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A data envelopment analysis based approach for target setting and resource allocation: application in gas companies

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Abstract

Resource allocation is an important application of data envelopment analysis and has been investigated by many researchers and managers from both economic and managerial field. In real managerial and economical decisions, situations often occur when extra productions of goods are utilized and the decision maker (DM) would like to determine the numbers of extra products that each unit can produced. In this paper, several methods based on the data envelopment analysis for resource allocation in such situations are introduced that can help the managers to make better decisions. The primary aim of this paper is to allocate resources such that the inefficient decision making units (DMUs) to become efficient as possible. For this aim, firstly several homogeneous units under the control of a central unit are considered and then the efficiency of each unit is determined. In addition, if the production of additional products seems logical, the DM wants to know how much of additional outputs should be produced by each unit such that the total outputs reach to a predetermined level. In this case the proposed algorithms determine quantities of the consumed input and produced output levels for each DMU to obtain the desirable output level. For using the whole power of system, the multi objective programming (MOP) problem has been used. A numerical example is given to show the solution process to improve the clarity of the proposed method. Finally, the real data of a gas company extracted from extant literature are used to demonstrate the proposed method.

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric frontier estimation methodology based on linear programming to measure the relative efficiency of a decision making unit (DMU) and provide DMUs with relative performance assessment on multiple inputs and outputs. It originated from Farell's seminal work [16] , was popularized by Charnes et al. [9], and has gained a wide range of applications measuring comparative efficiency. For instance, Amirteimori et al. [1] proposed a flexible slacks-based measure (FSBM) of efficiency in assessing UK higher education institutions. Ebrahimnejad and Tavana [13] showed used an interactive method for performance assessment in North Atlantic Treaty Organization (NATO) by establishing an equivalent relation between DEA and multi objective programming problem. Ebrahimnejad et al. [12] proposed a DEA model for banking with three stages. Maghbouli et al. [30] used the cooperative and non-cooperative game theories to assess the relative performance of 39 Spanish Airports based on a network DEA model with undesirable factors.

In empirical studies that examines scale economics and productive efficiency within a DEA framework, two of the most frequently used are resource allocation and target setting. Resource allocation is the setting of input- output levels for DMUs when the organization has limited input resources or output possibilities and plays a pivotal role in the management of corporations. Because of this, it has been an interesting topic to both business managers and researchers.

Several researchers have carried out investigations on resource allocation via DEA. Golany et al. [18] presented a five-step procedure for input resource allocation at an organizational level. Their procedure reduces to solving a linear program involving an objective function weighted according to DMU efficiencies. Their procedure does not compute output targets and only distributes input resources guided by current (weighted) DMU efficiencies. Their work uses the additive DEA model of Charnes et al. [8]. Golany and Tamir [19] presented a resource allocation model which simultaneously determines input and output targets based on maximizing total output. Their model is only applied in the case of a single output. For the multiple output case they suggest applying predetermined subjective weights to each output measure. Athanassopoulos [4] presented a goal programming model incorporating ideas from Thanassoulis and Dyson [32]. A significant feature of his work is that DMUs are linked together at the global level with respect to organizational input- output targets. In his model, proportional deviations from current input-output levels for each DMU, as well as proportional deviations from organizational input-output targets, are weighted together in a single linear objective which is to be minimized. In his work, the weights are specified by the DM. Thanassoulis [33] presented a paper dealing with the single input case. He presented a mixed-integer program to simultaneously cluster DMUs into k distinct sets and to determine a marginal resource level (MRL) for each output measure for each such cluster. MRLs were defined as the rate of (input) resource entitlement per unit of output. Once MRLs have been found, a logical numeric basis exists for future input resource allocation by DMs.

Thanassoulis [34] presented a paper concerned with estimating a single set of MRLs that apply to all DMUs in the single input case. He presented both the regression based method and the linear programming based method for such estimation. Both methods use a revised data set produced by replacing observed output levels for all DMUs by output levels that would have rendered each DMU DEA- efficient. Once this has been done, then MRLs (for each output measure) can be found: (a) in the regression based method from the regression coefficients associated with each output measure in the ordinary least-squares linear regression line (b) in the linear programming based method by solving a single linear program designed to ensure that the MRLs obtained will not enable any DMU to attain its DEA-efficient output levels using less resources than DEA suggests is needed for those levels. An efficiency based measure to enable a comparison to be made between alternative sets of MRLs is also presented. (Basso and Peccati [5]) introduced a dynamic programming algorithm to get optimal resource allocation with both minimum and maximum activation levels and fixed costs. Yan et al. [36] discussed a typical inverse optimization problem on the generalized DEA model to identify how to control or adjust the changes in the input and output such that the efficiency index of DMUs concerned is preserved. Beasley [6]developed a resource allocation model aiming to maximize the total efficiency of all DMUs. Lozano and Villa [28] and Lozano et al. [29] introduced the concept of centralized DEA models, which aim at optimizing the combined resource consumption by all units in an organization rather than considering the consumption by each unit separately. Asmild et al. [3] suggested modifying these centralized models to only consider adjustments of previously inefficient unit and showed how this new model formulation relate to a standard DEA model, namely as the analysis of the mean inefficient point. Fang [17] developed a new generalized centralized resource allocation model that extends and generalizes Lozano and Villa 's model [28] and Asmild et al.' s model [3] to a more general case. Hadi-Vencheh et al. [21] used an inverse DEA model for resource allocation in order to estimate increased requirements of the input vector when the output vector is increasing. Amireimoori and Mohaghegh Tabar [2] presented a DEA-based method for allocating fixed resources or costs across a set of decision making units and showed how output targets can be set at the same time as decisions are made about allocating input resources. Bi et al. [7] investigated the resource allocation and target setting for the organization consisting of production units, each of which has several parallel production lines. Wu et al. [35] proposed some new DEA models, which consider both economic and environmental factors in the allocation of a given resource. Li et al. [27] considered the model construction method for resource allocation considering undesirable outputs between different decision making units based on the DEA framework. They proposed some resource allocation models as a multiple objective linear problem which considers the input reduction, desirable output reduction and undesirable output reduction. Hosseinzadeh Lotfi et al. [24] proposed an allocation mechanism that is based on a common dual weights approach. Compared to alternative approaches, their model can be interpreted as providing equal endogenous valuations of the inputs and outputs in the reference set. Du et al. [11] used the cross-efficiency concept in DEA to approach cost and resource allocation problems. Hadi-Vencheh et al.[20] proposed a new method to find how much some inputs/outputs of each decision making unit (DMU) should be reduced such that the total efficiency of all DMUs after reduction being maximized.

It is worth noting that the assumptions that concern the unit's ability to change their input-output mix and efficiency are clearly the key factors affecting the results of the resource allocation. Although many valuable ideas have been proposed concerning these assumptions, the DMUs ability to change their input- output mix and efficiency has not be discussed thoroughly in the literature. In addition, the multiple criteria nature of the resource allocation problem has drawn only limited attention. DEA and multi-objective programming (MOP) can be used as tools in management control and planning. The structures of these two types of models have much in common but DEA is directed to assessing past performances as part of the management control function and MOP to planning further performances Cooper [10]. In order to find the most preferred allocation plan, Korhonen and Syrjanen [26] developed an interactive formal approach based on DEA and MOLP. Their approach concerns the modeling of units abilities to change their production. The authors considered two sets in their model: production possibility set and transformation possibility set. The first set describes all technically feasible production plans while the second describes the units ability to change its production within a planning period. They concluded that their approach can be applied to cases where DM controls only a part of the units.

Nasrabadi et al. [31] presented a model to investigate the resource allocation problem based on efficiency improvement. Their model uses parameters which are not necessarily unique in the case of alternative optimal solution. However, each optimal solution can be applied in the model to achieve performance improvement. This can be a shortcoming of their model, since finding all alternative optimal solutions and solving the model for each one seems unreasonable and time consuming.

Two kinds of factors which often have some relation with each other and play important roles in resource allocation models are economic factors and environmental factors. Economic factors usually refer to the desirable outputs generated in the production process, such as profit. Environmental factors usually refer to the undesirable outputs such as smoke pollution and waste. Jie and Qingxian [25] have proposed some new DEA models which consider not only economic but also environmental factors in the allocation of a given resource.

The purpose of this paper is to develop several algorithms based on MOP and DEA for resource allocation. Here, it is assumed that a central unit simultaneously controls all the units. If the production of additional products seems logical, the DM wants to know how much of additional outputs should be considered for each unit such that the total outputs reach a predetermined level. In the developed algorithms, the unit's abilities to change their production are modeled explicitly. The current input and output values are used to characterize a production possibility set. It is assumed that the units are able to modify their production plan within the production possibility set. These algorithms determine quantities of the consumed input and produced output levels for each DMU, such that the desirable output level is reached. Moreover, the number of efficient units is maximized, simultaneously.

It should be mentioned that in this paper we aim to allocate resources between units of a system which can be efficient but due to different problems such as: lack of proper supervision on the usage of the resources, wasting resources, inefficiency of workers and errors in production line, have become inefficient. These problems can be solved, therefore such units are first detected and then resource allocation is done among them. That is why it was not important to consider how inefficient the units are. The rest of this paper is organized as follows: In Section 2, first some fundamental models and definitions in DEA and MOP are reviewed and then the problem of resource allocation is stated. In Section 3, two algorithms to determine the input-output levels of each unit are introduced such that the maximum production capacity of the system can be used and the number of efficient units is maximized. Section 4, illustrates these algorithms using a numerical example. In Section 5, an empirical example of allocating experts to gas companies is presented to demonstrate the applicability of the proposed framework and exhibit the efficacy of the procedures. The paper ends with the conclusions and future research directions in Section 6.

2. Preliminaries and the statement of the problem

In this section, first a basic DEA model and also the concept of multiobjective linear programming problem are reviewed. Then, the main aim of problem under consideration is stated.

2.1. The input-oriented CCR model. Suppose there is a set of n, DMUs, $\{DMU_j, j = 1, 2, \cdots, n\}$ which produce multiple outputs $y_{rj}(r = 1, 2, \cdots, s)$ by utilizing multiple inputs $x_{ij}(i = 1, 2, \cdots, m)$. Let the inputs and outputs for DMU_j are $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^t$ and $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^t$, respectively. In addition $x_j \in R^m$, $y_j \in R^s$, $x_j > 0$ and $y_j > 0$, $j = 1, 2, \cdots, n$. We define the set of production possibility set (PPS), as $T = \{(x, y)|\ y\ can\ be\ produced\ by\ x\}$ and here we suppose that $T = T_{CCR}$ in which

$$T_{CCR} = \{((x_{1j}, x_{2j} \ldots, x_{mj}), (y_{1j}, y_{2j}, \ldots, y_{sj})) \mid x_{ij} \geq \sum_{j=1}^{n} x_{ij}\lambda_j,$$

$$y_{rj} \leq \sum_{j=1}^{n} y_{rj}\lambda_j,\ i : 1, ..., m;\ r : 1, ..., s; \forall j : \lambda_j \geq 0\}$$

The relative efficiency of the can be obtained by using the following linear programming (LP) model called input-oriented CCR primal model (Charnes et al. [9]):

$$\begin{aligned}
&\min_{\lambda,\theta} \quad \theta \\
&s.t \\
&\qquad \sum_{j=1}^{n} x_{ij}\lambda_j \leq \theta x_{io}, \quad i : 1, ..., m \\
&\qquad \sum_{j=1}^{n} y_{rj}\lambda_j \geq y_{ro}, \quad r : 1, ..., s \\
&\qquad \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{2.1}$$

Model (2.1) measures the efficiency under a constant return to scale (RTS) assumption of technology. In this model, the vector variable $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ exhibits the intensity vector variable. The components of this vector represent the contribution of efficient units to constructing the projection point of inefficient units. It should be noted that in this model, the feasible region is non-empty and the optimal value θ_o satisfies $0 < \theta_o \leqslant 1$.

2.1. Definition. The optimal value θ_o^* of model (2.1) is called the efficiency index of DMU_o. If $\theta_o^* = 1$ then DMU_o is called (at least) weakly efficient unit. If $\theta_o^* \neq 1$ then it is called an inefficient unit.

Now, the index sets of efficient and inefficient units are defined as $E = \{j; \theta_j^* = 1\}$ and $I = \{j; \theta_j^* \neq 1\}$, respectively.

2.2. Multi objective programming. Here, we review some fundamentals of MOP problems and Min-ordering method for solving them, which will be used throughout the remainder of this paper (Ehrgott and Galperin, [14]; Ehrgott,[15]; Hosseinzadeh Lotfi et al. [22], [23]).
The MOP problem can be presented as follows:

$$\begin{array}{ll} max & f(x) = (f_1(x), f_2(x), ..., f_p(x)) \\ s.t. & x \in S \end{array} \tag{2.2}$$

where is the feasible set of the optimization problem (2.2) and $f_k : S \rightarrow \mathbb{R}$ for $k = 1, ..., p$ are objective functions. The primary goal in MOP is to find the Pareto optimal (efficient) solutions and to help select the most preferred solution. In fact, a solution represented by a point in the decision variable space is a strictly Pareto optimal solution if it is not possible to move the point within the feasible region to improve an objective function value without deteriorating at least one of the other objectives. In other words, a feasible solution $\overline{x} \in S$ is called a Pareto optimal solution if there is no feasible solution $x \in S$ such that $f(x) \geq f(\overline{x})$ and $f(x) \neq f(\overline{x})$.

To generate any Pareto optimal solution, MOP(2.2) can be written as a following min-ordering problem:

$$(2.3) \qquad \max_{x \epsilon S} \quad \min_{1 \leq k \leq p} \quad f_k(x)$$

The min-ordering problem (2.3) can be solved as a single objective linear programming problem. If we introduce a variabl φ to the stand for $\min_{1 \leq k \leq p} \quad f_k(x)$ we can rewrite problem 2.3 as follows:

$$(2.4) \qquad \begin{aligned} &\max \ \ \varphi \\ &s.t. \ \ f_k(x) \ \geq \ \varphi; \quad k : 1, ..., p \\ &\qquad x \epsilon S \end{aligned}$$

2.3. Statement of the problem. Now, consider the following question: if the total output of a system wants to increase such that the efficient units remain unchanged, how much of the additional output must be produced by other DMUs?

To answer this question, we suppose the total output of a system, i.e. $Y = \sum_{j=1}^{n} Y_j$ should be increased from Y to β where $\beta \geqslant Y$ and $\beta \neq Y$.
Now, it is required to estimate the output vector Y_j^{new} for every $j \in I$ such that:

$$\sum_{j \in E} Y_j + \sum_{j \in I} Y_j^{new} = \beta$$

In this paper, we try to determine Y_j^{new} for every $j \in I$ such that the total consumed input is minimized by the system.

3. New Algorithms for Resource Allocation

In this section, several algorithms are proposed for solving the resource allocation problem stated in Section 2. In these algorithms, it is tried to determine the different levels of output by use of maximum production power of system. Note that there are systems that do not use their resources efficiently. In such systems, the DM is able to recognize some units as deficient, and if it is possible, change them. In fact we believe that the efficient units use their maximum power, and therefore they will be remained unchanged.

Consider a system where the deficient units are able to alter their input-output composition. Without loss of the generality, assume that that $Card\{I\} = q$, $DMU_1, DMU_2, ...DMU_q \in I$ and $I_o = I \setminus \{o\}$. In continue, several algorithms will be introduced for optimal resource allocation between such units. For achieving this, first the following model is solved to determine the minimum value of k-th output produced by inefficient unit o:

$$
\begin{aligned}
&\min \quad \widehat{y}_{ko}\\
(3.1)\qquad &s.t\\
&\sum_{r\in\{1,2,\ldots,s\}\setminus\{k\}} u_r^o y_{ro} + u_k^o \widehat{y}_{ko} = 1\\
&\sum_{i=1}^{m} v_i^o x_{io} = 1,\\
&\sum_{r=1}^{s} u_r^o y_{rj} - \sum_{i=1}^{m} v_i^o x_{ij} \leqslant\ 0, \qquad j \in I_o,\ E\\
&u_r^o \geq \varepsilon, v_i^o \geq \varepsilon, \quad \forall i, r,\\
&\widehat{y}_{kj} \geqslant y_{kj}, \quad \forall j \in I.
\end{aligned}
$$

In this model u_r^o and v_i^o are the variables of the problem. They represent the input and output weighs of the inefficient unit o, respectively. The constraints refer to the condition in which the inefficient unit o and the efficient unit j remain in the respective and the inefficient unit o is transformed into efficient ones.

Model (3.1) is a non-linear MOP problem. To convert this non-linear model into a linear one, let $p_{ko} = u_k^o \widehat{y}_{ko}$ for all $o \in I$. This leads to the following linear MOP problem:

$$
\begin{aligned}
&\min \quad \widehat{y}_{ko}\\
(3.2)\qquad &s.t\\
&\sum_{r\in\{1,2,\ldots,s\}\setminus\{k\}} u_r^o y_{ro} + p_{ko} = 1, \qquad o \in I\\
&\sum_{i=1}^{m} v_i^o x_{io} = 1, \qquad o \in I\\
&\sum_{r=1}^{s} u_r^o y_{rj} - \sum_{i=1}^{m} v_i^o x_{ij} \leqslant\ 0, \qquad j \in I_o,\ E\\
&u_r^o \geq \varepsilon, v_i^o \geq \varepsilon, \forall i, r\\
&\widehat{y}_{kj} \geqslant y_{kj}, \quad \forall j \in I\\
&p_{ko} \geqslant 0.
\end{aligned}
$$

3.1. Theorem. *Models (3.1) and (3.2) are equal to each other.*

Proof. Suppose $(u^o_{r\,r\neq k}, u^o{}_k, v_i{}^o, \widehat{y}_{ko})$ is the optimal solution of model (3.1), then $(u^o_{r\,r\neq k}, v_i^o,\ p_{ko} = u_k^o \widehat{y}_{ko}, \widehat{y}_{ko})$ is a feasible solution of model (3.2). Now we show that this solution is the optimal solution of model (3.2). By contradiction, suppose that $(u^o_{r\,r\neq k}, v_i^o,\ p_{ko} = u_k^o \widehat{y}_{ko}, \widehat{y}_{ko})$ is not the optimal solution of model (3.2) and there exists $(\widetilde{u}_r^o, r \neq k, \widetilde{v}_i^o,\ \widetilde{p}_{ko}, \widehat{\widehat{y}}_{ko})$ as the optimal solution of model (3.2) and $\widehat{\widehat{y}}_{ko} < \widehat{y}_{ko}$. Therefore, the solution $(\widetilde{u}_r^o, r \neq k,\ \widetilde{u}_k^o = \frac{\widetilde{p}_{ko}}{\widehat{\widehat{y}}_{ko}},\ \widetilde{v}_i^o, \widehat{\widehat{y}}_{ko})$ is a feasible solution of model (3.1). Since $\widehat{\widehat{y}}_{ko} < \widehat{y}_{ko}$ is in contrast with the optimality of $(u^o_{r\,r\neq k}, u_k^o, v_i^o, \widehat{y}_{ko})$, thus $(u^o_{r\,r\neq k}, v_i^o,\ p_{ko} = u_k^o \widehat{y}_{ko}, \widehat{y}_{ko})$ is also the optimal solution of model (3.2).

On the other hand, suppose $(u^o_{r\,r\neq k}, v^o_i,\ p_{ko}, \widehat{y}_{ko})$ is the optimal solution of model (3.2); then it is obvious that $(u^o_{r\,r\neq k},\ u^o_k = \frac{p_{ko}}{\widehat{y}_{ko}},\ v^o_i, \widehat{y}_{ko})$ is also a feasible solution of model (3.1). Lets suppose another solution like $(\overline{u}^o_{r\,r\neq k}, \overline{u}^o_k, \overline{v}^o_i, \overline{\widehat{y}}_{ko})$ as the optimal solution of model (3.1) so that $\overline{\widehat{y}}_{ko} < \widehat{y}_{ko}$. Therefore $(\overline{u}^o_{r\,r\neq k}, \overline{u}^o_k, p_{ko} = \overline{u}^o_k\overline{\widehat{y}}_{ko}, \overline{\widehat{y}}_{ko})$ is a feasible solution of model (3.2) and since $\overline{\widehat{y}}_{ko} < \widehat{y}_{ko}$, there is a contradiction with the previous solution. □

In a similar way, the model (3.2) gives the maximum value of output produced by inefficient unit o:

$$\begin{array}{lll} & \max & \widehat{y}_{ko} \\ (3.3) & s.t & \\ & \multicolumn{2}{l}{Constraints\ of\ model\ (3.2)} \end{array}$$

Suppose that $(\underline{y^*_{1o}}, \underline{y^*_{2o}}, ..., \underline{y^*_{so}})$ and $(\overline{y^*_{1o}}, \overline{y^*_{2o}}, ..., \overline{y^*_{so}})$and for all are optimal solutions of model (3.2) and model (3.3), respectively. We define the following index sets:

$$R_1 = \{r \in \{1,2,...,s\} \mid \beta_r - \sum_{j\in E} y_{rj} \leqslant \sum_{j\in I} \underline{y}^*_{rj}\}$$

$$R_2 = \{r \in \{1,2,...,s\} \mid \sum_{j\in I} \underline{y}^*_{rj} \leqslant \beta_r - \sum_{j\in E} y_{rj} \leqslant \sum_{j\in I} \overline{y}^*_{rj}\}$$

$$R_3 = \{r \in \{1,2,...,s\} \mid \beta_r - \sum_{j\in E} y_{rj} \geqslant \sum_{j\in I} \overline{y}^*_{rj}\}$$

In such situation, two cases may be occur: $R_3 = \phi$ or $R_3 \neq \phi$. These cases are investigated in the following sections, separately.

3.1. Resource allocation when $R_3 = \phi$. If $R_3 = \phi$ then the mentioned system produces the desired level of output without any need to extra resources. In this case, an algorithm is introduced to determine the output levels of each unit, such that the total output is equal to β. In such algorithm, the values of the extra output produced by inefficient units, is determined. The different levels of output are estimated in a way that the number of the inefficient units that are converted to efficient ones will be maximized.

In this algorithm the following symbols are applied:

K: The index set of outputs that amounts of their changes have been determined.

L: The iteration number of algorithm which has been run for the output under consideration.

S^L: The index set of units converted to inefficient unit in the iteration or previous iterations.

$J^i_{r'}$: The index set of units converted to inefficient unit in the iteration during the producing of output .

$O^{r'}$: The index set of units converted to inefficient unit during the producing of output .

$I^{r'}$: The index set of units that are able to change the output.

Algorithm1: Estimation output levels for inefficient units when $R_1 \cup R_2 \neq \phi$

Step 1: Set $K = \phi$ and $L = 1$.

Step 2: Assume that $r' = \arg max_{r \in R_1 \cup R_2 \setminus K} \{|\beta_r - \sum_{j \in E} y_{rj}|\}$.

Set $S^0 = J^0_{r'} = O^{r'} = \phi$. Define $Y^0_{r'} = \sum_{j \in I} \underline{y}^*_{r'j} + \sum_{j \in E} y'^j_r$ and

$$I^{r\prime} = \begin{cases} I; & if\ K = \phi\ or\ K \subset R_2 \\ \bigcup_{r \in K} O^{r\prime}; & if\ K \neq \phi \end{cases}$$

If $r\prime \in R_1$ go to Step 3; otherwise if $r\prime \in R_2$ go to Step 6.

Step 3: Since $r\prime \in R_1$, the level of output $r\prime$ is less than the sum of minimum $r\prime$-th outputs which could be generated by inefficient units. In this case, let $y_{r\prime j} = \underline{y}^*_{r\prime j}$ for $j \in I$ and compute $Y^{L,K}_{r\prime} = Y^{L-1}_{r\prime} - \underline{y}^*_{r\prime k} + y_{r\prime k}$ for every $k \in I^{r\prime}$.

Note that in this case for every $j \in I$, $y_{r\prime j}$ has been replaced with $\underline{y}^*_{r\prime j}$. Thus based on model (3.2) these DMUs are converted to efficient units.

By definition of $Y^{L,K}_{r\prime}$, we attempt to find those DMUs that by substituting $y_{r\prime j}$ instead of $\underline{y}^*_{r\prime j}$, the total output is reached to $\beta_{r\prime}$. If $Y^{L,K}_{r\prime} = \beta_{r\prime}$ then set $S^L = S^{L-1} \cup k$ and go to Step 4, otherwise; go to Step 5.

Step 4: For every $t \in S^L$, the $r\prime$-th output of DMUs is obtained as follows:

$$\widehat{y}_{r\prime j} = \begin{cases} y_{r\prime j}; & j \in E \\ \underline{y}^*_{r\prime j}; & j \in I^{r\prime} \setminus \{\cup^{L-1}_{i=0} J^i_{r\prime}, t\} \\ y_{r\prime j}; & j \in \{\cup^{L-1}_{i=0} J^i_{r\prime}, t\} \\ \underline{y}^*_{r\prime j}; & j \in I \setminus I^{r\prime} \end{cases}$$

Now set $O^{r\prime} := \{\cup^{L-1}_{i=0} J^i_{r\prime}, t\}$ and $K := K \cup \{r\prime\}$. If $K = \{1, 2, ..., s\}$ then stop, otherwise; if $R_1 \cup R_2 \neq \phi$ go to Step 2, else if $R_3 \neq \phi$ run Algorithm 2.
Step 5: Assume that $g_1 = argmin_{k \in I^{r\prime}} \{|\ \beta_{r\prime} - Y^{L,k}_{r\prime}\ |\}$. Consider the following cases:

Case1. $\beta_{r\prime} < Y^{L,g_1}_{r\prime}$: In this case set $Y^L_{r\prime} = Y^{L,g_1}_{r\prime}$ and consider the following sub cases:

Case1.1. $Y^L_{r\prime} < Y^{L-1}_{r\prime}$: In this case let

$$J_{r\prime}^{L} := J_{r\prime}^{L-1} \cup g_1, \quad O^{r\prime} := O^{r\prime} \cup \{\cup_{i=0}^{L-1} J_{r\prime}^{i}, g_1\}, \quad L := L+1, \quad K := K \cup \{r\prime\}$$

If $L > card(\cup_{r \in K} O^{r})$ then set $I^{r\prime} := I \setminus (\cup_{r \in K} O^{r})$ else set $I^{r\prime} := \cup_{r \in K} O^{r}$ and go to Step 3.

Case1.2. $Y_{r\prime}^{L} = Y_{r\prime}^{L-1}$: In this case compute $y_{r\prime j} + \beta_{r\prime} - Y_{r\prime}^{L,j}$ for all $j \in \{\cup_{i=0}^{L} J_{r\prime}^{i}\} \setminus \{\cup_{i=0}^{L-1} J_{r\prime}^{i}\}$.

If all of these values are positive then set $J_{r\prime}^{L} := J_{r\prime}^{L-1} \cup g_1$ and

$$\widehat{y}_{r\prime j} = \begin{cases} y_{r\prime j}; & j \in E \\ \{y_{r\prime j} + \beta_{r\prime} - Y_{r\prime}^{L,j} + (c-1)\underline{y}_{r\prime j}^{*}\} \times \dfrac{1}{c}; & j \in J_{r\prime}^{L} \\ \underline{y}_{r\prime j}^{*}; & j \in I^{r\prime} \setminus \{\cup_{i=0}^{L-1} J_{r\prime}^{i}\} \\ y_{r\prime j}; & j \in \{\cup_{i=0}^{L-1} J_{r\prime}^{i}\} \\ \\ \underline{y}_{r\prime j}^{*}; & j \in I \setminus I^{r\prime} \end{cases}$$

where $c = Card\{J_{r\prime}^{L}\}$. Now set $L := L+1$, $O^{r\prime} := O^{r\prime} \cup \{\cup_{i=0}^{L-1} J_{r\prime}^{i}\}$ and $K := K \cup \{r\prime\}$. If $K = \{1, 2, ..., s\}$ then stop, else go to Step 2.

If all values of $y_{r\prime j} + \beta_{r\prime} - Y_{r\prime}^{L,j}$ are not positive then there exist $h \in J_{r\prime}^{L}$ such that $\{y_{r\prime h} + \beta_{r\prime} - Y_{r\prime}^{L,h} + (c-1)\underline{y}_{r\prime h}^{*}\} \times \dfrac{1}{c} = 0$. In this case put $\widehat{y}_{r\prime h} = y_{r\prime h}$ and remove DMU_h from the system and substitute $\beta_{r\prime}$ and $O_{r\prime}$ by $\beta_{r\prime} - y_{r\prime h}$ and $O_{r\prime} \cup \{h\}$, respectively. Finally run Algorithm1 for the new system.

Case2. $\beta_{r\prime} > Y_{r\prime}^{L,g_1}$: In this case set

$$\widehat{y}_{r\prime j} = \begin{cases} y_{r\prime j}; & j \in E \\ \underline{y}_{r\prime j}^{*}; & j \in I^{r\prime} \setminus \{\cup_{i=0}^{L-1} J_{r\prime}^{i}, g_1\} \\ y_{r\prime g_1} + \beta_{r\prime} - Y_{r\prime}^{L,g_1}; & j = g_1 \\ y_{r\prime j}; & j \in \{\cup_{i=0}^{L-1} J_{r\prime}^{i}\} \\ \\ \underline{y}_{r\prime j}^{*}; & j \in I \setminus I^{r\prime} \end{cases}$$

Now, set $O^{r\prime} := O^{r\prime} \cup \{\cup_{i=0}^{L-1} J_{r\prime}^{i}, g_1\}$ and $K := K \cup \{r\prime\}$. If $K = \{1, 2, ..., s\}$ then stop, otherwise; if $R_1 \cup R_2 \neq \phi$ then set $L = 1$ go to Step 2, else if $R_1 \cup R_2 = \phi$ and $R_3 \neq \phi$ run Algorithm 2.

Step 6: Since $r' \in R_2$ then it is possible to determine the output level of all inefficient units such that they become efficient. To do this, the following model is solved. In this model without loss of the generality, we suppose that $\{o_1, o_2, ..., o_q'\} \in I^{r'}$ and $I_o^{r'} = I^{r'} \setminus \{o\}$.

$$
\begin{aligned}
&\max \quad \{\underline{y}^*_{r'o_1} + t_{r'o_1}(\overline{y}^*_{r'o_1} - \underline{y}^*_{r'o_1}),\ \underline{y}^*_{r'o_2} + t_{r'o_2}(\overline{y}^*_{r'o_2} - \underline{y}^*_{r'o_2}),\ \ldots,\ \underline{y}^*_{r'o'_q} + \\
&\qquad t_{r'o'_q}(\overline{y}^*_{r'o'_q} - \underline{y}^*_{r'o'_q})\} \\
&s.t \\
&\sum_{r \in K \setminus \{r'\}} u^o_r \widehat{y}_{ro} + u^o_{r'} \underline{y}^*_{r'o} + u^o_{r'} t_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o}) = 1; \quad o \in I^{r'} \\
&\sum_{i=1}^{m} v^o_i x_{io} = 1; \quad o \in I^{r'} \\
&\sum_{r \in K \setminus \{r'\}} u^o_r \widehat{y}_{rj} - \sum_{i=1}^{m} v^o_i x_{ij} \leqslant 0; \quad j \in I^{r'}_o \\
&\sum_{r \in K} u^o_r \widehat{y}_{rj} - \sum_{i=1}^{m} v^o_i x_{ij} \leqslant 0, \quad j \in E \\
&\sum_{r \in I^{r'}} \underline{y}^*_{r'o} + t_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o}) + \sum_{j \in E} y_{r'j} = \beta_{r'} \\
&u^o_r \geqslant \varepsilon, \quad v^o_i \geqslant \varepsilon \quad \forall i, r \quad, \forall o \in I^{r'} \\
&0 \leqslant t_{r'o} \leqslant 1; \quad \forall j \in I^{r'}.
\end{aligned}
\tag{3.4}
$$

In this model $\widehat{y}_{rj}$ and $\widehat{y}_{ro}$ for $r \in K \setminus \{r'\}$, $j \in I^{r'}_o$, $o \in I^{r'}$ are obtained from the previous steps. Note that for some values of $\widehat{y}_{rj}$ and $\widehat{y}_{ro}$ which are not obtained from the previous steps set $\widehat{y}_{rj} := \underline{y}^*_{rj}$ and $\widehat{y}_{ro} := \underline{y}^*_{ro}$.

If $K = \phi$ then set $\widehat{y}_{rj} := \underline{y}^*_{rj}$ and $\widehat{y}_{ro} := \underline{y}^*_{ro}$ for all $j, o \in I$. Also, set $\widehat{y}_{rj} := y_{rj}$ for all $j \in E$. By substituting $P^o_{r'j} = u^o_{r'} t_{r'j}$ the non-linear multi objective model (3.4) is converted to the following linear multi objective model:

$$
\begin{aligned}
&\max \quad \{\underline{y}^*_{r'o_1} + t_{r'o_1}(\overline{y}^*_{r'o_1} - \underline{y}^*_{r'o_1}), \underline{y}^*_{r'o_2} + t_{r'o_2}(\overline{y}^*_{r'o_2} - \underline{y}^*_{r'o_2}), \ldots, \underline{y}^*_{r'o'_q} + \\
&t_{r'o'_q}(\overline{y}^*_{r'o'_q} - \underline{y}^*_{r'o'_q})\} \\
&s.t \\
&\sum_{r \in K \setminus \{r'\}} u^o_r \widehat{y}_{ro} + u^o_{r'} \underline{y}^*_{r'o} + P^o_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o}) = 1; \quad o \in I^{r'} \\
&\sum_{i=1}^{m} v^o_i x_{io} = 1; \quad o \in I^{r'} \\
&\sum_{r \in K \setminus \{r'\}} u^o_r \widehat{y}_{rj} - \sum_{i=1}^{m} v^o_i x_{ij} \leqslant 0; \quad j \in I^{r'}_o
\end{aligned}
\tag{3.5}
$$

$$\sum_{r\in K} u_r^o \widehat{y}_{rj} - \sum_{i=1}^{m} v_i^o x_{ij} \leqslant\ 0, \qquad j\in E$$
$$\sum_{r\in I^{r'}} \underline{y}^*_{r'o} + t_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o}) + \sum_{j\in E} y_{r'j} = \beta_{r'}$$
$$u_r^o \geqslant \varepsilon, \quad v_i^o \geqslant \varepsilon \quad \forall i, r \quad , \forall o \in I^{r'}$$
$$0 \leqslant t_{r'o} \leqslant 1; \quad \forall j \in I^{r'}$$
$$0 \leqslant P^o_{r'j} \leqslant u^o_{r'}; \quad \forall j \in I^{r'}$$

3.2. Theorem. *Models (3.4) and (3.5) are equal to each other.*

Proof. It is similar to the proof of Theorem 3.1. □

Model (3.5) can be rewritten as follows:

$$\max\ \min\{\underline{y}^*_{r'o_1} + t_{r'o_1}(\overline{y}^*_{r'o_1} - \underline{y}^*_{r'o_1}), \underline{y}^*_{r'o_2} + t_{r'o_2}(\overline{y}^*_{r'o_2} - \underline{y}^*_{r'o_2}), ..., \underline{y}^*_{r'o'_q} + t_{r'o'_q}(\overline{y}^*_{r'o'_q} - \underline{y}^*_{r'o'_q})\}$$
$$s.t\ \ \textit{Constraints of model}\ (3.5). \tag{3.6}$$

Let us assume

$$t = \min\{\underline{y}^*_{r'o_1} + t_{r'o_1}(\overline{y}^*_{r'o_1} - \underline{y}^*_{r'o_1}), \underline{y}^*_{r'o_2} + t_{r'o_2}(\overline{y}^*_{r'o_2} - \underline{y}^*_{r'o_2}), ..., \underline{y}^*_{r'o'_q} + t_{r'o'_q}(\overline{y}^*_{r'o'_q} - \underline{y}^*_{r'o'_q})\}$$

Thus, model (3.6) can be rewritten as follows:

$$\begin{aligned} &\max\ t \\ &s.t \\ &t \leqslant \underline{y}^*_{r'o_1} + t_{r'o_1}(\overline{y}^*_{r'o_1} - \underline{y}^*_{r'o_1}), \\ &t \leqslant \underline{y}^*_{r'o_2} + t_{r'o_2}(\overline{y}^*_{r'o_2} - \underline{y}^*_{r'o_2}), \\ &\vdots \\ &t \leqslant \underline{y}^*_{r'o'_q} + t_{r'o'_q}(\overline{y}^*_{r'o'_q} - \underline{y}^*_{r'o'_q}) \\ &\textit{Constraints of model}\ (3.6). \end{aligned} \tag{3.7}$$

Assuming that $t^*_{r'o}$ are the optimal solutions of model (3.7), the values of output r' for inefficient units are determined as follows:

$$\widehat{y}_{r'o} = \underline{y}^*_{r'o} + t^*_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o}); \quad \forall o \in I$$

Now, set $K := K \cup \{r'\}$. If $K = \{1, 2, ..., s\}$ then stop, otherwise; if $R_1 \cup R_2 \neq \phi$ then set $L = 1$ and go to Step 2, else if$R_1 \cup R_2 = \phi$ and $R_3 \neq \phi$ run Algorithm 2.

3.2. Resource allocation when $R_3 \neq \phi$. If $R_3 \neq \phi$ then the system cannot generate the desired output even if all the inefficient units are converted to efficient units. In such situations, the extra resources must be distributed among the units in order to produce the desired output. In such cases, after the running of Algorithm 1 and finding $\widehat{y}_{rj}$ for $j \in I$ and $r \in R_1 \cup R_2$, there would be $r' \in \{1,2,...,s\}$ such that its desired level of output, with the present resources, cannot be generated by the system; even if all of the deficient unit are converted to efficient ones. As a result, Algorithm 2 would be run in order to determine the different levels of input and output for the various units, in a way that it becomes possible to generate the desired level of output.

Algorithm 2: Estimation input-output levels for inefficient units when $R_3 \neq \phi$

Note that in this case, a new unit as DMU_{n+1} is added to system such that its outputs in every level $r \in \{1,2,...,s\} \setminus \{\widetilde{r}\}$, is equal to zero but $y_{r,n+1}$ is equal to $\alpha_{\widetilde{r}}$, After solving the correspond model, the minimum inputs that DMU_{n+1} is required to produce $\alpha_{\widetilde{r}}$ are determined. The $\gamma_{i,n+1}$ for $i : 1,2,\cdots,m$ gives the minimum value of the i -th input consumed by system to produce $\alpha_{\widetilde{r}}$.

Step 1: Assume that

$$\alpha_{\widetilde{r}} = \arg max_{r' \in R_3}\{\beta_{r'} - (\sum_{j\in E} y_{r'j} + \sum_{j \in I} \overline{y}^*_{r'j})\}.$$

It is worth noting that $\alpha_{\widetilde{r}}$ is the maximum value of output that has not been generated yet.

Step 2: Solve the following model in order to determine the minimum required values of systems inputs for generating $\alpha_{\widetilde{r}}$:

$$\begin{aligned}
&\min \quad \{p_{1,n+1}, p_{2,n+1}, ..., p_{m,n+1}\}\\
&s.t\\
&\sum_{r=1}^{s} u_r^{n+1}\widehat{y}_{rj} - \sum_{i=1}^{m} v_i^{n+1}x_{ij} \leqslant 0; \quad j \in E\\
&\sum_{r\in R_1\cup R_2} u_r^{n+1}\widehat{y}_{rj} + \sum_{r \in R_3} u_r^{n+1}\overline{y}^*_{rj} + \sum_{r \notin \cup_{j=1}^{3} R_j} u_r^{n+1} y_{rj} - \sum v_i^{n+1}x_{ij} \leqslant 0; \quad j \in I\\
&u_r^{n+1}\alpha_{\widetilde{r}} - \sum_{i=1}^{m} p_{i,n+1} = 0\\
&u_r^{n+1} \geqslant \varepsilon, \quad v_i^{n+1} \geqslant \varepsilon, \quad p_{i,n+1} \geqslant \varepsilon, \quad r:1,2,...,s \quad i:1,2,...,m.
\end{aligned} \tag{3.8}$$

In model (3.8), the values of $\overline{y}^*_{rj}$ and $\widehat{y}^*_{rj}$ are given based on model (3.2) and Algorithm 1, respectively. If Algorithm (1) is not applied set $\widehat{y}_{rj} := y_{rj}(r:1,2,...,s)$. Assuming$(p^*_{1,n+1}, p^*_{2,n+1}, ..., p^*_{m,n+1}, v_1^{n+1*}, v_2^{n+1*}, ..., v_m^{n+1*}, u_1^{n+1*}, u_2^{n+1*}, ..., u_m^{n+1*})$ is a strongly efficient solution of model (3.8), set $\gamma^*_{i,n+1} = \dfrac{{p_{i,n+1}}^*}{v_i^{n+1*}}(i:1,...,m)$. In this case the minimum value of the required resources for generating the total output is given by:

$$\Gamma = (\sum_{j=1}^{n} x_{1j} + \gamma^*_{1,n+1}, \sum_{j=1}^{n} x_{2j} + \gamma^*_{2,n+1}, ..., \sum_{j=1}^{n} x_{mj} + \gamma^*_{m,n+1})$$

Step 3: Now we determine how the extra resources $\gamma_{i,n+1}$ for $i:1,2,\cdots,m$ should be allocated between the various units and how much extra output should be generated by each unit such that the amount of total output become β . To do this, solve the following model:

$$
\begin{aligned}
&\max \quad \{\theta'_1,\theta'_2,...,\theta'_\xi\}\\
&\max \quad \{\Delta y_{\widetilde{r}1},\Delta y_{\widetilde{r}2},...,\Delta y_{\widetilde{r}n}\}\\
&s.t \quad \sum_{j=1}^{n}\lambda_j^k\widehat{y}_{rj}\geqslant\widehat{y}_{rk},\quad r\in R_1\cup R_2,\ k:1,2,...,n\\
&\qquad \sum_{j\in E}\lambda_j^k y_{\widetilde{r}j}+\sum_{j\in I}\lambda_j^k\overline{y}^*_{\widetilde{r}j}+\xi^k_{\widetilde{r}j}\geqslant\overline{y}^*_{\widetilde{r}k}+\Delta y_{\widetilde{r}k},\quad k:1,2,...,n\\
&\qquad \sum_{j=1}^{n}\lambda_j^k\overline{y}^*_{rj}\geqslant\overline{y}^*_{rk},\quad r\in R_3\setminus\widetilde{r},\ k:1,2,...,n\\
&\qquad \sum_{j=1}^{n}\lambda_j^k y_{rj}\geqslant y_{rk},\quad r\notin R_1\cup R_2\cup R_3,\ k:1,2,...,n\\
&\qquad \sum_{j=1}^{n}\lambda_j^k x_{ij}+\delta^k_{ij}\leqslant\theta'_k x_{ik}+\mu_{ik},\quad k\in\bigcup_{r\in R_1}O^r,\ i:1,2,...,m\\
&\qquad \sum_{j=1}^{n}\lambda_j^k x_{ij}+\delta^k_{ij}\leqslant\theta_k(x_{ik}+\Delta x_{ik}),\quad k\in\{1,2,...,n\}\setminus\bigcup_{r\in R_1}O^r,\ i:1,2,...,m\\
&\qquad \sum_{j=1}^{n}(x_{ij}+\Delta x_{ij})=\Gamma_{i,n+1},\quad i:1,2,...,m\\
&\qquad \sum_{j=1}^{n}\Delta y_{\widetilde{r}j}=\alpha_{\widetilde{r}}\\
&\qquad \theta'_k\geqslant\theta_k,\quad k\in\bigcup_{r\in R_1}O^r\\
&\qquad \xi^k_{\widetilde{r}j}\geqslant 0,\quad \delta^k_{ij}\geqslant 0,\mu_{ik}\geqslant 0,\quad \lambda_j^k\geqslant 0,\quad lli,j,k.
\end{aligned}
\tag{3.9}
$$

where $\xi^k_{\widetilde{r}j}=\lambda_j^k\Delta y_{\widetilde{r}j}$, $\delta^k_{ij}=\lambda_j^k\Delta x_{ij}$ and $\mu_{ik}=\theta'_k\Delta x_{ik}$ for $i:1,2,...,m$ and $k,j:1,2,...,n$. It is worth noting that in model (3.7) $\xi=card(\bigcup_{r\in R_1}O^r)$ if $R_1\neq\phi$ and $\xi=card(I)$ if $R_1=\phi$.

In model (3.9), θ_k for $k\in\bigcup_{r\in R_1}O^r$ is the efficiency value of DMU_k given by model (2.1) by putting $y_{rj}:=\overline{y}^*_{rj}$ for $r\in R_3$ and $y_{rj}:=\widehat{y}_{rj}$ for $r\in R_1\cup R_2$. Set $R_3:=R_3\setminus\{\widetilde{r}\}$. If $R_3=\emptyset$then stop; otherwise go to Step 1.

The MOP models are valuable and useful since when under the same constraints they can address several objectives. In some models of the presented paper such as (3.8) and (3.9) where we want to find out the least amount of inputs to produce the intended outputs, it saves us some calculation and time if we solve a m objective problem instead of solving m different problems (for each input).

4. Numerical Example

4.1. Example. Consider the data reported in Table 1 with five DMUs that consume two inputs to produce two outputs.

Table 1. The raw data

DMU	Input 1	Input 2	Output 1	Output 2
1	19	131	150	50
2	27	168	180	72
3	55	255	230	90
4	31	206	152	80
5	50	268	250	100

The efficiency score of each DMU obtained by model (2.1) is reported in Table 2.

Table 2. Efficiency Scores

DMU	1	2	3	4	5
Efficiency	1	1	0.83	0.97	0.87

Thus, we have $E = \{1, 2\}$ and $I = \{3, 4, 5\}$. We first determine the minimum amount of kth output which can be produced by inefficient units according to model (3.2). For example, the corresponding model to find the minimum amount of the first output for DMU_3 is as follows

$$
\begin{aligned}
& \min\ \widehat{y}_{13} \\
(4.1)\quad & s.t \\
& 90u_2^3 + p_{13} = 1; \\
& 55v_1^3 + 255v_2^3 = 1; \\
& 152u_1^3 + 80u_2^3 - 31v_1^3 - 206v_2^3 \leqslant 0; \\
& 250u_1^3 + 100u_2^3 - 50v_1^3 - 268v_2^3 \leqslant 0; \\
& 150u_1^3 + 50u_2^3 - 19v_1^3 - 131v_2^3 \leqslant 0; \\
& 180u_1^3 + 72u_2^3 - 27v_1^3 - 168v_2^3 \leqslant 0; \\
& u_i^3 \geqslant 0.00001 \quad v_i^3 \geqslant 0.00001;\ i : 1, 2, \\
& \widehat{y}_{13} \geqslant 230;\ p_{13} \geqslant 0
\end{aligned}
$$

The optimal solution of model (4.1) is as follows:

$$
\begin{aligned}
& \widehat{y}_{13} = 291.9847 \quad p_{13}^1 = 1.00000000939939 \\
& u_2^3 = 0.00001 \quad u_1^3 = 0.3424837E - 02 \\
& v_1^3 = 0.00001 \quad v_2^3 = 0.3921569E - 02
\end{aligned}
$$

In addition, we can determine the maximum amount of kth output which can be produced by inefficient units according to model (3.3). For example, the corresponding model to find the maximum amount of the second output for DMU_4 is as follows:

$$
\begin{aligned}
& \max\ \widehat{y}_{24} \\
& s.t \\
& p_{24} + 152u_1^4 = 1; \\
& 31v_1^4 + 206v_2^4 = 1; \\
& 230u_1^4 + 90u_2^4 - 55v_1^4 - 255v_2^4 \leqslant 0; \\
& 250u_1^4 + 100u_2^4 - 50v_1^4 - 268v_2^4 \leqslant 0; \\
& 150u_1^4 + 50u_2^4 - 19v_1^4 - 131v_2^4 \leqslant 0; \\
& 180u_1^4 + 72u_2^4 - 27v_1^4 - 168v_2^4 \leqslant 0; \\
& u_i^4 \geqslant 0.00001 \quad v_i^4 \geqslant 0.00001;\ i: 1, 2, \\
& \widehat{y}_{24} \geqslant 80; p_{24} \geqslant 0
\end{aligned} \tag{4.2}
$$

The optimal solution of model (4.2) is as follows:

$$
\begin{aligned}
& \widehat{y}_{24} = 132.0593 \quad p_{24} = 0.5769581016676 \\
& u_2^4 = 0.4368932E-02 \quad u_1^4 = 0.2783172E-02 \\
& v_1^4 = 0.00001 \quad v_2^4 = 0.4854369E-02
\end{aligned}
$$

In a similar way it is possible to find the maximum and minimum amount of the all outputs for other units. The results are presented in Table 3.

Table 3. Maximum and minimum amount of outputs

DMU	Minimum output 1	Maximum output 1	Minimum output 2	Maximum output 2
3	291.985	12750	109.2858	150.5714
4	233.7211	300	82.6667	132.059
5	306.87	7500	114.857	138.519

Now suppose that decision makers want to increase the total output according to following case: The first output of total output is increased to $\beta_1 = 20000$ and the second output of total output is increased to $\beta_2 = 400$.

Now it is required to determine the amount of output produced by each DMU to reach the desired total output. Here, based on maximum and minimum amount of outputs of each DMU, the index sets of R_1, R_2 and R_3 are defined as follows:

$$
\begin{aligned}
R_1 &= \{r \in \{1,2\} \mid \beta_r - \sum_{j\in E} y_{rj} \leqslant \sum_{j\in I} \underline{y}^*_{rj}\} = \{2\} \\
R_2 &= \{r \in \{1,2\} \mid \sum_{j\in I} \underline{y}^*_{rj} \leqslant \beta_r - \sum_{j\in E} y_{rj} \leqslant \sum_{j\in I} \overline{y}^*_{rj}\} = \{1\} \\
R_3 &= \{r \in \{1,2\} \mid \beta_r - \sum_{j\in E} y_{rj} \geqslant \sum_{j\in I} \overline{y}^*_{rj}\} = \phi
\end{aligned}
$$

Since $R_3 = \emptyset$, Algorithm 1 is used for resource allocation. The steps of this algorithm are given as follows:

Step 1: $K = \emptyset$ $L = 1$.

Step 2:

$$r' = \arg max_{r\in\{1,2\}}\{|\beta_r - \sum_{j\in E} y_{rj}|\} = \arg max_{r\in\{1,2\}}\{|20000-330|, |400-122|\} = 1$$

$$S^0 = J_1^0 = O^1 = \emptyset, \quad Y_1^0 = 1162.5761, \; I^1 = \{3, 4, 5\}$$

Since $r' = 1$ then Step 6 is applied.

Step 6: In this step model (4.3) is solved to determine the output level of all inefficient units.

$$
\begin{aligned}
&\max \quad t \\
(4.3) \qquad & s.t \\
& t \leqslant 291.985 + t_{13}(12750 - 291.985) \\
& t \leqslant 233.7211 + t_{14}(300 - 233.7211) \\
& t \leqslant 306.87 + t_{15}(7500 - 306.87) \\
& 291.985u_1^3 + 90u_2^3 + p_{13}^3(12750 - 291.985) = 1 \\
& 233.7211u_1^4 + 80u_2^4 + p_{14}^4(300 - 233.7211) = 1 \\
& 306.87u_1^5 + 100u_2^5 + p_{15}^5(7500 - 306.87) = 1 \\
& 55v_1^3 + 255v_2^3 = 1 \\
& 31v_1^4 + 206v_2^4 = 1 \\
& 50v_1^5 + 268v_2^5 = 1 \\
& 233.7211u_1^3 + 80u_2^3 - 31v_1^3 - 206v_2^3 \leqslant 0 \\
& 306.87u_1^3 + 100u_2^3 - 50v_1^3 - 268v_2^3 \leqslant 0 \\
& 291.985u_1^4 + 90u_2^4 - 55v_1^4 - 255v_2^4 \leqslant 0 \\
& 306.87u_1^4 + 100u_2^4 - 50v_1^4 - 268v_2^4 \leqslant 0 \\
& 291.985u_1^5 + 90u_2^5 - 55v_1^5 - 255v_2^5 \leqslant 0 \\
& 233.7211u_1^5 + 80u_2^5 - 31v_1^5 - 206v_2^5 \leqslant 0 \\
& 150u_1^3 + 50u_2^3 - 19v_1^3 - 131v_2^3 \leqslant 0 \\
& 180u_1^3 + 72u_2^3 - 27v_1^3 - 168v_2^3 \leqslant 0 \\
& 150u_1^4 + 50u_2^4 - 19v_1^4 - 131v_2^4 \leqslant 0 \\
& 180u_1^4 + 72u_2^4 - 27v_1^4 - 168v_2^4 \leqslant 0 \\
& 150u_1^5 + 50u_2^5 - 19v_1^5 - 131v_2^5 \leqslant 0 \\
& 180u_1^5 + 72u_2^5 - 27v_1^5 - 168v_2^5 \leqslant 0 \\
& 291.985 + t_{13}(12750 - 291.985) + 233.7211 + t_{14}(300 - 233.7211) + \\
& 306.87 + t_{15}(7500 - 306.87) + 330 = 20000 \\
& u_i^j \geqslant 0.00001 \quad v_i^j \geqslant 0.00001;\ i:1,2, j:3,4,5 \\
& 0 \leqslant t_{1k} \leqslant 1; k:3,4,5 \\
& p_{1k}^k \geqslant 0; k:3,4,5
\end{aligned}
$$

The optimal solution of model (4.3) is as follows:

$t = 300.0000 \quad t_{13} = 0.9470239 \quad t_{14} = 1.000000 \quad t_{15} = 0.9694121$

$u_{13} = 0.8271283E-04 \quad u_{23} = 0.00001 \quad u_{14} = 0.333333E-02 \quad u_{24} = 0.00001$

$u_{15} = 0.1373631E-03 \quad u_{25} = 0.00001 \quad v_{13} = 0.00001 \quad v_{23} = 0.3921569E-02$

$v_{14} = 0.3225806E-01 \quad v_{24} = 0.00001 \quad v_{15} = 0.00001 \quad v_{25} = 0.3731343E-02$

$p_{13}^3 = 0.000078331026 \quad p_{14}^4 = 0.0033333 \quad p_{15}^5 = 0.00013316451233$

According to optimal solution of model (4.3) and according to Equation $\widehat{y}_{r'o} = \underline{y}^*_{r'o} + t^*_{r'o}(\overline{y}^*_{r'o} - \underline{y}^*_{r'o})$; $\forall o \in I$, the output levels of all inefficient units can be determined in this case. The results are presented in Table 4.

Table 4. Resource allocation of case 1

DMU:	1	2	3	4	5
*Output*1:	150	180	12090.0229515585	300	7279.977258873

Now, we set $K = \{1\}$. Since $R_1 \cup R_2 = \{1,2\} \neq \emptyset$ thus we go to Step 2.

Step 2:

$$r' = \arg max_{r \in \{1,2\}}\{|\beta_r - \sum_{j \in E} y_{rj}|\} = \arg max_{r \in \{2\}}\{|400 - 122|\} = 2$$

$$S^0 = J_2^0 = O^2 = \emptyset, \ \ Y_2^0 = 428.8095, \ I^2 = \{3,4,5\}$$

Since $r' \in R_1$ thus we go to Step 3.

Step 3: we set $y_{23} = 109.2858$ $y_{24} = 82.6667$, $y_{25} = 114.857$ and compute $Y_2^{1,k} = Y_2^0 - \underline{y}^*_{2k} + y_{2k}$ for every $k \in I^2$. We have:

$$Y_2^{1,3} = 428.8095 - 109.2858 + 90 = 409.5237$$
$$Y_2^{1,4} = 428.8095 - 82.6667 + 80 = 426.1428$$
$$Y_2^{1,5} = 428.8095 - 114.857 + 100 = 413.9525$$

Since $Y_2^{1,k} \neq 400$ for every $k \in I$ thus we go to Step 5.

Step 5: $g_1 = argmin_{k \in I^2}\{| \ \beta_2 - Y_2^{1,k} \ |\} = argmin\{9.5237, 26.1428, 13.9525\} = 3$. Since $Y_2^{1,3} = 409.5237 > 400 = \beta_2$ therefore we set $Y_2^1 := 409.5237$ and because of $Y_2^1 := 409.5237 < 428.8095 = Y_2^0$ let:

$$J_2^1 = J_2^0 \cup g_1 = \{3\}, \ O^2 := O^2 \cup \{\bigcup_{i=0}^{L-1} J_2^i, g_1\} = \{3\}, \ L := L+1, \ K = K \cup \{2\}$$

Since $L = 2 > 0 = card(\bigcup_{r \in \{1\}} O^1)$ then we set $I^2 := I \setminus (r \in \{1\}O^1) = \{3,4,5\}$ and go to Step3.

Step 3: By computing $Y_2^{2,k} = Y_2^1 - \underline{y}^*_{2k} + y_{2k}$ for every $k \in I^2$ we have:

$$Y_2^{2,3} = 409.5237 - 109.2858 + 90 = 390.2379$$
$$Y_2^{2,4} = 409.5237 - 82.6667 + 80 = 406.857$$
$$Y_2^{2,5} = 409.5237 - 114.857 + 100 = 394.6667$$

Since $Y_2^{2,k} \neq 400$ for every $k \in I$ thus we go to Step 5.

Step 5: $g_1 = argmin\{9.7621, 6.857, 5.3333\} = 5$.
$Y_2^{2,5} = 394.6667 < 400 = \beta_2$. Therefore we consider Case 2.

$$\widehat{y}_{2j} = \begin{cases} y_{2j}; & j \in E \\ \underline{y}^*_{r\prime j}; & j \in I^2 \setminus \{\cup_{i=0}^{1} J_2^i, 5\} \\ y_{25} + \beta_2 - Y_2^{2,5}; & j = 5 \\ y_{2j}; & j \in \{\cup_{i=0}^{1} J_2^i\} \\ & \\ \underline{y}^*_{2j}; & j \in I \setminus I^2 \end{cases}$$

and set $O^2 := O^2 \cup \{\bigcup_{i=0}^{1} J_2^i, 5\}$ and $K = K \cup \{2\}$. Since $K = \{1, 2\}$ then the process is stopped. The results are presented in Table 5.

Table 5. Results of Algorithm 1 for case 1

DMU:	1	2	3	4	5
*Output*2:	50	72	90	82.6667	105.3333

4.2. Example. Application in Gas Companies

In this section, we illustrate the resource allocation discussed in this paper with the analysis of gas companies activity. This example is taken from Amireimoori and Mohaghegh Tabar [2]. The data set consists of 20 gas companies located in 18 regions in Iran. The data for this analysis are derived from operations during 2005. There are six variables from the data set as inputs and outputs in this example. Inputs include capital (x_1), number of staff (x_2), and operational costs (excluding staff costs) (x_3) and outputs include number of subscribers (y_1), length of gas network (y_2) and the sold-out gas income (y_3). Table 6 contains a listing of the original data. In this example, the initial capital, number of the staff and the operation costs are considered as the resources while the number of gas subscribers, the length of gas network and the income from gas distribution are considered as the products.

Suppose that the DM wants to increase the number of subscribers to 9500000. How much of the additional output must be produced by each DMU? We apply the algorithms discussed in this paper to answer this question.
According to model (3.3) the maximum value of the first output which should produce by inefficient units is equal to 572609.81 numbers. Thus we have:

$$\sum_{j \in I} \overline{y}^*_{1j} + \sum_{j \in E} y_{1j} = 572609.81 + 349061 = 921670.81$$

Note that $\beta_1 = 9500000 > 921670.81 = \sum_{j \in I} \overline{y}^*_{1j} + \sum_{j \in E} y_{1j}$ therefore $y_1 \in R_3$.
Thus, Algorithm 2 is used to determine the additional output produced by each DMU

Table 6. Data and efficiency scores for Iranian gas companies

DMU	x_1	x_2	x_3	y_1	y_2	y_3	θ
DMU_1	124313	129	198598	30242	565	61836	1
DMU_2	67545	117	131649	14139	153	46233	0.7106
DMU_3	47208	165	228730	13505	211	42094	0.9015
DMU_4	43494	106	165470	8508	114	44195	0.5977
DMU_5	48308	141	180866	7478	248	45841	1
DMU_6	55959	146	194470	10818	230	136513	1
DMU_7	40605	145	179650	6422	127	70380	0.7044
DMU_8	61402	87	94226	18260	182	36592	1
DMU_9	87950	104	91461	22900	170	47650	1
DMU_{10}	33707	114	88640	3326	85	13410	0.5235
DMU_{11}	100304	254	292995	14780	318	79883	0.6679
DMU_{12}	94286	105	98302	19105	273	32553	1
DMU_{13}	67322	224	287042	15332	241	172316	0.9579
DMU_{14}	102045	104	18082	155514	441	30004	0.9939
DMU_{15}	177430	401	528325	77564	801	201529	1
DMU_{16}	221338	1094	1186905	44136	803	840446	1
DMU_{17}	267806	1079	1323325	27690	251	832616	0.9510
DMU_{18}	160912	444	648685	45882	816	251770	1
DMU_{19}	177214	801	909539	72676	654	341585	1
DMU_{20}	146325	686	545115	19839	177	341585	0.8911

to increase the number of subscribers to 9500000. Since $R_3 = \{1\}$ we have:

$$\alpha_{\tilde{r}} = \arg max_{\tilde{r} \in R_3} \{\beta_{\tilde{r}} - (\sum_{j \in E} y_{\tilde{r}j} + \sum_{j \in I} \overline{y}^*_{\tilde{r}j})\} = 9500000 - 921670.81 = 8578329.19$$

Now, model (3.8) is solved. The minimum value of the required resources for generating 9500000 numbers of subscribers is equal to:

$$\begin{aligned}\Gamma &= (\sum_{j=1}^{20} x_{1j} + \gamma^*_{1,21}, \sum_{j=1}^{20} x_{2j} + \gamma^*_{2,21}, \sum_{j=1}^{20} x_{3j} + \gamma^*_{3,21}) \\ &= (2125473 + 20000, 6446 + 35000, 7529507 + 42000)\end{aligned}$$

Regarding the results from the model (3.3), if we want to increase the number of the gas subscribers to 9500000, we need more initial resources. In this case, 20000 must be added to the initial capital, 35000 people should be added to the staff and the operation costs should increase by 42000. Now we solve model (3.9) for determining how the extra resources should be allocated between the various units and how much extra output should be generated by each unit. The obtained results are reported in Table7.

The results of the model show that among the inefficient units, units 17, 20 and 11 have higher positions in terms of both the inputs received and the outputs produced. Units 20, 14 and 10 have exactly the same position in terms of the inputs received and the outputs produced. In fact, the efficiency scores of units 20 and 14 are close to one and thus were able to produce more when they received more. On the other hand, since unit 10 has a low efficiency score, therefore it received fewer inputs and produced fewer outputs. Since

Table 7. Results of resource allocation for Iranian gas companies

DMU	x_1^*	x_2^*	x_3^*	y_1^*	$\sum_{r=2}^{3} \overline{y}_{rj} + y_{1j}^*$	$\sum_{i=1}^{3} x_{ij}^*$
DMU_1	124313	179.521	198598	271767.1	3341168.1	323090.521
DMU_2	68537.485	117	133781.552	111893.18	158279	202436.037
DMU_3	47208	165	230841.324	91358.74	133663.74	278214.324
DMU_4	44486.432	106	167599.944	100001	144310	210392.426
DMU_5	49300.485	141	182998.552	74484.49	120573.49	232440.037
DMU_6	56951.487	146	196602.764	77782.15	214525.15	253700.251
DMU_7	42944.011	145	181848.356	77700.97	148207.97	224937.367
DMU_8	62392.828	87	94226	94800.8	131574.8	156705.8275
DMU_9	88943.086	104	91461	86418.3	134238.3	180508.0855
DMU_{10}	33898.825	2645.827	94160.634	18992.4	32487.4	130705.2855
DMU_{11}	102944.869	32333.56	294488.088	597801.4	678002.4	429766.517
DMU_{12}	95278.485	105	100434.552	86111.49	118937.49	195818.0374
DMU_{13}	68318.133	224	294001.984	93194.32	265751.32	362544.1168
DMU_{14}	103037.485	104	157646.552	94201.23	124646.23	260788.0374
DMU_{15}	178422.485	401	528325	144570.49	346900.49	707148.4854
DMU_{16}	222330.558	1430.849	1189036.775	111330.71	952579.71	1412799.182
DMU_{17}	268788.176	1079	1325500.885	164111.23	996978.23	1595501.936
DMU_{18}	161853	444	651035.936	6896791	7149377	813332.936
DMU_{19}	178206.485	801	911671.552	139682.49	481921.49	1090679.037
DMU_{20}	147317.485	687.241	547247.552	167006.49	508768.49	695252.2784

unit 13 has a higher efficiency score among the inefficient units, it received more inputs compared to inefficient units 4 and 2 but produced less outputs compared to them; thus unit 13 has become efficient.
According to the optimal resource allocation plan, company 17 will receive more resource allocation in comparison with other companies. On the other hand, company 10 will receive the least allocated resource. Also, as Table 7 indicates, the most value of output 1 is set for company 18, whereas the least one is set for company 10. It is to be noted that the company 10 will receive the least allocated resource and so the least target is set for this company. For increasing the number of subscribers to 9500000, the first output of inefficient units will be increased to

$$\overline{y}_{1j}^* + \Delta y_{1j}^*; \quad j \in I = \{2, 3, 4, 7, 10, 11, 13, 14, 17, 20\}$$

where $\overline{y}_{1j}^*$ and Δy_{1j}^* are optimal solutions of model (3.3) and model (3.8), respectively. The efficient units i.e. DMU_j $j \in \{1, 5, 6, 8, 9, 12, 15, 16, 18, 19\}$ increase its first output to Δy_{1j}^* where are obtained from model (3.3).
The values of output 2 and output 3 of the DMUs are not changed but the inputs of all DMUs are increased as followes:

$$(\sum_{j=1}^{20} x_{1j}^*, \sum_{j=1}^{20} x_{2j}^*, \sum_{j=1}^{20} x_{3j}^*) = (2145473, 41446, 7571507)$$

5. Conclusion

In this paper, some algorithms for resource allocation are proposed. These algorithms help the DM in determining the input-output levels of each DMU, when the production of additional products seems to be desirable. DEA and MOP are applied in these algorithms. In fact these algorithms will allocate the resources between the units in a way that their maximum power will be applied for production. As an advantage of the method, it can be mentioned that it can be easily run, does not have complicated calculations and of course it tries to maximize the number of the efficient units. In this study, the data were considered as real, however, the algorithm can be expanded to the situations in which the data are of interval or fuzzy type. These cases will be investigated in future studies.

There are a number of challenges involved in the proposed research. These challenges provide a great deal of fruitful scope for future research. The practicality of this model can be further enhanced by developing the proposed framework into a decision support system to reduce the computation time and effort. Another future research direction, which could be an area of theoretical study, is extending the proposed method under a fuzzy environment.

The proposed algorithm can also be used to solve transportation problems, find the shortest route, obtain the maximum flow in a network, allocate people to jobs and etc. For example in transportation problems, in order to reduce the transfer cost up to a certain amount or to set the profit of transferring goods to a predetermined level, it is possible to consider routes as the DMUs and the goods as the resources allocated to each route. Finally, extending the proposed technique for resource allocation of the two-stage systems is an interesting research work. We hope that our study can inspire others to pursue further research.

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