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Investigation of spectral analysis of matrix quantum difference equations with spectral singularities

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Abstract

In this paper, we investigate the Jost solution, the continuous spectrum, the eigenvalues and the spectral singularities of a nonselfadjoint matrix-valued q-difference equation of second order with spectral singularities.

Keywords: Quantum difference equation, Discrete spectrum, Spectral theory, Spectral singularity, Eigenvalue.

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1. Introduction

Spectral analysis of nonselfadjoint differential equations including Sturm–Liouville, Schrödinger and Klein–Gordon equations has been treated by various authors since 1960 [23, 9, 11, 22, 12]. Study of spectral theory of nonselfadjoint discrete Schrödinger and Dirac equations were obtained in [1, 20, 8, 10, 7]. Also, spectral analysis of these equations in self-adjoint case is well-known [4, 5]. In addition to differential and discrete equations, spectral theory of q-difference equations has been investigated in recent years [2, 3], and important generalizations and results were given for dynamic equations including q-difference equations as a special case in [14, 13].

Some problems of spectral theory of differential and difference equations with matrix coefficients were studied in [15, 24, 18, 6]. But spectral analysis of the matrix q-difference equations with spectral singularities has not been investigated yet.

In this paper, we let $q > 1$ and use the notation $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us introduce the Hilbert space $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ consisting of all vector sequences $y \in \mathbb{C}^m$, $(y = y(t), \quad t \in q^{\mathbb{N}})$, such that $\sum_{t\in q^{\mathbb{N}}} \mu(t)\|y(t)\|^2_{\mathbb{C}^m} < \infty$ with the inner product $\langle y, z\rangle_q := \sum_{t\in q^{\mathbb{N}}} \mu(t)\,(y(t), z(t))_{\mathbb{C}^m}$, where $\mathbb{C}^m$ is m-dimensional $(m < \infty)$ Euclidean space, $\mu(t) = (q-1)t$ for all $t \in q^{\mathbb{N}}$, and $\|\cdot\|_{\mathbb{C}^m}$ and $(\cdot,\cdot)_{\mathbb{C}^m}$ denote

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the norm and inner product in $\mathbb{C}^m$, respectively. We denote by L the operator generated in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ by the q-difference expression

$$(ly)(t) := qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right), \quad t \in q^{\mathbb{N}},$$

and the boundary condition $y(1) = 0$, where $A(t)$, $t \in q^{\mathbb{N}_0}$ and $B(t)$, $t \in q^{\mathbb{N}}$ are linear operators (matrices) acting in $\mathbb{C}^m$. Throughout the paper, we will assume that $A(t)$ is invertible and $A(t) \neq A^*(t)$ for all $t \in q^{\mathbb{N}_0}$. Furthermore $B(t) \neq B^*(t)$ for all $t \in q^{\mathbb{N}}$, where $*$ denotes the adjoint operator. It is clear that L is a nonselfadjoint operator in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$. Related to the operator L, we will consider the matrix q-difference equation of second order

$$qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = \lambda y(t), \quad t \in q^{\mathbb{N}}, \tag{1.1}$$

where λ is a spectral parameter.

The set up of this paper is summarized as follows: Section 2 discusses the Jost solution of (1.1) and contains analytical properties and asymptotic behavior of this solution. In Section 3, we give the continuous spectrum of L, by using the Weyl compact perturbation theorem. In Section 4, we investigate the eigenvalues and the spectral singularities of L. In particular, we prove that L has a finite number of eigenvalues and spectral singularities with a finite multiplicity.

2. Jost solution of L

We assume that the matrix sequences $\{A(t)\}$ and $\{B(t)\}$, $t \in q^{\mathbb{N}}$ satisfy

$$\sum_{t \in q^{\mathbb{N}}} (\|I - A(t)\| + \|B(t)\|) < \infty, \tag{2.1}$$

where $\|\cdot\|$ denotes the matrix norm in $\mathbb{C}^m$ and I is identity matrix. Let $F(\cdot, z)$, denotes the matrix solution of the q-difference equation

$$qA(t)y(qt) + B(t)y(t) + A\left(\frac{t}{q}\right) y\left(\frac{t}{q}\right) = 2\sqrt{q}\cos z y(t), \quad t \in q^{\mathbb{N}}, \tag{2.2}$$

satisfying the condition

$$\lim_{t\to\infty} F(t,z) e^{i\frac{\ln t}{\ln q} z} \sqrt{\mu(t)} = I, \quad z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}. \tag{2.3}$$

The solution $F(\cdot, z)$ is called the Jost solution of (2.2).

2.1. Theorem. *Assume* (2.1). *Let the solution* $F(\cdot, z)$ *be the Jost solution of* (2.2). *Then*

$$F(t,z) = \frac{e^{i\frac{\ln t}{\ln q} z}}{\sqrt{\mu(t)}} I + \sum_{s \in [qt, \infty) \cap q^{\mathbb{N}}} \sqrt{\frac{s}{qt}} \, \frac{\sin\left(\frac{\ln s - \ln t}{\ln q}\right) z}{\sin z} H(s), \tag{2.4}$$

where

$$H(s) := \left[I - A\left(\frac{s}{q}\right)\right] F\left(\frac{s}{q}, z\right) - B(s)F(s,z) + q[I - A(s)]F(qs, z).$$

Proof. Using (2.2), we obtain

$$F\left(\frac{t}{q}\right) + qF(qt) - 2\sqrt{q}\cos z F(t) = H(t). \tag{2.5}$$

Since $\frac{\exp\left(i\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}I$ and $\frac{\exp\left(-i\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}I$ are linearly independent solutions of the homogeneous equation

$$F\left(\frac{t}{q}\right)+qF(qt)-2\sqrt{q}\cos zF(t)=0,$$

we get the general solution of (2.5) by

$$\begin{aligned}F(t,z)&=\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\alpha+\frac{e^{-i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\beta\\&+\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\sqrt{\frac{\mu(s)}{q}}\frac{1}{\sqrt{\mu(t)}}\quad\frac{\sin\left(\frac{\ln s-\ln t}{\ln q}\right)z}{\sin z}H(s),\end{aligned}\tag{2.6}$$

where α and β are constants in $\mathbb{C}^m$. Using (2.1), (2.3), and (2.6), we find $\alpha=I$ and $\beta=0$. This completes the proof, i.e., $F(t,z)$ satisfies (2.4). □

2.2. Theorem. *Assume* (2.1). *Then the Jost solution* $F(\cdot,z)$ *has a representation*

$$F(t,z)=T(t)\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(I+\sum_{r\in q^{\mathbb{N}}}K(t,r)e^{i\frac{\ln r}{\ln q}z}\right),\quad t\in q^{\mathbb{N}_0}\tag{2.7}$$

where $z\in\overline{\mathbb{C}}_+$, $T(t)$ *and* $K(t,r)$ *are expressed in terms of* $\{A(t)\}$ *and* $\{B(t)\}$.

Proof. If we put $F(\cdot,z)$ defined by (2.7) into (2.2), then we have the relations

$$\begin{aligned}&A(t)T(t)=T(qt),\quad K(t,q)-K(\frac{t}{q},q)=\frac{1}{\sqrt{q}}T^{-1}(t)B(t)T(t),\\&K(\frac{t}{q},q^2)-K(t,q^2)=T^{-1}(t)\left(T(t)-A^2(t)T(t)-\frac{1}{\sqrt{q}}B(t)T(t)K(t,q)\right),\\&K(t,rq^2)-K(\frac{t}{q},rq^2)=T^{-1}(t)\left(A^2(t)T(t)K(qt,r)+\frac{1}{\sqrt{q}}B(t)T(t)K(t,qr)\right)-K(t,r),\end{aligned}$$

and using these relations, we obtain

$$\begin{aligned}&T(t)=\prod_{p\in[t,\infty)\cap q^{\mathbb{N}}}[A(p)]^{-1},\quad K(t,q)=-\frac{1}{\sqrt{q}}\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}}T^{-1}(p)B(p)T(p),\\&K(t,q^2)=\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}}T^{-1}(p)\left[-\frac{1}{\sqrt{q}}B(p)T(p)K(p,q)+(I-A^2(p))T(p)\right],\\&K(t,rq^2)=K(qt,r)+\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}}T^{-1}(p)\left[I-A^2(p)\right]T(p)K(qp,r)\\&-\frac{1}{\sqrt{q}}\sum_{p\in[qt,\infty)\cap q^{\mathbb{N}}}T^{-1}(p)B(p)T(p)K(p,qr),\end{aligned}$$

for $r\in q^{\mathbb{N}}$ and $t\in q^{\mathbb{N}_0}$. Due to the condition (2.1), the infinite product and the series in the definition of $T(t)$ and $K(t,r)$ are absolutely convergent. □

Note that, in analogy to the Sturm–Liouville equation the function $F(1,z):=\frac{T(1)}{\sqrt{\mu(1)}}\left(I+\sum_{r\in q^{\mathbb{N}}}K(1,r)e^{i\frac{\ln r}{\ln q}z}\right)$ is called the Jost function.

2.3. Theorem. *Assume*

$$\sum_{t\in q^{\mathbb{N}}}\frac{\ln t}{\ln q}\left(\|I-A(t)\|+\|B(t)\|\right)<\infty.\tag{2.8}$$

Then the Jost solution $F(\cdot, z)$ is continuous in $\overline{\mathbb{C}}_+$ and analytic with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Proof. Using the equalities for $K(t,r)$ given in Theorem 2.2 and mathematical induction, we get

$$\|K(t,r)\| \leq C \sum_{p \in \left[tq^{\lfloor \frac{\ln r}{2\ln q} \rfloor}, \infty\right) \cap q^{\mathbb{N}}} \left(\|I - A(p)\| + \|B(p)\|\right), \tag{2.9}$$

where $C > 0$ is a constant and $\lfloor \frac{\ln r}{2\ln q} \rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$. From (2.8) and (2.9), we get that the series

$$\sum_{r \in q^{\mathbb{N}}} K(t,r) e^{i \frac{\ln r}{\ln q} z}, \quad \sum_{r \in q^{\mathbb{N}}} \frac{\ln r}{\ln q} K(t,r) e^{i \frac{\ln r}{\ln q} z}$$

are absolutely convergent in $\overline{\mathbb{C}}_+$ and in $\mathbb{C}_+$, respectively. This completes the proof. □

2.4. Theorem. *Under the condition* (2.8)*, the Jost solution satisfies*

$$F(t,z) = \frac{e^{i \frac{\ln t}{\ln q} z}}{\sqrt{\mu(t)}} \left(I + o(1)\right), \; z \in \overline{\mathbb{C}}_+, \; t \to \infty, \tag{2.10}$$

$$F(t,z) = T(t) \frac{e^{i \frac{\ln t}{\ln q} z}}{\sqrt{\mu(t)}} \left(I + o(1)\right), \; t \in q^{\mathbb{N}_0}, \; \operatorname{Im} z \to \infty. \tag{2.11}$$

Proof. It follows from the definition of $T(t)$, (2.8), and (2.9) that

$$\lim_{t \to \infty} T(t) = I, \tag{2.12}$$

and

$$\sum_{r \in q^{\mathbb{N}}} K(t,r) e^{i \frac{\ln r}{\ln q} z} = o(1), \; z \in \overline{\mathbb{C}}_+, \; t \to \infty. \tag{2.13}$$

From (2.7), (2.12), and (2.13), we get (2.10). Using (2.8) and (2.9), we have

$$\sum_{r \in q^{\mathbb{N}}} K(t,r) e^{i \frac{\ln r}{\ln q} z} = o(1), \; z \in \overline{\mathbb{C}}_+, \; \operatorname{Im} z \to \infty. \tag{2.14}$$

From (2.7) and (2.14), we get (2.11). □

3. Continuous spectrum of L

Let L_1 and L_2 denote the q-difference operators generated in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$ by the q-difference expressions

$$(l_1 y)(t) = q y(qt) + y\left(\frac{t}{q}\right)$$

and

$$(l_2 y)(t) = q\left[A(t) - I\right] y(qt) + B(t) y(t) + \left[A\left(\frac{t}{q}\right) - I\right] y\left(\frac{t}{q}\right)$$

with the boundary condition $y(1) = 0$, respectively. It is clear that $L = L_1 + L_2$.

3.1. Lemma. *The operator L_1 is self-adjoint in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$.*

Proof. Since

$$\|L_1 y\|_q \leq 2\sqrt{q}\|y\|_q$$

for all $y \in \ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$, L_1 is bounded in the Hilbert space $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$, and since

$$\begin{aligned} \langle l_1 y, z\rangle_q &= \sum_{t\in q^{\mathbb{N}}} \mu(t)(z(t))^* \left(qy(qt) + y\left(\frac{t}{q}\right)\right) \\ &= \sum_{t\in q^{\mathbb{N}}} \mu(t)\left(qz(qt) + z\left(\frac{t}{q}\right)\right)^* y(t) = \langle y, l_1 z\rangle_q, \end{aligned}$$

the operator L_1 is self-adjoint in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$. □

3.2. Theorem. *Assume* (2.8). *Then* $\sigma_c(L) = [-2\sqrt{q}, 2\sqrt{q}]$, *where* $\sigma_c(L)$ *denotes the continuous spectrum of* L.

Proof. It is easy to see that L_1 has no eigenvalues, so the spectrum of the operator L_1 consists only its continuous spectrum and

$$\sigma(L_1) = \sigma_c(L_1) = [-2\sqrt{q}, 2\sqrt{q}],$$

where $\sigma(L_1)$ denotes the spectrum of the operator L_1. Using (2.8), we find that L_2 is compact operator in $\ell_2(q^{\mathbb{N}}, \mathbb{C}^m)$[21]. Since $L = L_1 + L_2$ and $L_1 = (L_1)^*$, we obtain that

$$\sigma_c(L) = \sigma_c(L_1) = [-2\sqrt{q}, 2\sqrt{q}]$$

by using Weyl's theorem of a compact perturbation [19, p.13]. □

4. Eigenvalues and spectral singularities of L

If we define

(4.1) $$f(z) := detF(1, z),\ z \in \overline{\mathbb{C}}_+,$$

then the function f is analytic in $\mathbb{C}_+$, $f(z) = f(z + 2\pi)$ and is continuous in $\overline{\mathbb{C}}_+$. Let us define the semi-strips $P_0 = \{z \in \mathbb{C}_+ : 0 \leq \operatorname{Re} z \leq 2\pi\}$ and $P = P_0 \cup [0, 2\pi]$. We will denote the set of all eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definitions of eigenvalues and spectral singularities of nonselfadjoint operators[22, 23], we have

(4.2) $$\sigma_d(L) = \{\lambda \in \mathbb{C} :\ \lambda = 2\sqrt{q}\cos z,\ z \in P_0,\ f(z) = 0\},$$

(4.3) $$\sigma_{ss}(L) = \{\lambda \in \mathbb{C} :\ \lambda = 2\sqrt{q}\cos z,\ z \in [0, 2\pi],\ f(z) = 0\} \setminus \{0\}.$$

4.1. Theorem. *Assume* (2.8). *Then*

i) *the set* $\sigma_d(L)$ *is bounded and countable, and its limit points lie only in the interval* $[-2\sqrt{q}, 2\sqrt{q}]$,

ii) $\sigma_{ss}(L) \subset [-2\sqrt{q}, 2\sqrt{q}]$ *and the linear Lebesgue measure of the set* $\sigma_{ss}(L)$ *is zero.*

Proof. In order to investigate the quantitative properties of eigenvalues and spectral singularities of L, it is necessary to discuss the quantitative properties of zeros of f in P from (4.2) and (4.3). Using (2.11) and (4.1), we get

(4.4) $$f(z) = \det T(1)\frac{1}{\mu(1)}[I + o(1)],\ \operatorname{Im} z > 0,\ z \in P_0,\ \operatorname{Im} z \to \infty,$$

where $detT(1) \neq 0$. From (4.4), we get the boundedness of zeros of f in P_0. Since f is a 2π-periodic function and is analytic in $\mathbb{C}_+$, we obtain that f has at most a countable number of zeros in P_0. By the uniqueness of analytic functions, we find that the the limit points of zeros of f in P_0 can lie only in $[0, 2\pi]$. We get $\sigma_{ss}(L) \subset [-2\sqrt{q}, 2\sqrt{q}]$ using

(4.3). Since $f(z) \neq 0$ for all $z \in \mathbb{C}_+$, we get that the linear Lebesgue measure of the set of zeros of f on real axis is not positive, by using the boundary uniqueness theorem of analytic functions [17], i.e., the linear Lebesgue measure of the $\sigma_{ss}(L)$ is zero. □

4.2. Definition. The multiplicity of a zero of f in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

4.3. Theorem. *If, for some $\varepsilon > 0$,*

$$\sup_{t \in q^{\mathbb{N}}} \left\{ e^{\varepsilon \frac{\ln t}{\ln q}} \left(\|I - A(t)\| + \|B(t)\| \right) \right\} < \infty \tag{4.5}$$

then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Proof. Since $F(1,z) = \frac{T(1)}{\sqrt{q-1}} \left(I + \sum_{r \in q^{\mathbb{N}}} K(1,r) e^{i \frac{\ln r}{\ln q} z} \right)$, using (2.9) and (4.5), we get that

$$\|K(1,r)\| \leq D e^{-\frac{\varepsilon}{4} \frac{\ln r}{\ln q}}, \ r \in q^{\mathbb{N}}, \tag{4.6}$$

where $D > 0$ is a constant. From (4.1) and (4.6), we obtain that the function f has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{4}$. Because the series

$$\sum_{r \in q^{\mathbb{N}}} iK(1,r) \frac{\ln r}{\ln q} e^{i \frac{\ln r}{\ln q} z}$$

is uniformly convergent in $\operatorname{Im} z > -\frac{\varepsilon}{4}$. Since f is a 2π periodic function, the limit points of its zeros in P cannot lie in $[0, 2\pi]$. Using Theorem 4.1, we find that the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points, i.e., the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements. From the analyticity of f in $\operatorname{Im} z > -\frac{\varepsilon}{4}$, we get that all zeros of f in P have a finite multiplicity. □

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