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⊕-supplemented modules relative to an ideal

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Abstract

Let I be an ideal of a ring R and let M be a left R-module. A submodule L of M is said to be δ-small in M provided $M \neq L + X$ for any proper submodule X of M with M/X singular. An R-module M is called *I-⊕-supplemented* if for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K$ is δ-small in K. In this paper, we investigate some properties of I-⊕-supplemented modules. We also compare I-⊕-supplemented modules with ⊕-supplemented modules. The structure of I-⊕-supplemented modules and ⊕-δ-supplemented modules over a Dedekind domain is completely determined.

Keywords: δ-small submodules, ⊕-supplemented modules, ⊕-δ-supplemented modules, I-⊕-supplemented modules

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1. Introduction

All rings considered in this paper will be associative with an identity element and R will always denote a ring. We shall use $J(R)$ to denote the Jacobson radical of R. All modules will be unital left R-modules. Let M be an R-module. A submodule L of M is called *small* (δ-small) in M, denoted by $L \ll M$ ($L \ll_\delta M$), if $L + X \neq M$ for any proper submodule X of M ($L + X \neq M$ for any proper submodule X of M with M/X singular). Recall that M is called *⊕-supplemented* (*⊕-δ-supplemented*) if for every submodule $N \leq M$, there exists a direct summand K of M such that $N + K = M$ and $N \cap K \ll K$ ($N \cap K \ll_\delta K$).

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In Section 2, we study some special cases of submodules N of a module M for which $N \ll_\delta M$ is equivalent to $N \ll M$.

In Section 3, we introduce the notion of I-$\oplus$-supplemented R-modules, where I is an ideal of R. A module M will be called *I-$\oplus$-supplemented* if for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$. We shall compare this notion with the concept of $\oplus$-supplemented modules. Indecomposable I-$\oplus$-supplemented modules are characterized.

Section 4 is devoted to the study of some factor modules of an I-$\oplus$-supplemented module. Among other results, it is shown that if M is a direct sum of two hollow I-$\oplus$-supplemented modules, then any direct summand of M is I-$\oplus$-supplemented.

In Section 5, our main results (Theorems 5.4 and 5.13) describe the structure of I-$\oplus$-supplemented modules over Dedekind domains. It is also shown that over a Dedekind domain R, an R-module M is $\oplus$-δ-supplemented if and only if M is $\oplus$-supplemented.

2. Some properties of δ-small submodules

We begin with some results presenting some elementary properties of δ-small submodules which will be used in the sequel.

2.1. Lemma. ([19, Lemma 1.2]) *Let N be a submodule of a module M. The following are equivalent:*

(i) *N is δ-small in M;*

(ii) *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \leq N$.*

2.2. Lemma. (See [19, Lemma 1.3])

(i) *Let N and K be submodules of a module M with $K \subseteq N$. If $N \ll_\delta M$, then $K \ll_\delta M$.*

(ii) *Let M and M' be two modules. If $L \ll_\delta M$ and $f : M \to M'$ is a homomorphism, then $f(L) \ll_\delta M'$. In particular, if $K \ll_\delta M \leq M'$, then $K \ll_\delta M'$.*

(iii) *If N and L are submodules of a module M, then $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*

(iv) *Let M_1 and M_2 be two submodules of a module M such that $M = M_1 \oplus M_2$. Let $K_1 \leq M_1$ and $K_2 \leq M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Let N be a submodule of a module M. Recall that N is said to be *DM* in M (or N *decomposes* M) if there is a direct summand D of M such that $D \leq N$ and $M = D + X$, whenever $N + X = M$ for a submodule X of M (see [1, Definition 3.1]). Clearly, the following implications hold:

$$(N \ll M) \Rightarrow (N \ll_\delta M) \Rightarrow (N \text{ is } DM \text{ in } M).$$

Next, we exhibit some conditions under which $N \ll_\delta M$ is equivalent to $N \ll M$.

2.3. Proposition. *Let N be a proper submodule of an indecomposable module M. Then N is DM in M if and only if $N \ll_\delta M$ if and only if $N \ll M$.*

Proof. Assume that N is *DM* in M. Let X be a submodule of M such that $M = N + X$. Then there exists a direct summand D of M such that $D \leq N$ and $M = D + X$. Since M is indecomposable and $N \neq M$, we have $D = 0$ and $X = M$. Therefore, $N \ll M$. The rest of the proof is immediate. □

The next result was inspired by [16, Proposition 2.3(1)].

2.4. Proposition. *Let N be a submodule of a module M. Then $N \ll M$ if and only if $N \subseteq Rad(M)$ and $N \ll_\delta M$.*

Proof. It is enough to prove the sufficiency. Let X be a submodule of M such that $M = N + X$. Since $N \ll_\delta M$, there exists a projective semisimple submodule $P \leq N$ such that $M = P \oplus X$.

Assume that $P \neq 0$. Then P has a simple direct summand S. Since $S \subseteq Rad(M)$, $S \ll M$. Hence $S = 0$, a contradiction. Thus, $P = 0$. It follows that $N \ll M$. □

The following result is a direct consequence of Proposition 2.4.

2.5. Corollary. *Let M be a module with $Rad(M) = M$ and let N be a submodule of M. Then $N \ll_\delta M$ if and only if $N \ll M$.*

Let M be a module over a commutative integral domain R. Let $T(M)$ denote the set of all elements $x \in M$ for which there exists a nonzero element $r \in R$ such that $rx = 0$. It is well known that $T(M)$ is a submodule of M. This submodule is called the *torsion submodule* of M. If $T(M) = M$, then the module M is said to be a *torsion module*. The module M is said to be *torsion-free* if $T(M) = 0$.

2.6. Proposition. *Assume that R is a commutative integral domain. Let M be an R-module and N a submodule of M such that $N \subseteq T(M)$. Then $N \ll_\delta M$ if and only if $N \ll M$.*

Proof. Assume that $N \ll_\delta M$. Let X be a submodule of M such that $N + X = M$. Then there exists a projective submodule $P \leq N$ such that $P \oplus X = M$. Since P is projective, P is isomorphic to a direct summand of a free R-module. Hence, P is torsion-free. But P is a torsion module as $P \subseteq N$. Then $P = 0$ and $X = M$. It follows that $N \ll M$. The converse is obvious. □

Let N and K be submodules of a module M. Recall that K is said to be a *supplement* of N in M if $N + K = M$ and $N \cap K \ll K$. Let $M = \oplus_{i\in I} M_i$ be a decomposition of the module M. The next example shows that, in general, if $L = \oplus_{i\in I} L_i$ is a submodule of M such that $L_i \ll_\delta M_i$ for each $i \in I$, then L need not be δ-small in M.

2.7. Example. Let R be a discrete valuation ring with maximal ideal m. Let $M = \oplus_{i=1}^{\infty} R/m^i$. By [20, p. 48 The second corollary of Lemma 2.1], $Rad(M)$ does not have a supplement in M. Therefore, $Rad(M) = \oplus_{i=1}^{\infty} m/m^i$ is not small in M. Applying Proposition 2.6, it follows that $Rad(M)$ is not δ-small in M. On the other hand, it is clear that for each $i \geq 1$, $m/m^i \ll R/m^i$.

2.8. Proposition. *Let $M = \oplus_{i\in I} M_i$ be a decomposition of a module M. Assume that for every submodule $N \leq M$, we have $N = \oplus_{i\in I}(N \cap M_i)$. For each i, let L_i be a submodule of M_i. The following statements are equivalent:*

(i) *$L_i \ll_\delta M_i$ for every $i \in I$;*

(ii) *$L = \oplus_{i\in I} L_i \ll_\delta M$.*

Proof. (i) ⇒ (ii) Let X be a submodule of M such that $M = X + L$. By hypothesis, $X = \oplus_{i\in I}(X \cap M_i)$. So, $(X \cap M_i) + L_i = M_i$ for every $i \in I$. By assumption, for every $i \in I$, there exists a semisimple projective submodule P_i of L_i such that $(X \cap M_i) \oplus P_i = M_i$ (see Lemma 2.1). Let $P = \oplus_{i\in I} P_i$. Then $X \oplus P = M$. Note that P is a semisimple projective submodule of L. Therefore, $L \ll_\delta M$.

(ii) ⇒ (i) By Lemma 2.2(iv). □

3. I-$\oplus$-supplemented modules

Recall that a module M is called *$\oplus$-supplemented* (*$\oplus$-δ-supplemented*) if for every submodule $N \leq M$, there exists a direct summand K of M such that $N + K = M$ and $N \cap K \ll K$ ($N \cap K \ll_\delta K$).

Recall that a ring R is said to be *semilocal* provided $R/J(R)$ is a semisimple ring.

3.1. Proposition. *Let M be a module over a semilocal ring R. Then M is $\oplus$-supplemented if and only if for every submodule $N \leq M$, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq J(R)K$ and $N \cap K \ll_\delta K$.*

Proof. By Proposition 2.4 and [2, Corollary 15.18]. □

Motivated by the last proposition, we introduce the following notion:

3.2. Definition. Let M be an R-module and let I be an ideal of R. We say that M is I-$\oplus$-supplemented, provided for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$.

In this section we investigate some properties of I-$\oplus$-supplemented modules.

3.3. Remark. (i) It is clear that for every ideal I of R, every I-$\oplus$-supplemented module is $\oplus$-δ-supplemented.

(ii) Let M be an R-module. If I is an ideal of R such that $IM = 0$, then M is I-$\oplus$-supplemented if and only if M is semisimple.

Let M be an R-module. As in [19], let $\delta(M)$ denote the sum of all δ-small submodules of M. In the next proposition we provide a condition under which a $\oplus$-δ-supplemented module is I-$\oplus$-supplemented. To prove this result, we need the following elementary lemma.

3.4. Lemma. *Let M be an R-module and let I be an ideal of R. If K is a direct summand of M, then we have $IK = K \cap IM$.*

Proof. Let K' be a submodule of M such that $M = K \oplus K'$. Then $IM = IK \oplus IK'$. Hence $K \cap IM = IK$. □

3.5. Proposition. *Let M be an R-module and let I be an ideal of R such that $\delta(M) \subseteq IM$. Then M is I-$\oplus$-supplemented if and only if M is $\oplus$-δ-supplemented.*

Proof. The necessity is clear. Conversely, suppose that M is $\oplus$-δ-supplemented. Let N be a submodule of M. Then there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll_\delta K$. Note that $IK = K \cap IM$ by Lemma 3.4. Since $\delta(M) \subseteq IM$, we have

$$N \cap K \subseteq \delta(K) \subseteq K \cap \delta(M) \subseteq K \cap IM = IK.$$

Therefore M is I-$\oplus$-supplemented. This completes the proof. □

Recall that a nonzero module M is called *hollow* if every proper submodule is small in M. The module M is called *local* if it has a proper submodule which contains all other proper submodules. Note that the largest proper submodule of a local module M is $Rad(M)$. It is well known that every hollow module is $\oplus$-supplemented.

3.6. Example. (i) It is clear that every semisimple module is I-$\oplus$-supplemented for any ideal I of R.

(ii) Let p be a prime integer. It is well known that the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is hollow and injective. It is easily seen that $\mathbb{Z}(p^\infty)$ is I-$\oplus$-supplemented for every nonzero ideal I of $\mathbb{Z}$, but $\mathbb{Z}(p^\infty)$ is not 0-$\oplus$-supplemented.

(iii) It is easy to see that every $\oplus$-δ-supplemented module (in particular, every $\oplus$-supplemented module) is R-$\oplus$-supplemented (see Proposition 3.5).

3.7. Proposition. *Let M be an indecomposable R-module and let I be an ideal of R. The following conditions are equivalent:*

(i) *M is I-$\oplus$-supplemented;*

(ii) *M is hollow with $IM = M$ or $IM = Rad(M)$.*

Proof. (i) ⇒ (ii) Let N be a proper submodule of M. By hypothesis, there exists a direct summand K of M such that $N + K = M$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$. Since M is indecomposable, we have $K = M$. Hence, $N \subseteq IM$ and $N \ll_\delta M$. By Proposition 2.3, we have $N \ll M$. Thus, M is a hollow module. Moreover, note that if $IM \neq M$, then IM contains all other proper submodules of M. Hence M is a local module and $IM = Rad(M)$.

(ii) ⇒ (i) Let N be a proper submodule of M. Then $N+M = M$, $N \cap M = N \subseteq Rad(M) \subseteq IM$ and $N \cap M = N \ll_\delta M$. Therefore, M is I-⊕-supplemented. □

It follows from Proposition 3.7 that if I is an ideal of R, then every indecomposable I-⊕-supplemented R-module is ⊕-supplemented. Next, we present some examples of ⊕-supplemented modules which are not I-⊕-supplemented for an ideal I of R.

3.8. Example. (i) Let p and q be two different prime integers. Consider the local $\mathbb{Z}$-module $M = \mathbb{Z}/\mathbb{Z}p^3$. We have $Rad(M) = \mathbb{Z}p/\mathbb{Z}p^3$. Let $I_1 = \mathbb{Z}p$, $I_2 = \mathbb{Z}q$ and $I_3 = \mathbb{Z}p^2$. Then $I_1M = Rad(M)$, $I_2M = M$ and $I_3M = \mathbb{Z}p^2/\mathbb{Z}p^3$. By Proposition 3.7, M is I_i-⊕-supplemented for each $i = 1, 2$, but not I_3-⊕-supplemented. On the other hand, it is clear that M is ⊕-supplemented.

(ii) Let R be a discrete valuation ring with maximal ideal m. It is well known that the R-module ${}_RR$ is ⊕-supplemented. Let I be an ideal of R. From Proposition 3.7 it follows that ${}_RR$ is I-⊕-supplemented if and only if $I = m$ or $I = R$. Therefore, the module ${}_RR$ is not m^3-⊕-supplemented.

3.9. Proposition. *Let I be an ideal of R and let M be an R-module.*

(i) *Assume that for every submodule $N \leq M$, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \subseteq IM$. Then M/IM is semisimple.*

(ii) *If M is an I-⊕-supplemented R-module, then M/IM is semisimple.*

Proof. (i) Let N be a submodule of M such that $IM \subseteq N$. By assumption, there exists a submodule K of M such that $N + K = M$ and $N \cap K \subseteq IM$. Thus, $(N/IM) + [(K + IM)/IM] = M/IM$. Clearly, we have $N \cap (K + IM) = IM$. So, N/IM is a direct summand of M/IM. This completes the proof.

(ii) follows from (i). □

3.10. Proposition. *Let M be a module.*

(i) *If M is ⊕-δ-supplemented, then $M = M_1 \oplus M_2$ such that $Rad(M_1) \ll M_1$ and $Rad(M_2) = M_2$.*

(ii) *If M is I-⊕-supplemented, then $M = M_1 \oplus M_2$ such that $Rad(M_1) \subseteq IM_1$, $Rad(M_1) \ll M_1$ and $Rad(M_2) = M_2$.*

Proof. (i) Since M is ⊕-δ-supplemented, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $Rad(M) + M_1 = M$ and $Rad(M) \cap M_1 \ll_\delta M_1$. Note that $Rad(M) = Rad(M_1) \oplus Rad(M_2)$. Then $M_1 \oplus Rad(M_2) = M$ and $(Rad(M) \cap M_1) \oplus Rad(M_2) = Rad(M)$. Therefore $Rad(M_2) = M_2$ and $Rad(M) \cap M_1 = Rad(M_1)$. Moreover, we have $Rad(M_1) \ll M_1$ by Proposition 2.4. This completes the proof.

(ii) This follows by the same method as in (i) and adding the fact that $Rad(M) \cap M_1 \subseteq IM_1$. □

Combining Proposition 3.10(ii) and [2, Proposition 5.20(1)], we get the following result.

3.11. Corollary. *If M is an I-⊕-supplemented module with $Rad(M) \ll M$, then $Rad(M) \subseteq IM$.*

From the last corollary, we conclude that if I is an ideal of a left perfect ring R and M is an I-⊕-supplemented R-module, then $Rad(M) \subseteq IM$ (see [2, Remark 28.5(3)]).

An R-module M is said to be δ-*local* if $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M (see [4, Definition 3.1]). Next, we give an example of an R-⊕-supplemented module which is not ⊕-supplemented.

3.12. Example. Let $F = \mathbb{Z}/\mathbb{Z}2$ and let $A = F^{\mathbb{N}}$ be the ring of sequences over F, whose operations are pointwise multiplication and pointwise addition. Let $R \subseteq A$ be the subring generated by 1_A (the unit element of A) and all sequences that have only a finite number of nonzero entries. It is shown in [4, p. 318] that the ring R is not semilocal and the R-module ${}_RR$ is δ-local. Applying [15, Proposition 3.1], it is easily seen that ${}_RR$ is an R-$\oplus$-supplemented module. On the other hand, since the ring R is not semilocal, it is not semiperfect. Hence, the R-module ${}_RR$ is not $\oplus$-supplemented by [12, Corollary 4.42].

Next, we present conditions under which an I-$\oplus$-supplemented R-module is $\oplus$-supplemented.

3.13. Proposition. *Let M be an R-module with $Rad(M) = M$. Then M is $\oplus$-δ-supplemented if and only if M is $\oplus$-supplemented.*

Proof. As $Rad(M) = M$, we have $Rad(K) = K$ for every direct summand K of M. The result follows from Corollary 2.5. □

3.14. Proposition. *Assume that R is a commutative integral domain and let M be a torsion R-module. Then M is $\oplus$-δ-supplemented if and only if M is $\oplus$-supplemented.*

Proof. This follows from Proposition 2.6. □

3.15. Proposition. *Let I be an ideal of R and let M be an I-$\oplus$-supplemented R-module. If $IM \subseteq Rad(M)$, then M is $\oplus$-supplemented.*

Proof. Let N be a submodule of M. By hypothesis, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$. Since $IM \subseteq Rad(M)$, we have $IK = K \cap IM \subseteq K \cap Rad(M) = Rad(K)$ by Lemma 3.4 and [5, 20.4(7)]. So $N \cap K \ll K$ by Proposition 2.4. It follows that M is $\oplus$-supplemented. □

3.16. Corollary. *Let I be an ideal of R and let M be an I-$\oplus$-supplemented R-module. Assume that one of the following conditions is satisfied:*

(i) *$I \subseteq J(R)$, or*

(ii) *R is a local ring and $I \neq R$, or*

(iii) *$Rad(M) = M$, or*

(iv) *R is a commutative integral domain and M is a torsion R-module.*

Then M is $\oplus$-supplemented.

Proof. (i) follows from [2, Corollary 15.18] and Proposition 3.15.

(ii) follows from (i).

(iii) follows easily from Proposition 3.13.

(iv) is obvious by Proposition 3.14. □

Next, we focus on when a $\oplus$-supplemented R-module is I-$\oplus$-supplemented for an ideal I of R.

3.17. Proposition. *Let I be an ideal of R and let M be a $\oplus$-supplemented R-module such that $Rad(M) \subseteq IM$. Then M is I-$\oplus$-supplemented.*

Proof. Let N be a submodule of M. Then there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$. Thus, $N \cap K \ll_\delta K$. Moreover, we have $IK = K \cap IM$ by Lemma 3.4. Since $Rad(M) \subseteq IM$, it follows that

$$Rad(K) \subseteq K \cap Rad(M) \subseteq K \cap IM = IK.$$

Hence, $N \cap K \subseteq IK$. Therefore M is I-$\oplus$-supplemented. This completes the proof. □

The next corollary is a direct consequence of Proposition 3.17.

3.18. Corollary. *Let M be a $\oplus$-supplemented module such that $IM = M$. Then M is I-$\oplus$-supplemented.*

3.19. Corollary. *Let m be a maximal ideal of a commutative ring R and let M be an R-module. Assume that I is an ideal of R such that $IM = mM$. If M is a $\oplus$-supplemented R-module, then M is I-$\oplus$-supplemented.*

Proof. Note that $Rad(M) \subseteq mM$ by [7, Lemma 3]. The result follows from Proposition 3.17. □

Let R be a commutative integral domain. An R-module M is called *divisible* in case $rM = M$ for each nonzero element $r \in R$.

3.20. Corollary. *Let M be a divisible module over a commutative integral domain R. If M is $\oplus$-supplemented, then M is I-$\oplus$-supplemented for every nonzero ideal I of R.*

Proof. This follows from Corollary 3.18. □

Recall that a ring R is called a *left good* ring if $Rad(M) = J(R)M$ for every R-module M (see [18, 23.7]).

3.21. Corollary. *Let M be an R-module. Suppose further that either*
(i) *R is a left good ring, or*
(ii) *M is a projective module.*
Then M is $\oplus$-supplemented if and only if M is $J(R)$-$\oplus$-supplemented.

Proof. Note that $Rad(M) = J(R)M$ by [2, Proposition 17.10]. The result follows from Propositions 3.15 and 3.17. □

Combining Lemma 2.2 and the application of the same reasoning of [10, Proposition 3] to I-$\oplus$-supplemented modules, we obtain the following theorem.

3.22. Theorem. *Let I be an ideal of R. Then any finite direct sum of I-$\oplus$-supplemented R-modules is I-$\oplus$-supplemented.*

The next example shows that, in general, a direct sum of I-$\oplus$-supplemented modules is not I-$\oplus$-supplemented.

3.23. Example. Let p be a prime integer. Consider the $\mathbb{Z}$-module $M = \oplus_{i=1}^{\infty}\mathbb{Z}/\mathbb{Z}p^i$. Clearly, M is a torsion module. By [12, Propositions A.7 and A.8], M is not $\oplus$-supplemented. Therefore M is not $(\mathbb{Z}p)$-$\oplus$-supplemented by Corollary 3.16. On the other hand, note that for every $i \geq 1$, $\mathbb{Z}/\mathbb{Z}p^i$ is a $(\mathbb{Z}p)$-$\oplus$-supplemented $\mathbb{Z}$-module by Proposition 3.7.

The next result deals with a special case of a family of $\oplus$-δ-supplemented (I-$\oplus$-supplemented) modules $(M_\lambda)_{\lambda\in\Lambda}$ for which $M = \oplus_{\lambda\in\Lambda}M_\lambda$ is $\oplus$-δ-supplemented (I-$\oplus$-supplemented).

3.24. Proposition. *Let I be an ideal of R and let $M = \oplus_{\lambda\in\Lambda}M_\lambda$ be a direct sum of submodules M_λ ($\lambda \in \Lambda$) such that for every submodule N of M, we have $N = \oplus_{\lambda\in\Lambda}(N \cap M_\lambda)$. Assume that M_λ is $\oplus$-δ-supplemented (I-$\oplus$-supplemented) for every $\lambda \in \Lambda$. Then M is $\oplus$-δ-supplemented (I-$\oplus$-supplemented).*

Proof. Let N be a submodule of M. Then $N = \oplus_{\lambda\in\Lambda}(N \cap M_\lambda)$. For every $\lambda \in \Lambda$, there exists a direct summand K_λ of M_λ such that $(N \cap M_\lambda) + K_\lambda = M_\lambda$, ($N \cap K_\lambda \subseteq IK_\lambda$) and $N \cap K_\lambda \ll_\delta K_\lambda$. Set $K = \oplus_{\lambda\in\Lambda}K_\lambda$. Clearly, K is a direct summand of M and $N + K = M$. Also, we have ($N \cap K = \oplus_{\lambda\in\Lambda}(N \cap K_\lambda) \subseteq IK$) and $N \cap K \ll_\delta K$ by Proposition 2.8. This proves the proposition. □

4. Homomorphic images of I-$\oplus$-supplemented modules

We begin this section by an example showing that the I-$\oplus$-supplemented property does not always transfer from a module to each of its factor modules.

4.1. Example. Let F be a field. Consider the local ring $R = F[x^2, x^3]/(x^4)$ and let m be the maximal ideal of R. Let n be an integer with $n \geq 2$ and let $M = R^{(n)}$. By Proposition 3.7 and Theorem 3.22, M is m-$\oplus$-supplemented. Note that R is an artinian local ring which is not a principal ideal ring (see [3, Example on p. 91]). So, there exists a submodule K of M such that the factor module M/K is not $\oplus$-supplemented by [11, Example 2.2]. Therefore M/K is not m-$\oplus$-supplemented by Corollary 3.16.

Next, we show that under some conditions, a factor module of an I-$\oplus$-supplemented module is I-$\oplus$-supplemented.

Recall that a submodule N of a module M is called *fully invariant* if $f(N) \subseteq N$ for every endomorphism f of M. A module M is called *distributive* if $(A + B) \cap C = (A \cap C) + (B \cap C)$ for all submodules A, B, C of M (or equivalently, $(A \cap B) + C = (A + C) \cap (B + C)$ for all submodules A, B, C of M).

Analysis similar to the proofs of [6, Theorems 4.7 and 4.8] yields the following result. We give the first part of its proof for completeness.

4.2. Proposition. *Let I be an ideal of R and let M be an I-$\oplus$-supplemented module.*

(i) *Let $X \leq M$ be a submodule such that for every direct summand K of M, $(X + K)/X$ is a direct summand of M/X. Then M/X is I-$\oplus$-supplemented.*

(ii) *Let $X \leq M$ be a submodule such that for every decomposition $M = M_1 \oplus M_2$, we have $X = (X \cap M_1) \oplus (X \cap M_2)$. Then M/X is I-$\oplus$-supplemented.*

(iii) *If X is a fully invariant submodule of M, then M/X is I-$\oplus$-supplemented.*

(iv) *If M is a distributive module, then M/X is I-$\oplus$-supplemented for every submodule X of M.*

Proof. (i) Let N be a submodule of M such that $X \subseteq N$. Since M is I-$\oplus$-supplemented, there exists a direct summand K of M such that $N + K = M$, $N \cap K \subseteq IK$ and $N \cap K \ll_\delta K$. Therefore $(N/X) + ((X + K)/X) = M/X$ and $(N/X) \cap ((K + X)/X) = (X + (N \cap K))/X \subseteq ((X + IK)/X) \subseteq I((X + K)/X)$. Consider the natural epimorphism $\pi : K \to (X + K)/X$. Since $N \cap K \ll_\delta K$, we have $\pi(N \cap K) = (X + (N \cap K))/X \ll_\delta (X + K)/X$ by Lemma 2.2(ii). Note that by assumption, $(X + K)/X$ is a direct summand of M/X. It follows that M/X is I-$\oplus$-supplemented.

(ii), (iii) and (iv) These are consequences of (i). □

The next proposition was inspired by [11, Proposition 2.5].

4.3. Proposition. *Let M be an R-module and let I be an ideal of R. Let K be a fully invariant direct summand of M. Then the following assertions are equivalent:*

(i) *M is I-$\oplus$-supplemented;*

(ii) *K and M/K are I-$\oplus$-supplemented.*

Proof. (i) $\Rightarrow$ (ii) Let L be a submodule of K. By hypothesis, there exist submodules A and B of M such that $M = A \oplus B$, $M = A + L$, $A \cap L \subseteq IA$ and $A \cap L \ll_\delta A$. Clearly, we have $K = (A \cap K) + L$. Since K is fully invariant in M, we have $K = (A \cap K) \oplus (B \cap K)$. Hence, $A \cap K$ is a direct summand of M. Thus $I(A \cap K) = (A \cap K) \cap IM$ by Lemma 3.4. It follows that $(A \cap K) \cap L = A \cap L \subseteq (A \cap K) \cap IM = I(A \cap K)$. Since $A \cap L \ll_\delta A$ and $A \cap K$ is a direct summand of A, we have $A \cap L \ll_\delta A \cap K$ by Lemma 2.2(iv). Therefore, K is I-$\oplus$-supplemented. Moreover, M/K is I-$\oplus$-supplemented by Proposition 4.2(iii).

(ii) $\Rightarrow$ (i) This follows from Theorem 3.22. □

Let I be an ideal of R. An R-module M is called *completely* I-$\oplus$-*supplemented* ($\oplus$-*supplemented*) if every direct summand of M is I-$\oplus$-supplemented ($\oplus$-*supplemented*). Clearly, semisimple modules are completely I-$\oplus$-supplemented. Also, every I-$\oplus$-supplemented hollow module is completely I-$\oplus$-supplemented. The next result provides another example of completely I-$\oplus$-supplemented modules.

Recall that a module M is said to have *finite hollow dimension* $n \in \mathbb{N}$ if there exists a small epimorphism from M to a direct sum of n hollow modules. We denote this by $h.dim(M) = n$. It is well known that a module M is hollow if and only if $h.dim(M) = 1$ (see [5, p. 47 and p. 49]).

4.4. Proposition. *Let $M = H_1 \oplus H_2$ be a direct sum of hollow submodules H_1 and H_2. Then the following statements are equivalent:*

(i) *H_1 and H_2 are I-$\oplus$-supplemented modules;*

(ii) *The module M is completely I-$\oplus$-supplemented.*

Proof. (i) $\Rightarrow$ (ii) Let L be a nonzero direct summand of M. If $L = M$, then L is I-$\oplus$-supplemented by Theorem 3.22. Assume that $L \neq M$. Let K be a submodule of M such that $M = L \oplus K$. By [5, 5.4(1)], $h.dim(M) = 2 = h.dim(L) + h.dim(K)$. It follows that $h.dim(L) = 1$ and hence L is a hollow module. Let us prove that L is I-$\oplus$-supplemented. To see this, it suffices to show that $IL = L$ or $IL = Rad(L)$ by Proposition 3.7. Since M is I-$\oplus$-supplemented, $M/IM \cong (L/IL) \oplus (K/IK)$ is semisimple by Proposition 3.9. As L is a hollow module, $L/IL = 0$ or L/IL is simple. Hence $L = IL$ or L is a local module with maximal submodule IL. So $IL = L$ or $IL = Rad(L)$, as required.

(ii) $\Rightarrow$ (i) This is immediate. □

5. Modules over Dedekind domains

Our purpose in this section is to determine the structure of all I-$\oplus$-supplemented modules and all $\oplus$-δ-supplemented modules over Dedekind domains.

5.1. Proposition. *Let R be a Dedekind domain which is not a field. Then the following assertions are equivalent for an injective R-module M:*

(i) *M is $\oplus$-supplemented;*

(ii) *M is I-$\oplus$-supplemented for every nonzero ideal I of R;*

(iii) *M is I-$\oplus$-supplemented for some nonzero ideal I of R;*

(iv) *M is $\oplus$-δ-supplemented.*

Proof. (i) $\Rightarrow$ (ii) This follows from Corollary 3.20 since the module M is divisible.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) These are obvious.

(iv) $\Rightarrow$ (i) Since R is a Dedekind domain which is not a field and M is an injective R-module, we have $Rad(M) = M$. The result follows from Proposition 3.13. □

Let R be a Dedekind domain which is not a field. If M is an R-module, we will denote the sum of all divisible (injective) submodules of M by $d(M)$. It is well known that $d(M)$ is an injective R-module. Also, note that if f is an endomorphism of M, then $f(d(M))$ is isomorphic to a factor module of $d(M)$. So, $f(d(M))$ is injective as R is a Dedekind domain. Therefore, $f(d(M)) \subseteq d(M)$. It follows that $d(M)$ is a fully invariant submodule of M.

5.2. Proposition. *Let R be a Dedekind domain which is not a field. Let I be an ideal of R and let M be an R-module. Then the following are equivalent:*

(i) *M is $\oplus$-δ-supplemented (I-$\oplus$-supplemented);*

(ii) *M can be written as $M = M_1 \oplus M_2$ such that M_1 is injective, $Rad(M_2) \ll M_2$ and both of M_1 and M_2 are $\oplus$-δ-supplemented (I-$\oplus$-supplemented) modules.*

Proof. (i) $\Rightarrow$ (ii) Let $M_1 = d(M)$ and let M_2 be a submodule of M such that $M = M_1 \oplus M_2$. Note that M_2 has no submodules X with $Rad(X) = X$. Since M is $\oplus$-δ-supplemented (I-$\oplus$-supplemented), M_1 and M_2 are $\oplus$-δ-supplemented (I-$\oplus$-supplemented) by [14, Theorem 2.5] and Proposition 4.3. Moreover, we have $Rad(M_2) \ll M_2$ by Proposition 3.10.

(ii) $\Rightarrow$ (i) This follows by [14, Theorem 2.2] and Theorem 3.22. □

Next, we restrict our investigations about $\oplus$-δ-supplemented modules and I-$\oplus$-supplemented modules to the case of modules over discrete valuation rings.

5.3. Proposition. *Let M be a module over a discrete valuation ring R and let I be an ideal of R. Then M is $\oplus$-δ-supplemented if and only if M is $\oplus$-supplemented. In particular, every I-$\oplus$-supplemented R-module is $\oplus$-supplemented.*

Proof. Assume that M is $\oplus$-δ-supplemented. By Proposition 5.2, $M = M_1 \oplus M_2$ is a direct sum of a $\oplus$-δ-supplemented injective submodule M_1 and a submodule M_2 with $Rad(M_2) \ll M_2$. By Proposition 5.1, M_1 is $\oplus$-supplemented. In addition, M_2 is $\oplus$-supplemented by [20, Lemma 2.1] and [12, Proposition A.7]. Therefore, M is $\oplus$-supplemented by [8, Theorem 1.4]. The converse is immediate.

The remaining assertion is obvious. □

Let P be a nonzero prime ideal of a Dedekind domain R and let n be a nonzero natural number. We will use the notation $B_P(1, \dots, n)$ to denote the direct sum of arbitrarily many copies of R/P, R/P^2, …, R/P^n.

The next result provides a structure theorem for modules over a discrete valuation ring.

5.4. Theorem. *Assume that R is a discrete valuation ring with maximal ideal m, quotient field K and $Q = K/R$. Let I be an ideal of R and let M be an R-module.*

(1) *If $I = m$ or $I = R$, then the following are equivalent:*

(i) *M is I-$\oplus$-supplemented;*

(ii) *M is $\oplus$-δ-supplemented;*

(iii) *M is $\oplus$-supplemented;*

(iv) *$M \cong R^a \oplus K^b \oplus Q^c \oplus B_m(1, \dots, n)$ for some natural numbers a, b, c and n.*

(2) *If $I \notin \{m, R\}$, then the following are equivalent:*

(i) *M is I-$\oplus$-supplemented;*

(ii) *$M \cong K^b \oplus Q^c \oplus (R/m)^{(\Lambda)}$ for some natural numbers b and c and an index set Λ.*

Proof. (1) (i) $\Leftrightarrow$ (iii) By Corollaries 3.18 and 3.19 and Proposition 5.3.

(ii) $\Leftrightarrow$ (iii) By Proposition 5.3.

(iii) $\Leftrightarrow$ (iv) This follows from [12, Proposition A.7].

(2) (i) $\Rightarrow$ (ii) Assume that M is I-$\oplus$-supplemented. By Proposition 5.3, M is $\oplus$-supplemented. Applying [12, Proposition A.7], $M \cong R^a \oplus K^b \oplus Q^c \oplus B_m(1, \dots, n)$ for some natural numbers a, b, c and n. Since M/IM is semisimple (see Proposition 3.9) and $I \notin \{m, R\}$, we have $a = 0$ and for each $1 \le i \le n$, $R/(I + m^i)$ is semisimple. So, for each $1 \le i \le n$, we have $I + m^i = m$ or $I + m^i = R$. Therefore $n = 1$ because $I \subseteq m^2$. It follows that $B_m(1, \dots, n) = B_m(1)$ is semisimple, completing the proof.

(ii) $\Rightarrow$ (i) Note that $K^b \oplus Q^c$ is an injective $\oplus$-supplemented module by [12, Proposition A.7]. The result follows from Propositions 5.1 and 5.2. □

5.5. Remark. Let R be a discrete valuation ring with maximal ideal m, quotient field K and $Q = K/R$. Let I be an ideal of R.

(i) Assume that $I \notin \{m, R\}$. Theorem 5.4(2) and [12, Proposition A.7] provide many examples of $\oplus$-supplemented R-modules which are not I-$\oplus$-supplemented.

(ii) Note that [11, Corollary 4.5] shows that every $\oplus$-supplemented R-module is completely $\oplus$-supplemented.

Case 1. Assume that $I \in \{m, R\}$. Then every I-$\oplus$-supplemented R-module is completely I-$\oplus$-supplemented by Theorem 5.4.

Case 2. Suppose that $I \notin \{m, R\}$. Let M be an I-$\oplus$-supplemented R-module. Then $M = K^b \oplus Q^c \oplus (R/m)^{(\Lambda)}$ for some natural numbers b and c and an index set Λ. Let N and L be submodules of M such that $M = N \oplus L$ and let $d(M)$ be the sum of all injective submodules of M. It is clear that $d(M) = d(N) \oplus d(L) = K^b \oplus Q^c$. Then, $d(N) \cong K^{b'} \oplus Q^{c'}$ for some natural numbers b' and c' by [2, Corollary 12.7 and Lemma 25.4]. Therefore, $d(N)$ is I-$\oplus$-supplemented by Theorem 5.4. In addition, we have $(R/m)^{(\Lambda)} \cong M/d(M) \cong (N/d(N)) \oplus (L/d(L))$. Hence, $N/d(N)$ is semisimple. Thus, $N/d(N)$ is I-$\oplus$-supplemented. Since $d(N)$ is a direct summand of N, N is I-$\oplus$-supplemented by Theorem 3.22. Consequently, M is completely I-$\oplus$-supplemented.

Let L be a submodule of a module M. A submodule $K \leq M$ is called a *δ-supplement* of N in M if $M = L + K$ and $L \cap K \ll_\delta K$. The module M is called *δ-supplemented* if every submodule has a δ-supplement in M.

Our next goal is to describe $\oplus$-δ-supplemented modules and I-$\oplus$-supplemented modules over a nonlocal Dedekind domain R. The next proposition shows that every torsion-free δ-supplemented R-module is injective. First we prove the following lemma.

5.6. Lemma. *Let L be a proper submodule of a module M such that M/L is a cyclic module.*

(i) *If K is a δ-supplement of L in M, then $K = P \oplus Rx$, where P is a semisimple projective submodule of $L \cap K$ and $x \in K$. In this case, Rx is a δ-supplement of L in M.*

(ii) *If L has a δ-supplement that is a direct summand of M, then L has a cyclic δ-supplement that is a direct summand of M.*

Proof. (i) By assumption, we have $L + K = M$ and $L \cap K \ll_\delta K$. Thus, $M/L \cong K/(L \cap K)$ is cyclic. Let $x \in K$ such that $K = (L \cap K) + Rx$. Since $L \cap K \ll_\delta K$, there exists a semisimple projective submodule P of $L \cap K$ such that $K = P \oplus Rx$ by Lemma 2.1. Note that $L \cap K = L \cap (P \oplus Rx) = P \oplus (L \cap Rx) \ll_\delta P \oplus Rx$. By Lemma 2.2(iv), we have $P \ll_\delta P$ and $L \cap Rx \ll_\delta Rx$. Therefore P is a semisimple projective module by [15, Lemma 2.9]. Also, note that $L + Rx = M$. It follows that Rx is a δ-supplement of L in M.

(ii) follows from (i). □

5.7. Proposition. *Assume that R is a Dedekind domain which is not local. Let K denote the quotient field of R. If M is a δ-supplemented R-module, then $M/T(M) \cong K^{(\Lambda)}$ for some index set Λ.*

Proof. Assume that M has a maximal submodule L such that $T(M) \subseteq L$. Since M is δ-supplemented, there exists a cyclic submodule W of M such that $M = L + W$ and $L \cap W \ll_\delta W$ (see Lemma 5.6). Let A be an ideal of R such that $W \cong R/A$. Since W is not contained in L, W is not a torsion module. So $A = 0$ and $W \cong {}_RR$. Thus, W is an indecomposable R-module. Hence $L \cap W \ll W$ by Proposition 2.3. Since $W/(L \cap W) \cong M/L$, we conclude that W is a local submodule of M. This contradicts the fact that R is not a local ring. It follows that $Rad(M/T(M)) = M/T(M)$. Hence, the module $M/T(M)$ is injective. So there exists an index set Λ such that $M/T(M) \cong K^{(\Lambda)}$ by [9, Lemma 2.1]. □

5.8. Proposition. *Assume that R is a Dedekind domain which is not local. If M is a $\oplus$-δ-supplemented R-module with $Rad(M) \ll M$, then M is a torsion module.*

Proof. Since M is $\oplus$-δ-supplemented, there exist submodules A and B of M such that $M = A \oplus B = T(M) + B$ and $T(M) \cap B \ll_\delta B$. Since $T(M) = T(A) \oplus T(B)$, we have $M = T(A) \oplus B$ and $T(M) = T(A) \oplus (T(M) \cap B)$. Hence $T(A) = A$ and $T(B) = T(M) \cap B$. So, $T(B) \ll_\delta B$. By Proposition 2.6, we have $T(B) \ll B$. Note that $M/T(M) \cong B/T(B)$ is divisible by Proposition 5.7. It follows that for every nonzero element $r \in R$, we have $rB + T(B) = B$. So, $rB = B$ for every $0 \neq r \in R$. This implies that B is a divisible module, that is, $Rad(B) = B$ (see [9, Lemma 2.1]). But $Rad(B) \ll B$ since $Rad(M) \ll M$. Then $B = 0$ and $M = A$ is a torsion module, as required. □

5.9. Proposition. *Assume that R is a nonlocal Dedekind domain. If M is a $\oplus$-δ-supplemented R-module, then M is a torsion module.*

Proof. By Proposition 5.2, $M = M_1 \oplus M_2$ is a direct sum of $\oplus$-δ-supplemented submodules M_1 and M_2 such that $Rad(M_1) = M_1$ and $Rad(M_2) \ll M_2$. By Proposition 5.1, M_1 is $\oplus$-supplemented. So, M_1 is a torsion module by [12, Proposition A.8]. Moreover, M_2 is a torsion module by Proposition 5.8. Therefore M is a torsion module, as required. □

5.10. Corollary. *Assume that R is a nonlocal Dedekind domain. An R-module M is $\oplus$-δ-supplemented if and only if M is $\oplus$-supplemented.*

Proof. This follows easily from Propositions 3.14 and 5.9. □

5.11. Remark. Combining Proposition 5.3, Corollary 5.10 and [12, Propositions A.7 and A.8], we obtain the structure of $\oplus$-δ-supplemented modules over Dedekind domains.

5.12. Lemma. *Assume that R is a Dedekind domain which is not local. Let P be a maximal ideal of R and let i be a nonzero natural number. Then:*

(i) *$I + P = P$ if and only if $I \subseteq P$.*

(ii) *If $i \geq 2$, then $I + P^i = P$ if and only if $I \subseteq P$ and $I \nsubseteq P^2$.*

(iii) *$I + P^i = R$ if and only if $I \nsubseteq P$.*

Proof. (i) and (iii) are immediate.

(ii) ($\Rightarrow$) This is obvious.

($\Leftarrow$) By hypothesis, we have $I = PI'$, where I' is an ideal of R which is not contained in P (see [13, Theorem 6.14]). Since $I' + P^{(i-1)} = R$, we see that $PI' + P^i = P$. Hence, $I + P^i = P$. □

Let M be a module over a Dedekind domain R and let P be a nonzero prime ideal of R. We will denote by M_P the set $\{x \in M \mid P^n x = 0 \text{ for some integer } n \geq 0\}$ which is called the P-primary component of M. Note that if M is a torsion R-module, then M is a direct sum of its P-primary components. Let K be the quotient field of R. We will denote by $R(P^\infty)$ the P-primary component of the torsion R-module K/R. It is well known that $R(P^\infty)$ is a hollow module (see [9, Lemma 2.4]).

The next result describes the structure of I-$\oplus$-supplemented modules over nonlocal Dedekind domains. Recall that a module M is 0-$\oplus$-supplemented if and only if M is semisimple (see Remark 3.3(ii)).

5.13. Theorem. *Assume that R is a nonlocal Dedekind domain. Let I be a nonzero ideal of R. Then the following assertions are equivalent for an R-module M:*

(i) *M is I-$\oplus$-supplemented;*

(ii) *M is torsion and every P-primary component of M is I-$\oplus$-supplemented;*

(iii) *M is torsion and for every nonzero prime ideal P of R, there exist natural numbers a and n such that $M_P \cong (R(P^\infty))^a \oplus B_P(1, \ldots, n)$ with $n = 1$ if $I \subseteq P^2$.*

Proof. (i) $\Leftrightarrow$ (ii) It is well known that for every nonzero prime ideal P of R, M_P is a fully invariant submodule of M. The result follows from Propositions 3.24, 4.3 and 5.9.

(ii) $\Rightarrow$ (iii) Let P be a nonzero prime ideal of R. Since M_P is I-$\oplus$-supplemented, M_P is $\oplus$-supplemented by Corollary 5.10. Thus, there exist natural numbers a and n such that $M_P \cong (R(P^\infty))^a \oplus B_P(1,\ldots,n)$ by [12, Propositions A.7 and A.8]. Let $1 \leq i \leq n$. Since M/IM is semisimple (see Proposition 3.9), $(R/P^i)/((I+P^i)/P^i) \cong R/(I+P^i)$ is semisimple. As R/P^i is a local R-module, we have $I+P^i = R$ or $I+P^i = P$. Note that if $I \subseteq P^2$ and $i \geq 2$, then $I+P^i \subseteq P^2$. In this case we have $I+P^i \neq R$ and $I+P^i \neq P$. This shows that $I \subseteq P^2$ forces $n = 1$.

(iii) $\Rightarrow$ (ii) Let P be a nonzero prime ideal of R. Note that M_P and $(R(P^\infty))^a$ are $\oplus$-supplemented by [12, Propositions A.7 and A.8]. We divide the rest of the proof into three cases:

Case 1. Assume that $I \subseteq P^2$. By hypothesis, $n = 1$. Therefore $B_P(1,\ldots,n) = B_P(1)$ is semisimple. Hence $M_P \cong (R(P^\infty))^a \oplus B_P(1)$ is I-$\oplus$-supplemented (see Proposition 5.1 and Theorem 3.22).

Case 2. Suppose that $I \nsubseteq P^2$ and $I \nsubseteq P$. Then, $IM_P = M_P$ by Lemma 5.12(iii). Therefore, M_P is I-$\oplus$-supplemented by Corollary 3.18.

Case 3. Assume that $I \nsubseteq P^2$ and $I \subseteq P$. In this case we have $IM_P = PM_P$ by Lemma 5.12. Applying Corollary 3.19, we conclude that M_P is I-$\oplus$-supplemented. This completes the proof. □

5.14. Remark. Let I be an ideal of a nonlocal Dedekind domain R. Using Theorem 5.13, [17, Theorem 1] and an analysis similar to that in Remark 5.5, we conclude that every I-$\oplus$-supplemented R-module is completely I-$\oplus$-supplemented.

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