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A generalization of supplemented modules

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Abstract

Let M be a left module over a ring R and I an ideal of R. M is called an I-supplemented module (finitely I-supplemented module) if for every submodule (finitely generated submodule) X of M, there is a submodule Y of M such that $X + Y = M$, $X \cap Y \subseteq IY$ and $X \cap Y$ is PSD in Y. This definition generalizes supplemented modules and δ-supplemented modules. We characterize I-semiregular, I-semiperfect and I-perfect rings which are defined by Yousif and Zhou [12] using I-supplemented modules. Some well known results are obtained as corollaries.

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1. Introduction and Preliminaries

It is well known that supplemented modules play an important role in characterizing semiperfect, semiregular and perfect rings. Recently, some authors had worked with various extensions of these rings (see for examples [1, 6, 7, 12, 13]). As generalizations of semiregular rings, semiperfect rings and perfect rings, the notions of I-semiregular rings, I-semiperfect rings and I-perfect rings were introduced by Yousif and Zhou [12]. Our purposes of this paper is to chracterize I-semiregular rings, I-semiperfect rings and I-perfect rings by defining I-supplemented modules.

Let R be a ring and I an ideal of R, M a module and $S \leq M$. S is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. M is said to be singular if $M = Z(M)$, where $Z(M) = \{x \in M : l_R(x)$ is essential in ${}_RR\}$. As a proper generalization of small submodules, the concept of δ-small submodules was introduced by Zhou[13]. N is said to be δ-*small* in M if, whenever $N + X = M$, M/X singular, we have $X = M$. $\delta(M) = Rej_M(\wp) = \cap\{N \leq M \mid M/N \in \wp\}$, where $\wp$ be the class of all singular simple modules. Let $N, L \leq M$. N is called a *supplement* of L in M if

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$N + L = M$ and N is minimal with respect to this property. Equivalently, $M = N + L$ and $N \cap L \ll N$. M is called *supplemented* if every submodule of M has a supplement in M. M is said to be *lifting* if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently, for every submodule N of M, M has a decomposition with $M = M_1 \oplus M_2$, $M_1 \leq N$ and $M_2 \cap N$ is small in M_2. N is called a *δ-supplement* [4] of L if $M = N + L$ and $N \cap L \ll_\delta N$. M is called a *δ-supplemented module* if every submodule of M has a δ-supplement. M is said to be *δ-lifting* [4] if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll_\delta M/K$, equivalently, for every submodule N of M, M has a decomposition with $M = M_1 \oplus M_2$, $M_1 \leq N$ and $M_2 \cap N$ is δ-small in M_2. M is (δ-) *semiregular* [10] if for any cyclic submodule N of M, there is a decomposition $M = P \oplus Q$ such that $P \leq N$ and $N \cap Q$ is a (δ-) small submodule of Q. An element m of M is called *I-semiregular* [1] if there exists a decomposition $M = P \oplus Q$ where P is projective, $P \subseteq Rm$ and $Rm \cap Q \subseteq IM$. M is called *an I-semiregular module* if every element of M is I-semiregular. R is called *I-semiregular* if ${}_RR$ is an I-semiregular module. Note that I-semiregular rings are left-right symmetric and R is ($\delta-$) semiregular if and only if R is ($\delta({}_RR)-$) $J(R)$-semiregular. M is called (δ)-*semiperfect* [7] if every factor module of M has a projective (δ-)cover. M is called *an I-semiperfect module* [7] if for every submodule K of M, there is a decomposition $M = A \oplus B$ such that A is projective, $A \subseteq K$ and $K \cap B \subseteq IM$. R is called *I-semiperfect* if ${}_RR$ is an I-semiperfect module. Note that R is ($\delta-$) semiperfect if and only if R is ($\delta({}_RR)-$) $J(R)$-semiperfect. R is called *a left I-perfect ring* [12] if, for any submodule X of a projective module P, X has a decomposition $X = A \oplus B$ where A is a direct summand of P and $B \subseteq IP$. By [7, Proposition 2.1], R is a left I-perfect ring if and only if every projective module is an I-semiperfect module. For other standard definitions we refer to [2, 3, 11].

In this note all rings are associative with identity and all modules are unital left modules unless specified otherwise. Let R be a ring and M a module. We use $Rad(M)$, $Soc(M)$, $Z(M)$ to indicate the Jacobson radical, the socle, the singular submodule of M respectively. $J(R)$ is the radical of R and I is an ideal of R.

2. PSD submodules and I-supplemented modules

In this section, we give some properties of PSD submodules and use PSD submodules to define (finitely) I-supplemented modules and I-lifting modules which are generalizations of some well-known supplemented modules and lifting modules. Some properties of I-supplemented modules are discussed. We begin this section with the following definitions.

2.1. Definition. Let I be an ideal of R and $N \leq M$. N is PSD in M if there exists a projective summand S of M such that $S \leq N$ and $M = S \oplus X$ whenever $N + X = M$ for any submodule $X \leq M$. M is PSD for I if any submodule of IM is PSD in M. R is a left PSD ring for I if any finitely generated free left R-module is PSD for I.

2.2. Lemma. Let M and N be modules.

(1) If K is PSD in M and $f : M \to N$ is an epimorphism, then $f(K)$ is PSD in N.
(2) If $L \leq N \leq M$ and L is PSD in N, then L is PSD in M.
(3) If $L \leq N \leq M$ and N is PSD in M, then L is PSD in M.
(4) Let $M = M_1 \oplus M_2$. If N_1 is PSD in M_1 and N_2 is PSD in M_2, then $N_1 \oplus N_2$ is PSD in M.
(5) Let N be a direct summand of M and $A \leq N$. Then A is PSD in M if and only if A is PSD in N.

Proof. (1) Let $f(K) + L = N$ with $L \leq N$. Then $K + f^{-1}(L) = M$. Since K is PSD in M, there is a projective summand H of M with $H \leq K$ such that $H \oplus f^{-1}(L) = M$. So $f(H) \oplus L = N, f(H) \subseteq f(K)$. It is easy to see that $f(H) \cong H$ is projective.

(2) Let $M = L + X$ with $X \leq M$. Then $N = L + (N \cap X)$. Since L is PSD in N, there is a projective summand H of N with $H \leq L$ such that $N = H \oplus (N \cap X)$, and hence $L = H \oplus (L \cap X)$. So $M = H \oplus X$.

(3) Let $M = L + K$ with $K \leq M$, then $M = N + K$. Since N is PSD in M, there is a projective summand H of M with $H \leq N$ such that $M = H \oplus K$, and hence $M/K \cong H$ is projective. Thus the natural epimorphism $f : L \to M/K$ splits and $Kerf = L \cap K$ is a direct summand of L. Write $L = (L \cap K) \oplus Q$ with $Q \leq L$, we have $M = Q \oplus K$. The rest is obvious.

(4) Let $M = N_1 \oplus N_2 + L$ with $L \leq M$. Since N_1 is PSD in M_1, N_1 is PSD in M. Thus there is a projective summand S_1 of M with $S_1 \subseteq N_1$ such that $M = S_1 \oplus (N_2 + L)$. Similarly, there exists a projective summand S_2 of M with $S_2 \subseteq N_2$ such that $M = S_1 \oplus S_2 \oplus L$. The rest is obvious.

(5) " $\Rightarrow$ " Since N is a direct summand of M, $M = N \oplus K$ for some submodule $K \leq M$. Suppose that $N = A + X$ with $X \leq N$, then $M = A + (X \oplus K)$. Since A is PSD in M, there is a projective direct summand Y of M such that $Y \leq A$ and $M = Y \oplus X \oplus K$, and hence $N = N \cap M = X \oplus Y$.

" $\Leftarrow$ " Let $M = A + L$ with $L \leq M$. Then $N = N \cap M = A + N \cap L$. Since A is PSD in N, there is a projective summand K of N with $K \leq A$ such that $N = K \oplus (N \cap L)$. It is easy to see that $K \cap L = 0$. Next we only show that $M = K + L$. Let $m \in M$, then $m = a + l$, $a \in A, l \in L$. Since $a = k + s, k \in K, s \in N \cap L$, $m = k + s + l$. Note that $s + l \in L$, so $m \in K + L$, and hence $M = K + L$, as required.

□

2.3. Proposition. Let M be a module and $N \leq M$.

(1) $N \ll M$ if and only if $N \subseteq Rad(M)$, N is PSD in M.
(2) $N \ll_\delta M$ if and only if $N \subseteq \delta(M)$, N is PSD in M.

Proof. (1) " $\Rightarrow$ " is clear.

" $\Leftarrow$ " Let $M = N + L$ with $L \leq M$. Since N is PSD in M, there is a projective summand H of M with $H \subseteq N \subseteq Rad(M)$ such that $M = H \oplus L$. So $Rad(H) \oplus Rad(L) = Rad(M) = H \oplus Rad(L)$. Thus $Rad(H) = H$. Since H is projective, $H = 0$, and hence $L = M$.

(2) " $\Rightarrow$ " is clear by [13, Lemma 1.2].

" $\Leftarrow$ " Let $M = N + L$ with $L \leq M$. Since N is PSD in M, there is a projective summand H of M with $H \subseteq N \subseteq \delta(M)$ such that $M = H \oplus L$. So $\delta(H) \oplus \delta(L) = \delta(M) = H \oplus \delta(L)$. Thus $\delta(H) = H$. Since H is projective, H is semisimple by [7, Proposition 2.13]. Thus $N \ll_\delta M$ by [13, Lemma 1.2].

□

2.4. Corollary. Let M be a module. Then

(1) M is (δ-) supplemented if and only if for every submodule X of M, there is a submodule Y of M such that $X + Y = M$, $X \cap Y \subseteq (\delta(Y))$ $Rad(Y)$ and $X \cap Y$ is PSD in Y.
(2) M is (δ-) lifting if and only if for every submodule X of M, there is a decomposition $M = A \oplus B$ such that $A \subseteq X$ and $X \cap B \subseteq (\delta(B))$ $Rad(B)$ and $X \cap B$ is PSD in B.

2.5. Definition. Let R be a ring and I an ideal of R, M a module. M is called an I-supplemented module (finitely I-supplemented module) if for every submodule (finitely generated submodule) X of M, there is a submodule Y of M such that $X+Y=M$, $X\cap Y\subseteq IY$ and $X\cap Y$ is PSD in Y. In this case, we call Y is an I-supplement of X in M. M is called I-lifting if for every submodule X of M, there is a decomposition $M=A\oplus B$ such that $A\subseteq X$ and $X\cap B\subseteq IB$ and $X\cap B$ is PSD in B.

2.6. Example. It is easy to see that a module M is 0-supplemented if and only if M is semisimple, and so the supplemented module $\mathbb{Z}(p^\infty)$ is not 0-supplemented, where p is a prime integer. However, $\mathbb{Z}(p^\infty)$ is I-supplemented for every nonzero ideal I of $\mathbb{Z}$.

2.7. Theorem. Consider the following statements for a module M.

(1) M is a $J(R)$-supplemented module (a $\delta({}_RR)$-supplemented module, respectively).
(2) M is a supplemented module (a δ-supplemented module, respectively).

Then "(1) $\Rightarrow$ (2)", "(2) $\Rightarrow$ (1)" if M is projective or R satisfies $J(R)M=Rad(M)$ $(\delta({}_RR)M=\delta(M))$ for any module M over R.

Proof. "(1) $\Rightarrow$ (2)" By Proposition 2.3.

"(2) $\Rightarrow$ (1)" Let M be a supplemented module. Then for every submodule X of M, there is a submodule Y of M such that $X+Y=M$ and $X\cap Y\ll Y$. Since M is projective, Y is a direct summand of M, and hence Y is projective. It is clear that $X\cap Y\subseteq Rad(Y)=J(R)Y$ and $X\cap Y$ is PSD in Y. (Let M be a δ-supplemented module. Since M is projective, M is δ-lifting. Thus for every submodule X of M, there is a direct summand Y of M such that $M=X+Y$ and $X\cap Y\ll_\delta Y$. The rest is obvious.) When R satisfies $J(R)M=Rad(M)$ $(\delta({}_RR)M=\delta(M))$ for any module M over R, the proof is similar.

□

Similar to the proof of Theorem 2.7, we have the following.

2.8. Theorem. Consider the following statements for a module M.

(1) M is a finitely $J(R)$-supplemented module (a finitely $\delta({}_RR)$-supplemented module, respectively).
(2) M is a finitely supplemented module (a finitely δ-supplemented module, respectively).

Then "(1) $\Rightarrow$ (2)", "(2) $\Rightarrow$ (1)" if M is projective or R satisfies $J(R)M=Rad(M)$ $(\delta({}_RR)M=\delta(M))$ for any module M over R.

2.9. Theorem. Consider the following statements for a module M.

(1) M is a $J(R)$-lifting module (a $\delta({}_RR)$-lifting module, respectively).
(2) M is a lifting module (a δ-lifting module, respectively).

Then "(1) $\Rightarrow$ (2)", "(2) $\Rightarrow$ (1)" if M is projective or R satisfies $J(R)M=Rad(M)$ $(\delta({}_RR)M=\delta(M))$ for any module M over R.

We know that if a ring R is left $(\delta-)$semiperfect ring, then $(\delta({}_RR)M=\delta(M))$ $J(R)M=Rad(M)$ for any module M over R. So "(1) $\Leftrightarrow$ (2)" in Theorem 2.7, 2.8 and 2.9 if R is left $(\delta-)$semiperfect ring.

2.10. Lemma. Let M be a module and $K,L,H\leq M$. If K is an I-supplement of L in M, L is an I-supplement of H in M, then L is an I-supplement of K in M.

Proof. Let $M=K+L=L+H$ with $K\cap L\subseteq IK, L\cap H\subseteq IL$ and $K\cap L$ be PSD in K, $L\cap H$ be PSD in L. We only show that $K\cap L\subseteq IL$ and $K\cap L$ is PSD in L. It is easy to see that $K\cap L\subseteq IK\cap L$. Let $l=\Sigma_{i=1}^n p_ik_i\in IK\cap L$, $p_i\in I, k_i\in K$ and $k_i=l'_i+h_i(i=1,2,\cdots,n)$, $l'_i\in L, h_i\in H$. Since $L\cap H\subseteq IL$, $l\in IL$, and hence

$K \cap L \subseteq IL$. Next, we shall prove that $K \cap L$ is PSD in L. Let $K \cap L + X = L$ with $X \leq L$, then $M = L + H = K \cap L + X + H$. Since $K \cap L$ is PSD in K, $K \cap L$ is PSD in M by Lemma 2.2. Thus there is a projective summand Y of M with $Y \subseteq K \cap L$ such that $M = Y \oplus (X + H)$. Since $L = L \cap M = L \cap (Y \oplus (X + H)) = Y \oplus (X + L \cap H)$ and $L \cap H$ is PSD in L, there is a projective summand Y' of L with $Y' \subseteq L \cap H$ such that $L = Y \oplus X \oplus Y'$. Since $L/X \cong Y \oplus Y'$ is projective, the natural epimorphism $f : K \cap L \to L/X$ splits, and hence $Kerf = K \cap X$ is a direct summand of $K \cap L$. Write $K \cap L = (K \cap X) \oplus Q, Q \leq K \cap L$. So $L = Q \oplus X$, as required.

□

2.11. Lemma. Let M be a π-projective module. If N and K are I-supplement of each other in M, then $N \cap K$ is projective. If in addition M is projective, then N and K are projective.

Proof. Let $f : N \oplus K \to N + K = M$ with $(n, k) \mapsto n + k$ for $n \in N, k \in K$. Since M is a π-projective module, f splits, and so $Kerf = \{(n, -n) | n \in N \cap K\}$ is a direct summand of $N \oplus K$. Write $N \oplus K = Kerf \oplus U, U \cong M$. Since $N \cap K$ is PSD in N and K, $Kerf$ is PSD in $N \oplus K$ by Lemma 2.2. Thus there is a projective summand Y of $N \oplus K$ with $Y \subseteq Kerf$ such that $N \oplus K = Y \oplus U$, so $Y = Kerf \cong N \cap K$ is projective. If M is projective, $Y \oplus U$ is projective. So N and K are projective. □

Recall that a pair (P, f) is called a projective I-cover of M [9] if P is projective, f is an epimorphism from P to M such that $Kerf \leq IP$, and $Kerf$ is PSD in P.

We end this section with the following lemma.

2.12. Lemma. Let $M = A + B$. If M/A has a projective I-cover, then B contains an I-supplement of A.

Proof. Let $\pi : B \to M/A$ be the canonical homomorphism and $f : P \to M/A$ a projective I-cover. Since P is projective, there is a homomorphism $g : P \to B$ such that $\pi g = f$. Thus $M = A + g(P)$ and $A \cap g(P) = g(Kerf)$. Since $Kerf \subseteq IP$ and $Kerf$ is PSD in P, $A \cap g(P) \subseteq Ig(P)$ and $A \cap g(P)$ is PSD in $g(P)$ by Lemma 2.2. So $g(P)$ is an I-supplement of A contained in B. □

3. Characterizations of I-semiregular, I-semiperfect and I-perfect rings in terms of I-supplemented modules

We shall characterize I-semiregular rings, I-semiperfect rings and I-perfect rings by I-supplemented modules in this section. We begin this section with the following.

3.1. Theorem. Let R be a ring and I an ideal of R, P a projective module. Consider the following conditions:

(1) P is an I-supplemented module.
(2) P is an I-semiperfect module.

Then (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (1) if P is PSD for I.

Proof. "(1) $\Rightarrow$ (2)" Let P be an I-supplemented module and $N \leq P$. Then there exists $X \leq P$ such that $P = N + X$, $N \cap X \subseteq IX$ and $N \cap X$ is PSD in X. Let $\pi : P \to P/N$ and $\pi \mid_X : X \to P/N$ be the canonical epimorphisms. Since P is projective, there is a homomorphism $g : P \to X$ such that $\pi \mid_X g = \pi$. We have $P = g(P) + N$ and $X = g(P) + N \cap X$. Since $N \cap X$ is PSD in X, there is a projective summand Y of X with $Y \subseteq N \cap X$ such that $X = g(P) \oplus Y$. It is easy to verify that $g(P) \cap N \subseteq Ig(P)$. Since $g(P) \cap N \subseteq N \cap X$ and $N \cap X$ is PSD in X, $g(P) \cap N$ is PSD in X by Lemma 2.2, and so $g(P) \cap N$ is PSD in $g(P)$ by Lemma 2.2. Thus $g(P)$ is an I-supplement of N in

P. Since P is an I-supplemented module, $g(P)$ has an I-supplement Q in P. Thus $g(P)$ is also an I-supplement of Q in P by Lemma 2.10, and so $g(P)$ is projective by Lemma 2.11. Since $g(P) \cap N \subseteq Ig(P)$ and $g(P) \cap N$ is PSD in $g(P)$, the canonical epimorphism $g(P) \to P/N$ is a projective I-cover of P/N. So P is an I-semiperfect module by [9, Lemma 2.9].

"(2) $\Rightarrow$ (1)" Let P be an I-semiperfect module, then for every submodule X of P, there is a decomposition $P = A \oplus Y$ such that A is projective, $A \subseteq X$ and $X \cap Y \subseteq IP$. Thus $P = X + Y$, $X \cap Y \subseteq IY$. Since P is PSD for I, $X \cap Y$ is PSD in Y by Lemma 2.2, as desired.

□

By Theorem 3.1, we know that if a module M is projective and PSD for I, then M is an I-supplemented module if and only if M is I-lifting if and only if M is an I-semiperfect module.

3.2. Corollary. Let M be a projective module with $Rad(M) \ll M$ ($\delta(M) \ll_\delta M$). Then M is a (δ-)supplemented module if and only if M is a (δ-)semiperfect module if and only if M is a (δ-)lifting module.

3.3. Theorem. Let I be an ideal of R. Consider the following conditions:

(1) Every finitely generated R-module is I-supplemented.
(2) Every finitely generated projective R-module is I-supplemented.
(3) Every finitely generated projective R-module is I-lifting.
(4) ${}_RR$ is I-lifting.
(5) ${}_RR$ is I-supplemented.
(6) R is I-semiperfect.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (6) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold; if R is a left PSD ring for I, (2) $\Rightarrow$ (3) and (6) $\Rightarrow$ (1) also hold.

Proof. "(1) $\Rightarrow$ (2) $\Rightarrow$ (5)" and "(3) $\Rightarrow$ (4) $\Rightarrow$ (5)" are clear.

"(5) $\Rightarrow$ (6)" By Theorem 3.1.

If R is a left PSD ring for I, then (2) $\Rightarrow$ (3) is obvious by Theorem 3.1 and [9, Corollary 2.4].

"(6) $\Rightarrow$ (1)" Let M be a finitely generated module and $N \leq M$. Then $M = N + M$ and M/N has a projective I-cover by [9, Theorem 2.13], so M contains an I-supplement of N by Lemma 2.12. Hence M is I-supplemented. □

Let $I = J(R)$ or $\delta({}_RR)$ in Theorem 3.3, since R is a left PSD ring for I and R is ($\delta-$) semiperfect if and only if ($\delta({}_RR)-$)$J(R)$-semiperfect , we use Theorem 2.7 and Theorem 2.9 to obtain the following.

3.4. Corollary. ([5, Theorem 4.41]) The following statements are equivalent for a ring R.

(1) R is semiperfect.
(2) Every finitely generated R-module is supplemented.
(3) Every finitely generated projective R-module is supplemented.
(4) Every finitely generated projective R-module is lifting.
(5) ${}_RR$ is lifting.
(6) ${}_RR$ is supplemented.

3.5. Corollary. ([4, Theorem 3.3]) The following statements are equivalent for a ring R.

(1) R is δ-semiperfect.

(2) Every finitely generated R-module is δ-supplemented.
(3) Every finitely generated projective R-module is δ-supplemented.
(4) Every finitely generated projective R-module is δ-lifting.
(5) ${}_RR$ is δ-lifting.
(6) ${}_RR$ is δ-supplemented.

Since if R is $Z({}_RR)$-semiregular, then $Z({}_RR) = J(R) \subseteq \delta({}_RR)$ by [1, Theorem 3.2], and so R is a left PSD ring for $Z({}_RR)$. Thus we have the following result.

3.6. Corollary. The following statements are equivalent for a ring R.

(1) R is $Z({}_RR)$-semiperfect.
(2) Every finitely generated R-module is $Z({}_RR)$-supplemented.
(3) Every finitely generated projective R-module is $Z({}_RR)$-supplemented.
(4) Every finitely generated projective R-module is $Z({}_RR)$-lifting.
(5) ${}_RR$ is $Z({}_RR)$-lifting.
(6) ${}_RR$ is $Z({}_RR)$-supplemented.

Let M be a projective module, then $Soc(M) = Soc({}_RR)M$. So if $I \leq Soc({}_RR)$, then $IM \subseteq Soc(M)$, and hence R is a left PSD ring for I. Thus we have

3.7. Corollary. The following statements are equivalent for a ring R.

(1) R is $Soc({}_RR)$-semiperfect.
(2) Every finitely generated R-module is $Soc({}_RR)$-supplemented.
(3) Every finitely generated projective R-module is $Soc({}_RR)$-supplemented.
(4) Every finitely generated projective R-module is $Soc({}_RR)$-lifting.
(5) ${}_RR$ is $Soc({}_RR)$-lifting.
(6) ${}_RR$ is $Soc({}_RR)$-supplemented.

3.8. Theorem. Let R be a left PSD ring and I an ideal of R. Then R is an I-semiregular ring if and only if ${}_RR$ is a finitely I-supplemented module if and only if R_R is a finitely I-supplemented module.

Proof. Similar to Theorem 3.3. □

3.9. Corollary. ([8, Proposition 19.1]) The following statements are equivalent for a ring R.

(1) R is semiregular.
(2) ${}_RR$ is a finitely supplemented module.
(3) R_R is a finitely supplemented module.

3.10. Corollary. The following statements are equivalent for a ring R.

(1) R is δ-semiregular.
(2) ${}_RR$ is a finitely δ-supplemented module.
(3) R_R is a finitely δ-supplemented module.

3.11. Corollary. A ring R is $Soc({}_RR)$-semiregular if and only if ${}_RR$ is a finitely $Soc({}_RR)$-supplemented module if and only if R_R is a finitely $Soc(R_R)$-supplemented module.

3.12. Corollary. A ring R is $Z({}_RR)$-semiregular if and only if ${}_RR$ is a finitely $Z({}_RR)$-supplemented module if and only if R_R is a finitely $Z(R_R)$-supplemented module.

Next we use I-supplemented modules to characterize I-perfect rings.

3.13. Definition. A ring R is called a strongly left PSD ring for I if any projective left R-module is PSD for I.

3.14. Example. It is easy to verify that a ring R is perfect if and only if R is a semiperfect ring and a strongly PSD ring for $J(R)$. Let $R = \mathbb{Z}_{(p)}$ (integers localized at the prime p). It is well known that R is a commutative, semiperfect ring that is not perfect, and so R is a PSD ring for $J(R)$ that is not a strongly PSD ring for $J(R)$ (Since if R is a strongly PSD ring for $J(R)$, then R is perfect. This is a contradiction.).

3.15. Theorem. Let I be an ideal of R. Consider the following conditions:

(1) Every R-module is I-supplemented.
(2) Every projective R-module is I-supplemented.
(3) Every projective R-module is I-lifting.
(4) Every free R-module is I-lifting.
(5) Every free R-module is I-supplemented.
(6) R is left I-perfect.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (5) $\Rightarrow$ (6) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold; if R is a strongly left PSD ring for I, (2) $\Rightarrow$ (3) and (6) $\Rightarrow$ (1) also hold.

Proof. "(1) $\Rightarrow$ (2) $\Rightarrow$ (5)" and "(3) $\Rightarrow$ (4) $\Rightarrow$ (5)" are clear.

"(5) $\Rightarrow$ (6)" By Theorem 3.1.

When R is a strongly left PSD ring for I, "(2) $\Rightarrow$ (3)" is obvious.

"(6) $\Rightarrow$ (1)" Let M be a module. Then there is a free module F such that $\eta : F \to M$ is epic. Since F is I-semiperfect, there is a decomposition $F = F_1 \oplus F_2$ such that $F_1 \subseteq Ker\eta$ and $F_2 \cap Ker\eta \subseteq IF_2$. Since F is PSD for I, $F_2 \cap Ker\eta$ is PSD in F. By Lemma 2.2, $F_2 \cap Ker\eta$ is PSD in F_2, so $\eta|_{F_2} : F_2 \to M$ is a projective I-cover of M. Thus we prove that an arbitrary module has a projective I-cover, and so for $N \leq M$, M/N has a projective I-cover. The rest follows by Lemma 2.12. □

Let $I = J(R)$ or $\delta(_RR)$ in Theorem 3.15. Since R is ($\delta-$)perfect if and only if R is ($\delta(_RR)-$) $J(R)$-perfect and if a ring R is ($\delta-$)perfect, then for every module M, ($\delta(M) \ll_\delta M$) $Rad(M) \ll M$, R is a strongly left PSD ring for I. So we have the following.

3.16. Corollary. ([5, Theorem 4.41])The following statements are equivalent for a ring R.

(1) R is left perfect.
(2) Every R-module is supplemented.
(3) Every projective R-module is supplemented.
(4) Every projective R-module is lifting.
(5) Every free R-module is lifting.
(6) Every free R-module is supplemented.

3.17. Corollary. ([4, Theorem 3.4]) The following statements are equivalent for a ring R.

(1) R is left δ-perfect.
(2) Every R-module is δ-supplemented.
(3) Every projective R-module is δ-supplemented.
(4) Every projective R-module is δ-lifting.
(5) Every free R-module is δ-lifting.
(6) Every free R-module is δ-supplemented.

Since if $I \leq Soc(_RR)$, then R is a strongly left PSD ring for I, and hence we have

3.18. Corollary. The following statements are equivalent for a ring R.

(1) R is left $Soc(_RR)$-perfect.

(2) Every R-module is $Soc(_RR)$-supplemented.
(3) Every projective R-module is $Soc(_RR)$-supplemented.
(4) Every projective R-module is $Soc(_RR)$-lifting.
(5) Every free R-module is $Soc(_RR)$-lifting.
(6) Every free R-module is $Soc(_RR)$-supplemented.

Since if R is $Z(_RR)$-perfect, then $Z(_RR) = J(R) \subseteq \delta(_RR)$ by [1, Theorem 3.2], and so we have the following result.

3.19. Corollary. The following statements are equivalent for a ring R.
(1) R is left $Z(_RR)$-perfect.
(2) Every R-module is $Z(_RR)$-supplemented.
(3) Every projective R-module is $Z(_RR)$-supplemented.
(4) Every projective R-module is $Z(_RR)$-lifting.
(5) Every free R-module is $Z(_RR)$-lifting.
(6) Every free R-module is $Z(_RR)$-supplemented.

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