC model by method of maximum likelihood. The Bayesian analysis of the model is carried out using Jeffreys prior. The Binomial discrete logistic model for the relation between sensory and consumer preference have been presented by [8]. It is also concluded that no preference is better to model as a function than considered as ties for the sensory data. The Thurstone-Mosteller model for PCs has been modified by [10] which allows for widely differing proportions of draws. Data relating to games between the 64 greatest chess player of the world is analyzed for the model. [11] has discussed the technique of iterative maximum likelihood estimates algorithms for the generalization of the Bradley Terry model. [13] has presented that PC allow a large number of draws and variability of draw percentages among the players of chess or soccer matches. The results are based on matching the number of home wins, home draws, away wins and away draws for each team with their expected values. Glenn- David model is used for the estimation. [16] have recommended the procedure of lasso that categorized the contestants with similar aptitudes. The standard maximum likelihood method is used for the prediction of rankings. The teams ranking of National football league 2010-2011 and the American college hockey men's division I 2009-2010 have been used for the analysis.

In Section 2, the method of the Rayleigh model development and notations for the model is discussed. The Bradley-Terry model is given in Section 3.The prior distributions and the Bayesian analysis is provided in Sections 4 and 5. Concluding remarks are provided in Section 6.

2. The Rayleigh Paired Comparisons Model

By considering PC experiments of the Rayleigh random variables with same shape parameter and different scale (α_i) parameter, the Rayleigh model is derived on the basis of the Stern's model criteria . The Rayleigh random variables are used to examine wind velocity. The data of MRI images is also Rayleigh distributed. As the Rayleigh distribution can be used in communication theory, so in paired comparison, perception of the preference one object is communicated to the other object in a pair, for this reason, Rayleigh distribution may be considered for PC model. The probability that the

preference of T_i over T_j is denoted by $\phi_{i.ij}$ and defined as:

$$\phi_{i.ij} = P(T_j \leq T_i)$$

$$\phi_{i.ij} = \int_0^\infty \int_{t_j}^\infty \frac{t_i}{\alpha_i^2} e^{-\frac{t_i^2}{2\alpha_i^2}} \frac{t_j}{\alpha_j^2} e^{-\frac{t_j^2}{2\alpha_j^2}} \, dt_i dt_j,$$

$$(2.1) \qquad \phi_{i.ij} = \frac{\alpha_i^2}{\alpha_i^2 + \alpha_j^2}$$

and $\phi_{j.ij}$ is the probability that T_j is preferred over T_i and is obtained as:

$$\phi_{j.ij} = 1 - \phi_{i.ij}$$

$$(2.2) \qquad \phi_{j.ij} = \frac{\alpha_j^2}{\alpha_i^2 + \alpha_j^2}$$

where α_i; $(i < j) = 1, 2, ...m$ are the treatment parameters. The (2.1) and (2.2) represent the model named as the Rayleigh model for PC.
We define the notations for the model. Let w_{ij} be the random variable associated with the rank of the treatments in the kth repetition of the treatment pair (T_i, T_j), where $(i \neq j; i \geq 1, j \leq m; k = 1, 2, ..., r_{ij})$ and m is the number of observation.

$w_{i.ijk} = 1$ or 0 accordingly as treatment T_i is preferred to treatment T_j

or not in the kth repetition of comparison.

$w_{j.ijk} = 1$ or 0 accordingly as treatment T_j is preferred to treatment T_i

or not in the kth repetition of comparison.

$$w_{i.ij} = \sum_k w_{i.ijk} = \qquad \text{the number of times treatment } T_i \text{ is preferred to}$$

treatment T_j.

$$w_{j.ij} = \sum_k w_{j.ijk} = \qquad \text{the number of times treatment } T_j \text{ is preferred to}$$

treatment T_i.

$r_{ij} =$ the number of times treatment T_i is compared with treatment T_j.

$$r_{ij} = w_{i.ij} + w_{j.ij}.$$

The likelihood function of the observed outcomes of the trial $\boldsymbol{w}$ and the parameters $\boldsymbol{\alpha} = \alpha_1, \alpha_2, ..., \alpha_m$ is :

$$(2.3) \qquad l(\boldsymbol{w}, \boldsymbol{\alpha}) = \prod_{i<j=1}^m \frac{r_{ij}!}{w_{ij}! \ (r_{ij} - w_{ij})!} \frac{\alpha_i^{2w_{i.ij}} \alpha_j^{2w_{j.ij}}}{(\alpha_i^2 + \alpha_j^2)^{w_{i.ij} + w_{j.ij}}} \qquad , \alpha_i > 0$$

A constraint is imposed on parameters of the model i.e., $\sum_{i=1}^m \alpha_i = 1$. This condition confirms that parameters are well defined.

3. The Bradley-Terry Model

The Bradley-Terry model is the basic PC model. [7] proposed a model of PCs assuming the Logistic density instead of the standard normal density using [15] and [18]. The difference between two latent variables (T_i, T_j) has a Logistic density with location parameters $(\ln \alpha_i, \ln \alpha_j)$. The probability that treatment T_i is preferred to treatment T_j according to Bradley and Terry is given as:

$$(3.1) \qquad \phi_{i.ij} = \frac{\alpha_i}{\alpha_i + \alpha_j}$$

$$(3.2) \qquad \phi_{j.ij} = \frac{\alpha_j}{\alpha_i + \alpha_j},$$

where (3.1) and (3.2) is known as the Bradley-Terry model for PC.

4. Non-Informative Prior Distributions

The non-informative (Uniform and Jeffreys) priors are assumed for the Bayesian analysis.

4.1. Uniform Prior. The Bayesian analysis of the unknown parameter using Uniform prior is suggested by [6] and [14]. We use the Uniform U(0,1) as the prior distribution, defined as:

$$(4.1) \qquad p_U(\boldsymbol{\alpha}) \propto 1$$

where α is defined in (2.3) and $\alpha_i > 0$. It is the improper prior.

4.2. Jeffreys Prior. The Jeffreys prior is defined as the density function proportional to partially differentiating twice the log likelihood function and taking the square root of the expected value, i.e.

$$(4.2) \qquad p_J(\boldsymbol{\alpha}) \propto det[I(\boldsymbol{\alpha})]^{\frac{1}{2}}$$

$$\text{where } det[I(\boldsymbol{\alpha})] = (-1)^2 \begin{vmatrix} E[\frac{\partial^2 \log l(.)}{\partial \alpha_1^2}] & E[\frac{\partial^2 \log l(.)}{\partial \alpha_1 \alpha_2}] \\ E[\frac{\partial^2 \log l(.)}{\partial \alpha_2 \alpha_1}] & E[\frac{\partial^2 \log l(.)}{\partial \alpha_2^2}] \end{vmatrix}$$

for $m = 3$ and $\alpha_3 = 1 - \alpha_1 - \alpha_2$.
So, the Jeffreys prior for the parameters is derived as:

$$p_J(\boldsymbol{\alpha}) \propto \sqrt{\frac{A_1}{A_2}}$$

$$\begin{aligned} \text{where } A_1 =& 2\alpha_1^6 - 6\alpha_1^5 + 6\alpha_1^5\alpha_2 + 15\alpha_1^4\alpha_2^2 - 14\alpha_1^4\alpha_2 + 7\alpha_1^4 - 4\alpha_1^3 + 12\alpha_1^3\alpha_2 + \\ & 20\alpha_1^3\alpha_2^3 - 28\alpha_1^3\alpha_2^2 - 4\alpha_2\alpha_1^2 + \alpha_1^2 - 28\alpha_1^2\alpha_2^3 + 15\alpha_1^2\alpha_2^4 + 18\alpha_1^2\alpha_2^2 + \\ & 12\alpha_1\alpha_2^3 - 4\alpha_1\alpha_2^2 + 6\alpha_1\alpha_2^5 - 14\alpha_1\alpha_2^4 + \alpha_2^2 + 7\alpha_2^4 - 4\alpha_2^3 - 6\alpha_2^5 + \\ & 2\alpha_2^6 \\ A_2 =& (2\alpha_2^2 + 1 - 2\alpha_1 - 2\alpha_2 + \alpha_1^2 + 2\alpha_1\alpha_2)^2(2\alpha_2^2 + 1 - 2\alpha_1 - 2\alpha_2 + \alpha_2^2 \\ & + 2\alpha_1\alpha_2)^2(\alpha_1^2 + \alpha_2^2)^2 \end{aligned}$$

Maple-15 package is used for the mathematical derivation of Jeffreys prior.

5. Bayesian Analysis of the Model for m=3

The joint posterior distribution of the Rayleigh model parameters given data using the (2.3) and $p(\boldsymbol{\alpha})$ (prior distribution) is:

$$p(\alpha_i, \alpha_j | \boldsymbol{w}) = \frac{1}{K} \ p(\boldsymbol{\alpha}) \prod_{i<j=1}^{m=3} \frac{r_{ij}!}{w_{ij}! \ (r_{ij} - w_{ij})!} \ \frac{\alpha_i^{2w_{i.ij}} \alpha_j^{2w_{j.ij}}}{(\alpha_i^2 + \alpha_j^2)^{w_{i.ij} + w_{j.ij}}}$$

where K is the normalizing constant, defined as:

$$K = \int_0^1 \int_0^{1-\alpha_i} p(\boldsymbol{\alpha}) \prod_{i<j=1}^{m=3} \frac{r_{ij}!}{w_{ij}! \ (r_{ij} - w_{ij})!} \ \frac{\alpha_i^{2w_{i.ij}} \alpha_j^{2w_{j.ij}}}{(\alpha_i^2 + \alpha_j^2)^{w_{i.ij} + w_{j.ij}}} d\alpha_j d\alpha_i$$

The marginal posterior distribution of the Rayleigh model parameter α_i given data under Uniform prior using the (4.1) and Sec.5.1 is:

$$(5.1) \qquad p(\alpha_i | \boldsymbol{w}) = \frac{1}{K} \int_0^{1-\alpha_i} p_U(\boldsymbol{\alpha}) \prod_{i<j=1}^{m=3} \frac{r_{ij}!}{w_{ij}! \ (r_{ij} - w_{ij})!} \ \frac{\alpha_i^{2w_{i.ij}} \alpha_j^{2w_{j.ij}}}{(\alpha_i^2 + \alpha_j^2)^{w_{i.ij} + w_{j.ij}}} d\alpha_j,$$

$$\alpha_i > 0, \ \textstyle\sum_{i=1}^{m=3} \alpha_i = 1.$$

The marginal posterior distribution of the Rayleigh model parameter α_1 given data under Jeffreys prior using the (4.2) and Sec. 5.2 is:

$$(5.2) \qquad p(\alpha_i | \boldsymbol{w}) = \frac{1}{K} \int_0^{1-\alpha_i} p_J(\boldsymbol{\alpha}) \prod_{i<j=1}^{m=3} \frac{r_{ij}!}{w_{ij}! \ (r_{ij} - w_{ij})!} \ \frac{\alpha_i^{2w_{i.ij}} \alpha_j^{2w_{j.ij}}}{(\alpha_i^2 + \alpha_j^2)^{w_{i.ij} + w_{j.ij}}} d\alpha_j,$$

$$\alpha_i > 0, \ \textstyle\sum_{i=1}^{m=3} \alpha_i = 1.$$

The posterior distribution is not in closed form but can be used numerically using package like SAS.

For illustrative purposes, two real data sets (r_{ij}) of 5 and 30 respondents is collected from the students of the Quaid-i-Azam University Pakistan. These data sets comprise of the three different brands of cigarettes (Benson & Hedges (BH), Marlboro (ML) and Dunhill (DH)) which are commonly used among students. Bayesian analysis for the data sets in Table 1 is carried out using non-informative priors.

Table 1. Data of Cigarette Brands

Pairs	Data 1			Pairs	Data 2		
	$w_{i.ij}$	$w_{j.ij}$	r_{ij}		$w_{i.ij}$	$w_{j.ij}$	r_{ij}
(BH , ML)	1	4	5	(BH , ML)	11	13	24
(BH , DH)	2	3	5	(BH , DH)	12	15	27
(ML , DH)	4	1	5	(ML , DH)	16	11	27

5.1. Posterior Estimates. The posterior means are used as the estimates of the parameters. In the Table 2, the posterior means of the Rayleigh and the Bradley-Terry models are given.

Table 2. Posterior Means under Non-Informative Priors

Parameters	Data 1				Data 2			
	Bradley-Terry		Rayleigh		Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform
α_1	0.15965	0.18604	0.23427	0.23063	0.28907	0.29027	0.31080	0.31051
α_2	0.63030	0.57543	0.49461	0.49991	0.39659	0.39470	0.36483	0.36523
α_3	0.21005	0.23853	0.27112	0.26946	0.31435	0.31502	0.32437	0.32426

From the Table 2, it is concluded that the cigarette brand Marlboro may be ranked number one among the brands and commonly used by students. Dunhill is ranked number two. Benson & Hedges has the lowest rank. Further observes that ranking of the brands (of cigarettes) have same order under both the models and data sets using non-informative priors.

5.2. Graphs of the Marginal Posterior Distribution. The graphs of the marginal posterior distribution of the Bradley-Terry and the Rayleigh model using both data sets for non-informative priors are drawn below.
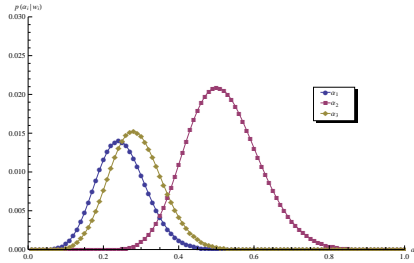

Figure 1. The Marginal Posterior Distributions for α_i of the Rayleigh Model using Uniform Prior (Data-1)
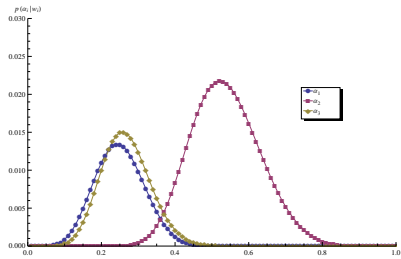
Figure 2. The Marginal Posterior Distributions for α_i of the Rayleigh Model using Jeffreys Prior (Data-1)
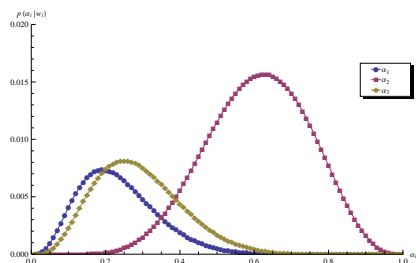
Figure 3. The Marginal Posterior Distributions for α_i of the Bradley-Terry Model using Uniform Prior (Data-1)
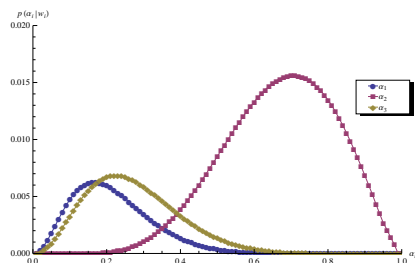

Figure 4. The Marginal Posterior Distributions for α_i of the Bradley-Terry Model using Jeffreys Prior (Data-1)
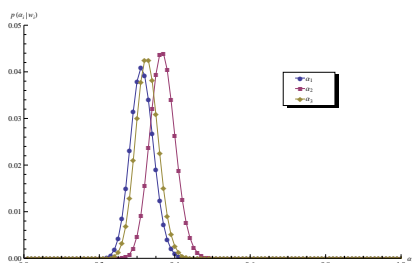

Figure 5. The Marginal Posterior Distributions for α_i of the Rayleigh Model using Uniform Prior (Data-2)
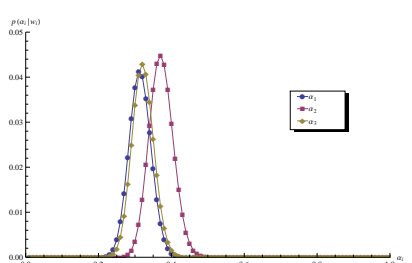

Figure 6. The Marginal Posterior Distributions for α_i of the Rayleigh Model using Jeffreys Prior (Data-2)
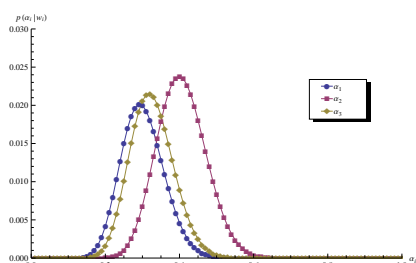

Figure 7. The Marginal Posterior Distributions for α_i of the Bradley-Terry Model using Uniform Prior (Data-2)
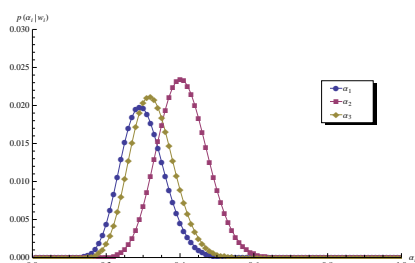

Figure 8. The Marginal Posterior Distributions for α_i of the Bradley-Terry Model using Jeffreys Prior (Data-2)

The Figures 1, 2, 3 and 4 have skewed marginal posterior distributions for the Rayleigh and the Bradley-Terry models under non-informative priors for the data set-1. Where as figures 5, 6, 7 and 8 have symmetrical marginal posterior distributions for the Rayleigh and the Bradley-Terry models under non-informative priors for the data set-2. Due to large data set shows symmetrical graphs.

5.3. Credible Intervals. The 95 % credible intervals are constructed for the Bradley-Terry and the Rayleigh models.

Table 3. 95% Credible Intervals under Non-Informative Priors

Parameters	Data-1			
	Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform
α_1	(0.11902, 0.20029)	(0.14547, 0.22662)	(0.20679, 0.26175)	(0.20296, 0.25830)
α_2	(0.56769, 0.69291)	(0.51878, 0.63207)	(0.45733, 0.53189)	(0.46271, 0.53712)
α_3	(0.16432, 0.25577)	(0.19459, 0.28247)	(0.24349, 0.29875)	(0.24151, 0.29741)
Parameters	Data-2			
	Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform
α_1	(0.26477, 0.31336)	(0.26623, 0.31431)	(0.29806, 0.32354)	(0.29770, 0.32332)
α_2	(0.37023, 0.42295)	(0.36881, 0.42060)	(0.35191, 0.37776)	(0.35222, 0.37823)
α_3	(0.29121, 0.33748)	(0.29216, 0.33789)	(0.31254, 0.33619)	(0.31236, 0.33616)

From the Table 3, it is observed that 95 % interval are narrower for the data-2. Further more it is concluded that the credible intervals for the Rayleigh model are narrower than the Bradley Terry model under non-informative priors.

5.4. Preference Probability. The term preference probability is used for the superiority of probability of T_i over T_j on some defined attribute or characteristic. Using the posterior means of the Rayleigh and the Bradley-Terry model provided in the Table 2, the preference probabilities are calculated using (2.1), (2.2), (3.1) and (3.2) presented in the Table 4.

Table 4. Preference Probabilities under Non-Informative Priors

$\phi_{i.ij}$	Data-1				Data-2			
	Bradley-Terry		Rayleigh		Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform
$\phi_{1.12}$	0.20210	0.24431	0.18323	0.17549	0.42159	0.42377	0.42054	0.41955
$\phi_{1.13}$	0.43184	0.43818	0.42747	0.42282	0.47905	0.47956	0.47865	0.47835
$\phi_{2.23}$	0.75004	0.70695	0.76895	0.77487	0.55784	0.55613	0.55850	0.55921

From the Table 4, it is perceived that the preference probabilities implies the same ranking order as the posterior means for both the models and data sets under non-informative priors.

5.5. Predictive Probability. The predictive probabilities is used to predict the future single preference of one treatment T_i over treatment T_j. It is denoted by $P_{i.ij}$ and defined as:

$$P_{i.ij} = \int_{\alpha_i=0}^{1} \int_{\alpha_j=0}^{1-\alpha_i} \phi_{ij} \; p(\alpha_i, \alpha_j | \boldsymbol{w}) \; d\alpha_j \; d\alpha_i$$

Table 5. Predictive Probabilities under Non-Informative Priors

$P_{i.ij}$	Data-1				Data-2			
	Bradley-Terry		Rayleigh		Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform
$P_{1.12}$	0.20875	0.24877	0.20870	0.20078	0.42222	0.42434	0.42222	0.42128
$P_{1.13}$	0.43393	0.43960	0.43388	0.43003	0.47895	0.47945	0.47895	0.47865
$P_{2.23}$	0.74477	0.70352	0.74482	0.75080	0.55721	0.55556	0.55721	0.55788

The predictive probabilities are closed to the preference probabilities and favors the same ranking order for both the models and data sets under non-informative priors.

5.6. Bayesian Hypotheses Testing. In Bayesian analysis, the task of deciding between the hypotheses is conceptually more straightforward. One merely calculates the posterior probabilities and decides between hypotheses accordingly.

$$H_{ij} : \alpha_i \geq \alpha_j \quad Vs. \quad H_{ji} : \alpha_i < \alpha_j,$$

The posterior probability for the hypothesis H_{ij} is:

$$p_{ij} = \int_{\zeta=0}^{1} \int_{\eta=\zeta}^{(1+\zeta)/2} p(\zeta, \eta | w) d\eta d\zeta,$$

The posterior probability for the hypothesis H_{ji} is:

$$q_{ij} = 1 - p_{ij}$$

where $\eta = \alpha_i$ and $\zeta = \alpha_i - \alpha j$.

The decision rule for the hypotheses is based on Bayes factor. It is denoted by 'B' and the most general form of the Bayes factor can be described as follows.:

$$B = \frac{\text{Posterior odd ratios}}{\text{Prior odd ratios}}$$

The central notion of Bayes factor is that prior and posterior information should be combined in a ratio that provides evidence of one model specification over another.

It can be interpreted as the 'odds for H_{ij} to H_{ji} that are given by the data. [12] gives the following typology for comparing H_{ij} Vs. H_{ji}

$$B \geq 1 \qquad \text{support } H_{ij}$$
$$10^{-0.5} \leq B \leq 1 \quad \text{minimal evidence against } H_{ij}$$
$$10^{-1} \leq B \leq 10^{-0.5} \text{substantial evidence against } H_{ij}$$
$$10^{-2} \leq B \leq 10^{-1} \text{ strong evidence against } H_{ij}$$
$$B \leq 10^{-2} \text{decisive evidence against } H_{ij}$$

Table 6. Posterior Probability under Non-Informative Priors

Pairs	Data-1							
	Bradley-Terry				Rayleigh			
	Jeffreys		Uniform		Jeffreys		Uniform	
	p_{ij}	B	p_{ij}	B	p_{ij}	B	p_{ij}	B
$\alpha_1 > \alpha_2$	0.02872	0.02957	0.04139	0.04318	0.02872	0.02957	0.04139	0.04318
$\alpha_1 > \alpha_3$	0.33206	0.49714	0.34256	0.52105	0.33206	0.49714	0.34256	0.52105
$\alpha_2 > \alpha_3$	0.92737	12.76842	0.90968	10.07174	0.92737	12.76842	0.90968	10.07174
Pairs	Data-2							
	Bradley-Terry				Rayleigh			
	Jeffreys		Uniform		Jeffreys		Uniform	
	p_{ij}	B	p_{ij}	B	p_{ij}	B	p_{ij}	B
$\alpha_1 > \alpha_2$	0.15254	0.17999	0.15566	0.18436	0.14174	0.16515	0.14048	0.16344
$\alpha_1 > \alpha_3$	0.37300	0.59490	0.37428	0.59816	0.35536	0.55125	0.35479	0.54988
$\alpha_2 > \alpha_3$	0.75584	3.09567	0.75208	3.03356	0.74108	2.86219	0.74275	2.88727

The Bayes factor in the Table 6 signify substantial evidence against H_{12}, minimal evidence against H_{13} and H_{23} is supported for both the models and data sets under non-informative priors. The preference order of treatments is confirmed through testing of hypotheses.

5.7. Appropriateness of the Model. It is used to compare the discrepancies of the observed preferences among the expected preferences. The Chi-square test is used for the appropriateness of the models. The hypothesis is defined as:
H_0: The model is true for some values of $\boldsymbol{\alpha} = \alpha_0$
H_1: The model is not true for any values of the parameters.
where $\boldsymbol{\alpha}=\alpha_1, \alpha_2, ..., \alpha_m$ is the vector of the unknown parameters, $\alpha_i > 0$.
The χ^2 has the following form:

$$\chi^2 = \sum_{i<j}^{m} \left\{ \frac{(w_{ij} - \hat{w}_{ij})^2}{\hat{w}_{ij}} + \frac{(w_{ji} - \hat{w}_{ji})^2}{\hat{w}_{ji}} \right\}$$

with $(m-1)(m-2)/2$ degrees of freedom [4].
The expected number of preferences are obtained by the following form:

$$\hat{w}_{i.ij} = r_{ij} \frac{\alpha_i^2}{\phi_{ij}} \quad \text{and} \quad \hat{w}_{j.ij} = r_{ij} \frac{\alpha_j^2}{\phi_{ij}}, \text{where} \quad \phi_{ij} = \alpha_i^2 + \alpha_j^2.$$

w_{ij} and w_{ji} are the observed number of preferences from the data set given in the Table 1.

Table 7. Appropriateness of the Rayleigh Model

	Data-1				Data-2			
	Bradley-Terry		Rayleigh		Bradley-Terry		Rayleigh	
	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform	Jeffreys	Uniform
χ^2	0.0873	0.2918	0.0519	0.0495	0.3947	0.3963	0.3945	0.3947
$P-value$	0.2324	0.4109	0.1802	0.1761	0.4702	0.4710	0.4701	0.4702

From the Table 7 , the values of χ^2 for the Rayleigh and the Bradley-Terry model for both data sets under non-informative priors have high P-values as P-values > 0.05 . It is evident from the P-values that both the models have good fit. Furthermore, the Rayleigh model is considered to be better fit for the small data set in the Table 1 than the Bradley-Terry model under both the non-informative priors.

6. Conclusion

A new model for paired comparison is developed, named as the Rayleigh paired comparison model. The Rayleigh paired comparison model is analyzed in the Bayesian framework using non-informative (Uniform and Jeffreys) priors. The results are also compared with the existing Bradley-Terry model. For the analysis, we use the data sets of the preferences of cigarette brands (Benson & Hedges, Marlboro and Dunhill) used by university students. It is noticed that the cigarette brand Marlboro is highly preferred among the students of university. Benson & Hedges is the lowest preferred. The graphs, preference probabilities, predictive probabilities and hypotheses testing also confirm the same preference. The credible intervals for the Rayleigh model are narrower than the Bradley Terry model under non-informative priors. The appropriateness of the models (the Bradley-Terry and the Rayleigh model) through χ^2- statistic suggests that the fit is good but the proposed Rayleigh model is better fit for small data set than the Bradley-Terry as the P-value of χ^2- statistic under the Rayleigh model is smaller than the Bradley-Terry model.

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