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Equivariant estimation of common location parameter of two exponential populations using censored samples

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Abstract

In this paper, we consider the problem of estimating common location parameter of two exponential populations using type-II censored samples when the scale parameters are unknown. The loss function is taken as the quadratic loss. First, we derive a class of affine equivariant estimators, which includes the maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE). A sufficient condition for improving estimators in the class is derived. Consequently, estimators dominating the MLE and the UMVUE in terms of the risk values are obtained. An example is given to compute the estimates using our result. Finally a simulation study has been carried out to numerically compare the risk functions of all the estimators.

Keywords: Brewster - Zidek technique, Equivariant estimators, Inadmissibility, Quadratic loss function, Relative risk performances, Type-II censoring.

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1. Introduction

Suppose $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ $(2 \leq r \leq m)$ and $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$ $(2 \leq s \leq n)$ be the ordered observations taken from two exponential populations $Ex(\mu, \sigma_1)$ and $Ex(\mu, \sigma_2)$ respectively. Here $Ex(\mu, \sigma_i)$ denotes the exponential distribution with density function

$$f(t, \mu, \sigma_i) = \frac{1}{\sigma_i} \exp\{-(t-\mu)/\sigma_i\}, \quad t > \mu, \sigma_i > 0, \qquad -\infty < \mu < \infty;\ i = 1, 2. \tag{1.1}$$

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The problem is to estimate the common location parameter μ (minimum guarantee time) when the scale parameters σ_1, σ_2 (residual life times) are unknown, with respect to the loss function,

$$L(d, \underline{\alpha}) = \left(\frac{d-\mu}{\sigma_1}\right)^2, \tag{1.2}$$

where d is an estimate for μ and $\underline{\alpha} = (\mu, \sigma_1, \sigma_2)$.

The model (1.1) under consideration arises naturally in the study of reliability, life testing and survival analysis and has applications in industry, engineering, business and social science. For example, two brands of electronic devices having $m(\geq 2)$ and $n(\geq 2)$ number of units respectively put for a life testing experiment. Due to some constraints (may be time or cost) the experimenter could able to observe only the $r(\leq m)$ and $s(\leq n)$ failure times respectively. It is assumed that, the life times of each units are random and follow exponential distributions having same minimum guarantee time. The problem we consider, comes under the umbrella of estimation problems "estimation of parameters of a distribution function using censored samples". For some more examples on related model one may refer to Suresh [13]. Basically, the censoring schemes available are type-I (number of failures are random), type-II (censoring time is random) and random censoring (both may be random) or some modifications of these. We consider the conventional type-II right censoring sampling scheme which is a particular case of progressive type-II censoring scheme. For some results on estimation of parameters of exponential distributions using various such conventional censoring schemes one may refer to Lawless [9] and Johnson et al. [8]. For some reference on estimation of parameters using progressive type-II censored samples one may refer to Chandrasekar et al. [4], Madi [12] and Wang et al. [14] and the references cited there in. Some applications of these types of models have been discussed in Balakrishnan and Aggarwala [1] and Balakrishnan and Cramer [2]. It is very surprising to see in the literature that, a very little attention has been paid for estimation of a common mean/location (or in general common parameter) when incomplete data (censored samples) are available from the population. In that regard, Chiou and Cohen [5] considered the model in (1.1) under type-II censoring scheme and estimate the common location parameter μ, when the scale parameters are unknown. They obtained the maximum likelihood estimate (MLE) and the uniformly minimum variance unbiased estimate (UMVUE) for μ. They have also generalized the results to $k = 3$ exponential populations. Elfessi and Pal [6] considered the problem of estimation of common scale parameter of several exponential populations under type-II censoring scheme. They provided stein type testimators for the common scale parameter and used this to construct estimators for the location parameters.

In the case of full sample (that is $r = m$ and $s = n$) probably, Ghosh and Razmpour [7] was the first to consider the problem of estimation of μ. They obtained the MLE, a modification to the MLE (MMLE) and the UMVUE for μ. Asymptotic and numerical comparisons of these estimators were done in terms of bias and mean squared error. Their simulation study shows that, the MLE is dominated by both the MMLE and the UMVUE. Jin and Pal [11] considered the problem of estimation of common location parameter of several exponential populations and suggested a class of estimators which dominates the MLE under a class of convex loss functions. For some early results on the estimation of common location of exponential populations we refer to Jin and Crouse [10] and the references there in.

In this article, we consider the model in (1.1) under the conventional type-II censoring, which was considered earlier by Chiou and Cohen [5] and estimated the common location parameter μ with respect to a quadratic loss function. The aim of the present work is twofold, one is to propose a wide class of estimators which include the MLE, the MMLE (we propose in next section) and the UMVUE for μ. Secondly, we derive a sufficient condition that helps in obtaining estimators which dominate estimators belonging to this class. The rest of the work is organized as follows. In Section 2, we present the model and discuss some basic results. In Section 3, a general class of estimators has been proposed and sufficient conditions for improving estimators in the class has been derived. This class contains the MLE, MMLE and the UMVUE for μ. Using the results of section 3, estimators dominating the MLE and the UMVUE have been obtained. In Section 4, a massive simulation study has been carried out to numerically compare the risk performances of all these estimators.

2. Some Basic Results

In this section, we discus the model and derive some basic estimators such as the MLE, a modification to the MLE (MMLE) and the uniformly minimum variance unbiased estimator (UMVUE) for the common location parameter μ, when the scale parameters are unknown.

Specifically, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $(2 \leq r \leq m)$ be the r ordered observations taken from a random sample of size m which follows $Ex(\mu, \sigma_1)$ as in (1.1). Similarly, let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(s)}$, $(2 \leq s \leq n)$ be the s ordered observations from a random sample of size n which follows $Ex(\mu, \sigma_2)$ as in (1.1). We assume that the two random samples drawn are stochastically independent. The joint probability density function of $\underline{X}_r = (X_{(1)}, X_{(2)}, \cdots, X_{(r)})$ and $\underline{Y}_s = (Y_{(1)}, Y_{(2)}, \cdots, Y_{(s)})$ is given by

$$f(\underline{x}_r, \underline{y}_s) = M \exp\left\{ -\frac{\sum_{i=1}^{r}(x_{(i)} - \mu) + (m-r)(x_{(r)} - \mu)}{\sigma_1} - \frac{\sum_{j=1}^{s}(y_{(j)} - \mu) + (n-s)(y_{(s)} - \mu)}{\sigma_2} \right\}, \tag{2.1}$$

where, $\mu \leq x_{(1)} \leq x_{(2)} \cdots \leq x_{(r)}$; $\mu \leq y_{(1)} \leq y_{(2)} \cdots \leq y_{(s)}$; $-\infty < \mu < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$ and

$$M = \frac{m(m-1)\cdots(m-r+1)n(n-1)\cdots(n-s+1)}{\sigma_1^r \sigma_2^s}.$$

It can be observed that, the maximum likelihood estimator (MLE) of μ is $Z = \min(X_{(1)}, Y_{(1)}) = d_{ML}$ (say). The MLEs of both σ_1 and σ_2 can be obtained by differentiating the log-likelihood function with respect to σ_i $(i = 1, 2)$ and equating to zero. These are obtained as,

$$\begin{aligned} \hat{\sigma}_1 &= \frac{\sum_{i=1}^{r}(X_{(i)} - Z) + (m-r)(X_{(r)} - Z)}{r}, \\ \hat{\sigma}_2 &= \frac{\sum_{j=1}^{s}(Y_{(j)} - Z) + (n-s)(Y_{(s)} - Z)}{s}. \end{aligned}$$

Let us introduce the new random variables

$$U_1 = \frac{\sum_{i=1}^{r} X_{(i)} + (m-r)X_{(r)}}{m}, \quad \text{and} \quad U_2 = \frac{\sum_{j=1}^{s} Y_{(j)} + (n-s)Y_{(s)}}{n}.$$

For our model, a sufficient statistic is (U_1, U_2, Z) (see Chiou and Cohen [5]). Further, the joint probability density function of (U_1, U_2, Z) is given by,

$$f_{U_1,U_2,Z}(u_1, u_2, z) = K\Big[\frac{(u_1 - z)^{r-2}(u_2 - z)^{s-1}}{\Gamma s \Gamma(r-1)} + \frac{(u_1 - z)^{r-1}(u_2 - z)^{s-2}}{\Gamma r \Gamma(s-1)}\Big] \exp\left\{ -\frac{m(u_1 - \mu)}{\sigma_1} - \frac{n(u_2 - \mu)}{\sigma_2} \right\}, \quad u_1 > x_{(1)}, u_2 > y_{(1)}, z > \mu \tag{2.2}$$

where

$$K = \frac{m^r n^s}{\sigma_1^r \sigma_2^s}.$$

It should be noted that the details of derivation of the joint probability density function of (U_1, U_2, Z) has been omitted here for brevity, however for equal sample sizes one may refer to Chiou and Cohen [5].

The probability density function of Z is given by

$$f_Z(z) = p\exp\{-p(z-\mu)\}, \quad z > \mu, \tag{2.3}$$

where $p = \frac{m}{\sigma_1} + \frac{n}{\sigma_2}$. It can be noted that, $E(Z) = \mu + \frac{1}{p}$. Motivated by Ghosh and Razmpour [7], we propose a modification to the MLE d_{ML} as,

$$d_{MM} = Z - \frac{1}{\hat{p}}, \tag{2.4}$$

where, we have the MLE for p as $\hat{p} = \frac{m}{\hat{\sigma}_1} + \frac{n}{\hat{\sigma}_2}$. It can be further noticed that the statistics $(U_1 - Z, U_2 - Z)$ and Z are independent. Using the complete and sufficient statistic $(U_1 - Z, U_2 - Z, Z)$, it is easy to observe that the UMVUE of μ is given by,

$$d_{MV} = Z - \frac{(U_1 - Z)(U_2 - Z)}{(r-1)(U_2 - Z) + (s-1)(U_1 - Z)}, \tag{2.5}$$

(see Chiou and Cohen [5] for $m = n$ and $r = s$).

In the next section, we prove a general inadmissibility result for affine equivariant class of estimators and as a consequence, estimators dominating the MLE d_{ML} and the UMVUE d_{MV} in terms of risk values have been obtained.

3. A Sufficient Condition for Improving Equivariant Estimators

In this section, we introduce the concept of invariance to our problem and obtain some inadmissibility conditions for estimators belonging to the affine equivariant class.

Let $G = \{g_{a,b} : g_{a,b}(x) = ax + b, a > 0, -\infty < b < \infty\}$ be an affine group of transformations. Let us use the notation $V_1 = U_1 - Z$, $V_2 = U_2 - Z$. Under the transformation $g_{a,b}$, the sufficient statistics, $V_1 \to aV_1$, $V_2 \to aV_2$ and $Z \to aZ + b$. The set of parameters being transformed as $\mu \to a\mu + b$, $\sigma_i \to a\sigma_i, i = 1, 2$. In order that, the loss function (2.1) to be invariant, the decision rule d must satisfy the equation,

$$d(aZ + b, aV_1, aV_2) = ad(Z, V_1, V_2) + b.$$

Taking choice for $b = -aZ$, where $a = \frac{1}{V_1}$, and rearranging the terms, we obtain the form of an affine equivariant estimator based on (Z, V_1, V_2) for estimating μ as,

$$\begin{aligned} d(Z, V_1, V_2) &= Z + V_1\Psi(V), \\ &= d_\Psi, \quad \text{(say)}, \end{aligned} \tag{3.1}$$

where $\Psi(V)$ is any function of $V = \frac{V_2}{V_1}$.

Further, define a function Ψ_0, for the affine equivariant estimator d_Ψ (as given in (3.1)) as,

$$\Psi_0(v) = \begin{cases} -\frac{1}{r+s}\max(v,1), & \text{if } \Psi(v) < -\frac{1}{r+s}\max(v,1) \\ \Psi(v), & \text{if } -\frac{1}{r+s}\max(v,1) \le \Psi(v) \le -\frac{1}{r+s}\min(v,1), \\ -\frac{1}{r+s}\min(v,1), & \text{if } \Psi(v) > -\frac{1}{r+s}\min(v,1) \end{cases} \tag{3.2}$$

Next, we present the main result of this section which helps in obtaining the improved estimators for μ.

3.1. Theorem. *For the affine equivariant estimator d_Ψ given in (3.1), define the function Ψ_0 as given in (3.2) and the loss function be the affine invariant loss (1.2). The estimator d_Ψ is inadmissible and is improved by d_{Ψ_0}, if there exist some values of parameters $(\mu, \sigma_1, \sigma_2)$ such that, $P(d_\Psi \neq d_{\Psi_0}) > 0$.*

Proof. The proof of this theorem can be done by using a result of Brewster and Zidek [3]. So, consider the conditional risk function of d_Ψ given $V = v$.

$$\begin{aligned} R((d_\Psi, \underline{\alpha})|V = v) &= \frac{1}{\sigma_1^2} E[(d_\Psi - \mu)^2 | V = v], \\ &= \frac{1}{\sigma_1^2} E[(Z + V_1\Psi(V) - \mu)^2 | V = v]. \end{aligned} \tag{3.3}$$

The above risk function (3.3) is a convex function in Ψ. Hence, the minimizing value of $\Psi(V)$ for fixed values of V is obtained as,

$$\hat{\Psi}(v, \sigma_1, \sigma_2) = -\frac{1}{p}\frac{E(V_1|V = v)}{E(V_1^2|V = v)}. \tag{3.4}$$

To evaluate the above expression in (3.4), we have the joint probability density function of (U_1, U_2, Z) as given in (2.2). Let us use the transformation $V_1 = U_1 - Z$, $V_2 = U_2 - Z$ and $Z = Z$. The inverse transformation is given by $U_1 = V_1 + Z$, $U_2 = V_2 + Z$, and $Z = Z$. The jacobian is obtained as $J = 1$. Hence, the joint probability density function of (Z, V_1, V_2) is obtained as,

$$\begin{aligned} f_{V_1,V_2,Z}(v_1, v_2, z) = \frac{m^r n^s}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r-1}v_2^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r-2}v_2^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\{-\frac{m}{\sigma_1}(v_1 + z - \mu) - \\ \frac{n}{\sigma_2}(v_2 + z - \mu)\}, \\ v_1 > 0, v_2 > 0, z > \mu. \end{aligned}$$

Using the independence of (V_1, V_2) and Z one can easily write the joint probability density function of (V_1, V_2) and is given by,

$$\begin{aligned} f_{V_1,V_2}(v_1, v_2) = \frac{m^r n^s p^{-1}}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r-1}v_2^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r-2}v_2^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\Big\{-\frac{m}{\sigma_1}v_1 - \frac{n}{\sigma_2}v_2\Big\}, \\ v_1 > 0, v_2 > 0. \end{aligned}$$

We need to calculate the conditional density of V_1 given V. Let us use the transformation, $V = \frac{V_2}{V_1}$, $V_1 = V_1$. The inverse transformation is given by $V_2 = VV_1$, $V_1 = V_1$. The jacobian of this transformation is obtained as V_1. Hence the joint probability density function of (V_1, V) is obtained as,

$$\begin{aligned} f_{V_1,V}(v_1, v) = \frac{m^r n^s p^{-1}}{\sigma_1^r \sigma_2^s}\Big[\frac{v_1^{r+s-2}v^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v_1^{r+s-2}v^{s-1}}{\Gamma s\Gamma(r-1)}\Big] \exp\Big\{-\frac{m}{\sigma_1}v_1 - \frac{n}{\sigma_2}vv_1\Big\}, \\ v_1 > 0, v > 0. \end{aligned}$$

The marginal density function of V is given by

$$\begin{aligned} f_V(v) = \frac{m^r n^s p^{-1}\Gamma(r+s-1)}{\sigma_1^r \sigma_2^s}\Big(\frac{m}{\sigma_1} + \frac{n}{\sigma_2}v\Big)^{1-r-s}\Big[\frac{v^{s-2}}{\Gamma r\Gamma(s-1)} + \frac{v^{s-1}}{\Gamma s\Gamma(r-1)}\Big], \\ v > 0. \end{aligned}$$

It is easy to observe that, the conditional probability density function of V_1 given $V = v$, is a gamma distribution with shape parameter $r + s - 1$ and scale parameter $\frac{\sigma_1\sigma_2}{m\sigma_2 + n\sigma_1 v}$.

Here the gamma probability density function with a shape parameter α and a scale parameter β is defined as,

$$g(x,\alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\beta}},\quad x>0,\alpha>0,\beta>0.$$

So, the conditional expectations are calculated and obtained as

$$E(V_1|V=v)=\frac{(r+s-1)\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v},\tag{3.5}$$

and

$$E(V_1^2|V=v)=(r+s-1)(r+s)\Big(\frac{\sigma_1\sigma_2}{m\sigma_2+n\sigma_1 v}\Big)^2.\tag{3.6}$$

Substituting these conditional expectations from (3.5) and (3.6) in (3.4), and simplifying, we have the minimizing choice of $\hat{\Psi}(\tau,v)$ for fixed v as,

$$\hat{\Psi}(\tau,v)=-\frac{m+n\tau v}{(r+s)(m+n\tau)},\tag{3.7}$$

where $\tau=\frac{\sigma_1}{\sigma_2}>0$, and $v>0$.

In order to apply the Brewster Zidek orbit-by-orbit improvement technique (see Brewster and Zidek [3]), we need to find the supremum and infimum of $\hat{\Psi}(\tau,v)$ with respect to τ for fixed v. Let $h(\tau)=-\frac{m+n\tau v}{m+n\tau}$. It can be easily seen that, $h(\tau)$ is an increasing function in τ if and only if $v<1$ and decreasing if and only if $v\geq 1$. We consider two separate cases for obtaining the supremum and infimum of $\hat{\Psi}(v)$, depending upon $v<1$ or $v\geq 1$.

Case-I: $v<1$. In this case, the supremum and infimum of the function $\Psi(\tau,v)$ for fixed values of v, are obtained as,

$$\sup_{\tau>0}\hat{\Psi}(v,\tau)=-\frac{v}{r+s},\quad\text{and}\quad \inf_{\tau>0}\hat{\Psi}(v,\tau)=-\frac{1}{r+s}.$$

Case-II: $v\geq 1$. For this case we have the supremum and infimum of $\Psi(\tau,v)$ as,

$$\sup_{\tau>0}\hat{\Psi}(v,\tau)=-\frac{1}{r+s},\quad\text{and}\quad \inf_{\tau>0}\hat{\Psi}(v,\tau)=-\frac{v}{r+s}.$$

Utilizing the results from Case-I and II, we can easily define the function $\Psi_0(v)$ as in (3.2). Now applying the orbit by orbit improvement technique of [3] (see Theorem 3.1.1 in Brewster and Zidek [3]), the proof follows. □

Next, our target is to apply the results of Theorem 3.1, and provide improved estimators for μ which will perform better than the MLE d_{ML} and the UMVUE d_{MV} in terms of risk values. The class considered above contains the MLE d_{ML}, the modified MLE d_{MM} and the UMVUE d_{MV}. Hence, expressing d_{ML} and d_{MV} in the form (3.1), we have

$$\begin{aligned} d_{ML} &= Z+V_1\Psi_{ML}(V),\text{where}\ \ \Psi_{ML}(V)=0,\\ d_{MV} &= Z+V_1\Psi_{MV}(V),\text{where}\ \ \Psi_{MV}(V)=-\frac{V}{(r-1)V+(s-1)}. \end{aligned}$$

Let us define the new estimators for μ as,

$$d_{MLI}=\begin{cases} Z-\frac{V_2}{r+s}, & \text{if } V_1>V_2,\\ Z-\frac{V_1}{r+s}, & \text{if } V_1\leq V_2\end{cases}\tag{3.8}$$

and

$$d_{MVI}=\begin{cases} Z-\frac{V_1}{r+s}\max(V,1), & \text{if } \Psi_{MV}(V)<-\frac{1}{r+s}\max(V,1),\\ d_{MV}, & \text{otherwise.}\end{cases}\tag{3.9}$$

Next, we present the result in the form of a theorem, regarding improvement over the MLE d_{ML} and the UMVUE d_{MV}, for estimating μ.

3.2. Theorem. *Let the loss function be the quadratic loss as in (1.2).*

- *The estimator d_{ML} (MLE) is inadmissible and is improved by d_{MLI}.*
- *The estimator d_{MVI} improves upon d_{MV} (UMVUE), if there exists some values of parameters $(\mu, \sigma_1, \sigma_2)$ such that, $P(d_{MV} \neq d_{MVI}) > 0$.*

Proof. The proof follows by an application of Theorem 3.1. The choice of $\Psi_{ML} = 0 > -\frac{v}{r+s}$ (when $v < 1$) and also $\Psi_{ML} = 0 > -\frac{1}{r+s}$ (when $v \geq 1$). Hence, replacing these choices by their respective supremum values, we get the estimator defined in (3.8), which has smaller risk values than d_{ML} by an application of Theorem 3.1. Also we note that, for estimator d_{MV}, the choice $P(\Psi_{MV}(V) < -\frac{1}{r+s}) > 0$ (when $v < 1$) and $P(\Psi_{MV}(V) < -\frac{v}{r+s}) > 0$ (when $v \geq 1$). Replacing $\Psi_{MV}(V)$ by these extreme values in d_{MV} we get the required estimator d_{MVI} as given in (3.9), which has smaller risk values than d_{MV}. $\square$

Let us define $\Psi_1 = -\frac{1}{r+s}\max(v, 1)$ and $\Psi_2 = -\frac{1}{r+s}\min(v, 1)$.

3.1. Remark. Though the estimator d_{MM} is a member of the class considered in (3.1) (we can write $d_{MM} = Z + V_1\Psi_{MM}(V)$, where $\Psi_{MM}(V) = -\frac{V}{rV+s}$), it can not be improved by using our result in Theorem 3.1, as it can be seen that, $P(\Psi_{MM}(V) \in [\Psi_1, \Psi_2]) = 1$.

3.2. Remark. The class of estimators $D_\Psi = \{d_\Psi : \Psi_1 \leq \Psi \leq \Psi_2\}$ form a complete class for estimating common location parameter μ when the loss is (1.2).

Next, we present an example where our model fits well and compute the estimates for the minimum guarantee time.

3.1. Example. (Simulated Data) Suppose two brands of electronic devices each having 30 units are placed for a life testing experiment. It is known that, the lifetimes (in hours) of each unit follows an exponential distribution with same minimum guarantee time. The experimenter could able to observe only 15 failures (in hours) from each brands of devices because of some constraints. The data for both the brands are obtained as,

Brand 1: 1417.70, 1458.49, 2963.76, 3969.39, 5995.44, 6939.76, 7048.85, 7768.59, 7844.87, 8824.96, 9190.34, 9321.34, 9434.04, 10793.03, 12881.22.

Brand 2: 462.71, 659.86, 1187.35, 1295.99, 1370.69, 2050.36, 2305.46, 2633.27, 3176.41, 3297.63, 3413.95, 3806.01, 4571.04, 4639.71, 6059.09.

On the basis of above data, we have computed the statistic values as $Z = 462.7199$, $T_1 = 9506.285$, and $T_2 = 3931.15$. The various estimates for μ have been computed as $d_{ML} = 462.7199$, $d_{MLI} = 331.6816$, $d_{MM} = 277.3143$, $d_{MV} = 264.0711$ and $d_{MVI} = 264.0711$. It can be noted that the condition for improvement over d_{MV} (that is $\Psi_{MV} < -\frac{1}{r+s}\max(w, 1)$) is not satisfied. So, in this case we will not get improved estimator for d_{MV}. In this situation, we recommend to use d_{MM}.

4. Numerical Comparisons

In this section, we compare numerically the simulated risk values of all the estimators proposed in previous sections for estimating μ. For this purpose, we have generated 20,000 type-II censored random samples each from two exponential populations (1.1) with a common location parameter μ and different scale parameters σ_1, σ_2. The loss function is taken as (1.2). We use Monte-Carlo simulation method to compute the risk values of each estimator. The accuracy of simulation has been checked and the error is of the order of 10^{-3}. It can be easily seen that with respect to the loss (1.2), the risk functions of all the estimators are function of τ (> 0) for fixed sample sizes. Though the values of τ can lie in the interval $(0, \infty)$ theoretically, to avoid simulation error we present the risk values for τ up to 4. Let us define the percentage of relative risk improvements (RRI) of all estimators with respect to the MLE as,

$$\begin{aligned} R(MLI) &= \frac{d_{ML} - d_{MLI}}{d_{ML}} \times 100, \ R(MM) = \frac{d_{ML} - d_{MM}}{d_{ML}} \times 100 \\ R(MV) &= \frac{d_{ML} - d_{MV}}{d_{ML}} \times 100, \ R(MVI) = \frac{d_{ML} - d_{MVI}}{d_{ML}} \times 100. \end{aligned}$$

Further we define the censoring factors ($CF1$, $CF2$) for both the populations as the ratio of number of observed samples to total number of samples. That is for first population $CF1 = r/m$ and for second population $CF2 = s/n$. It can be noticed that the censoring factors $CF1$ and $CF2$ always lie between 0 and 1. A massive simulation study has been carried out by considering various combinations of sample sizes. However, for illustration purpose, we present (in Table 4.1-4.3) the percentage of relative risk performances of d_{MLI}, d_{MM}, d_{MV} and d_{MVI} over d_{ML} for equal and unequal sample sizes. Specifically in Table 4.1 we present the percentage of relative risk performances for sample sizes $(16, 16)$ and $(24, 24)$. The first column gives the values of τ. Corresponding to one value of τ, there corresponds three values of relative risk performances for an estimators. These three values are obtained for $CF1 = CF2 = 0.25, 0.50, 0.75$ respectively. Similarly in Tables 4.2-4.3 the relative risk performances have been presented for unequal sample sizes. We have also plotted the graph of the RRI values of the improved estimators with respect to MLE in Figures 1 and 2 for $CF1 = CF2 = 0.25$ and $CF1 = CF2 = 0.5$ respectively. It can be seen that, as the values of $CF1$ and $CF2$ become close to 1, the amount of improvements for d_{MVI} over d_{MV} is marginal.

The following conclusions can be made from our simulation study and Table 4.1-4.3.

(i) The risk values of all the estimators are decreasing as τ increases, with respect to the loss function (1.2). Further, as τ becomes large the risk values of all the estimators converge to some constant value. The percentage of relative risk performances of each estimator with respect to MLE increases as censoring factors ($CF1$ and $CF2$) increase for fixed sample sizes.

(ii) When the sample sizes are small, and for small values of τ, the percentage of relative risk improvement for d_{MM} is maximum (near about 46%). For moderate values of τ the estimator d_{MVI} has the best percentage of relative risk performance (near about 46.5%). For large values of τ the estimator d_{MM} performs the best (near about 45%).

(iii) For moderate sample sizes, and for small values of τ the estimator d_{MM} performs the best(about 47%). When τ values are moderate the estimators d_{MM} and d_{MVI} are good competitors of each other. For large values of τ the estimator d_{MM} performs the best (about 48.5%).

(iv) For large sample sizes, and for small values of τ the estimator d_{MVI} has the best performance (48%). For moderate values of τ the estimators d_{MM} and d_{MVI} are competing each other. For large values of τ the estimator d_{MM} has the best percentage of relative risk performance (48%).

(vi) As the sample sizes increase for fixed censoring factors ($CF1$ and $CF2$) the amount of percentage of improvements of d_{MLI} over d_{ML} increases. Also the amount of improvement of d_{MVI} over d_{MV} increases as sample sizes increase. The percentage of risk improvement of d_{MVI} over d_{MV} is near about 2.5% and this value decreases as $CF1$ and $CF2$ become close to 1. The percentage of improvement for d_{MLI} over d_{ML} has been seen near about 45.5%. This validates the findings of theoretical results in Section 3.

(vii) It has been also noticed that for small and large values of τ (that is when the standard deviations vary significantly) the percentage of relative risk improvements for d_{MVI} is very marginal. A similar type of observations were made for other combinations of m, n and r, s and we omit the tables here.

(viii) Combining the facts (ii)-(iv), we recommend using d_{MM} for all sample sizes. Though the estimator performs better theoretically (around 2.5% improvement from simulation study) than d_{MM}, we do not recommend using it as it is not a smooth estimator.

5. Conclusions

In this article, we have considered the model that was earlier considered by Chiou and Cohen [5] for exponential populations. Specifically, we have considered the estimation of common location parameter μ of two exponential populations when the samples are type-II right censored in a decision theoretic approach. First we propose a broad class of estimators (which are equivariant under an affine group of transformations) for the common location parameter μ. Interestingly, this class contains the MLE and the UMVUE for μ. Then we provide a sufficient condition which may be useful for improving certain estimators in this class. Using our results of Theorem 3.1, we have obtained an estimator which dominates the MLE significantly (the percentage of relative risk improvement is between 28% to 46%). However, the improved estimator obtained for the UMVUE has marginal percentage of risk improvements. The theoretical results are well supported by a simulation study. It should be noted that, a very little attention has been given by the researchers in the recent past for the problem considered in this article. Our work revisits the problem and will definitely help the researchers to search new estimators which may work better than the usual one.

Table 4.1: Relative risk performances of different estimators for μ with CF1=CF2=0.25,0.50,0.75

$\tau\downarrow$	m=n=16 and r=s=4,8,12				m=n=24 and r=s=6,12,18			
	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$
0.25	33.53	40.45	37.89	38.28	36.95	44.43	43.73	43.78
	38.07	45.33	44.88	44.90	39.62	47.01	46.90	46.90
	39.74	47.04	46.92	46.92	40.54	47.75	47.70	47.70
0.50	36.14	41.51	40.22	41.34	40.33	44.93	44.38	44.63
	42.32	46.77	46.60	46.70	43.48	47.42	47.29	47.30
	42.81	47.06	47.03	47.07	44.71	48.30	48.21	48.22
0.75	36.95	41.19	39.84	41.28	41.06	44.75	44.71	45.08
	43.37	46.40	46.38	46.58	44.76	47.12	47.21	47.25
	45.40	47.66	47.56	47.64	46.91	48.74	48.73	48.76
1.00	36.91	41.13	40.54	41.84	41.36	44.61	44.24	44.80
	43.05	45.57	45.32	45.61	45.68	47.54	47.62	47.67
	46.24	48.07	48.03	48.11	47.17	48.40	48.37	48.41
1.25	37.04	41.11	39.91	41.40	41.69	45.15	44.95	45.35
	43.31	46.07	46.08	46.22	45.66	47.69	47.67	47.71
	45.66	47.77	47.74	47.81	47.00	48.60	48.62	48.63
1.50	36.94	41.47	40.52	41.78	40.81	44.64	44.51	44.84
	43.30	46.55	46.41	46.56	45.85	48.58	48.57	48.61
	44.08	46.73	46.71	46.76	45.93	48.43	48.48	48.49
1.75	36.55	41.29	40.00	41.09	40.19	44.35	43.84	44.23
	42.78	46.62	46.38	46.54	44.19	47.39	47.29	47.31
	44.06	47.46	47.43	47.46	45.06	48.22	48.24	48.25
2.00	36.31	41.67	40.28	41.48	39.85	44.55	44.17	44.45
	41.34	45.79	45.66	45.80	43.58	47.44	47.35	47.36
	43.12	47.04	46.94	46.96	43.79	47.76	47.84	47.84
2.25	35.56	41.56	40.84	41.65	39.05	44.24	43.88	44.10
	41.15	46.22	46.08	46.13	42.34	46.86	46.76	46.77
	42.19	47.11	47.14	47.14	44.53	48.81	48.74	48.74
2.50	35.49	41.43	40.01	40.72	38.77	44.60	44.42	44.56
	40.18	45.91	45.92	45.95	42.26	47.50	47.42	47.42
	42.22	47.26	47.12	47.12	43.32	48.38	48.35	48.35
2.75	34.69	40.89	39.52	40.17	38.29	44.12	43.57	43.66
	40.42	46.30	46.05	46.08	41.04	46.71	46.64	46.64
	41.84	47.48	47.33	47.34	42.76	48.28	48.24	48.24
3.00	34.69	41.25	39.37	40.01	37.65	44.02	43.49	43.61
	39.39	45.78	45.62	45.65	41.31	47.33	47.19	47.19
	41.43	47.70	47.66	47.66	42.30	47.90	47.78	47.78
3.25	33.76	40.31	38.73	39.18	37.41	43.81	43.07	43.16
	39.01	45.49	45.08	45.12	41.10	47.42	47.30	47.30
	40.58	46.69	46.46	46.46	41.26	47.63	47.62	47.62
3.50	33.60	40.59	39.24	39.73	36.49	43.66	43.39	43.46
	38.88	45.76	45.45	45.46	40.77	47.60	47.44	47.45
	39.85	46.48	46.35	46.35	41.16	47.74	47.70	47.70
3.75	33.37	40.29	38.57	38.95	36.26	43.28	42.58	42.65
	38.39	45.22	44.71	44.72	40.13	47.33	47.26	47.26
	40.37	47.54	47.41	47.41	41.13	47.95	47.88	47.88
4.00	33.25	40.55	38.77	39.17	36.25	43.72	43.12	43.17
	38.84	45.89	45.35	45.36	39.91	47.02	46.82	46.82
	40.19	47.51	47.35	47.35	40.77	47.89	47.82	47.82

Table 4.2: Relative risk performances of different estimators for μ with CF1=CF2=0.25,0.50,0.75

$\tau\downarrow$	m=12, n=16 and r=3,6,9;s=4,8,12				m=16,n=24 and r=4,8,12;s=6,12,18			
	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$
0.25	29.25 35.04 36.97	37.89 44.60 46.18	34.50 44.27 45.94	35.38 44.29 45.94	31.45 36.23 37.14	41.49 46.53 47.65	40.45 46.27 47.62	40.63 46.28 47.62
0.50	32.74 39.42 41.94	39.09 44.97 47.14	37.52 44.80 46.96	39.11 44.89 47.01	36.07 40.79 42.25	42.83 46.90 47.81	42.52 46.94 47.79	43.10 47.00 47.80
0.75	34.76 42.23 44.05	39.84 45.88 47.15	38.18 45.60 47.23	40.33 45.91 47.29	39.25 43.45 45.70	43.96 46.94 48.37	43.31 47.04 48.32	44.16 47.11 48.36
1.00	35.28 42.72 45.23	39.89 45.92 47.48	38.37 45.90 47.53	40.58 46.24 47.61	39.79 44.68 46.89	43.63 46.95 48.56	43.20 46.86 48.58	44.07 47.01 48.63
1.25	36.23 42.67 45.66	40.66 45.58 47.63	39.31 45.63 47.45	41.34 45.82 47.57	39.76 45.22 46.41	43.25 47.16 47.99	42.80 46.95 47.99	43.49 47.06 48.04
1.50	35.91 42.88 45.20	40.14 45.88 47.53	38.42 45.75 47.47	40.25 45.98 47.52	40.43 44.96 46.86	44.05 47.30 48.62	43.65 47.34 48.57	44.32 47.42 48.58
1.75	35.92 42.04 44.36	40.24 45.31 47.13	38.61 45.13 47.06	40.13 45.32 47.10	40.30 43.86 46.41	43.92 46.20 47.99	43.42 46.00 47.99	43.95 46.11 48.04
2.00	35.56 41.86 43.84	40.27 45.49 47.06	38.44 45.20 47.01	40.20 45.32 47.04	39.64 44.45 44.86	43.22 47.31 47.42	42.43 47.22 47.37	42.85 47.24 47.37
2.25	35.96 41.39 43.64	40.75 45.43 47.33	38.80 45.25 47.26	40.09 45.35 47.28	40.18 43.16 45.15	44.16 46.47 48.31	43.59 46.39 48.35	43.89 46.42 48.35
2.50	35.56 41.00 43.34	40.52 45.26 47.24	38.81 44.94 47.08	39.93 45.03 47.10	39.32 42.95 44.75	43.34 46.28 47.94	42.48 46.01 47.84	42.83 46.02 47.84
2.75	34.77 41.15 42.67	39.67 45.70 46.74	37.73 45.34 46.49	38.64 45.41 46.49	38.48 42.86 44.14	42.70 46.61 47.98	41.93 46.39 47.98	42.24 46.40 47.98
3.00	35.79 40.39 42.31	41.03 45.18 46.84	39.09 44.86 46.68	39.89 44.90 46.69	38.51 42.14 44.36	43.07 46.35 48.21	42.41 46.29 48.15	42.64 46.31 48.15
3.25	34.58 40.11 42.24	39.71 45.23 47.07	37.22 44.88 46.86	38.10 44.95 46.87	38.88 42.37 43.46	43.40 46.53 47.78	42.37 46.30 47.77	42.59 46.31 47.77
3.50	34.35 39.99 42.06	39.71 44.93 47.33	37.56 44.35 47.23	38.19 44.37 47.24	38.68 42.19 43.44	43.24 46.74 47.60	42.33 46.56 47.47	42.46 46.56 47.47
3.75	34.03 39.45 40.85	39.71 44.74 46.03	37.80 44.28 45.86	38.46 44.29 45.86	38.35 41.49 42.85	43.18 46.00 47.12	42.32 45.80 46.96	42.45 45.80 46.96
4.00	33.97 39.61 40.81	39.39 45.10 46.32	36.59 44.54 46.20	37.23 44.57 46.20	38.19 41.79 43.13	43.11 46.23 47.75	42.12 45.85 47.63	42.27 45.85 47.63

Table 4.3: Relative risk performances of different estimators for μ with CF1=CF2=0.25,0.50,0.75

$\tau \downarrow$	m=16, n=12 and r=4,8,12;s=3,6,9				m=24,n=16 and r=6,12,18;s=4,8,12			
	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$	$R(MLI)$	$R(MM)$	$R(MV)$	$R(MVI)$
	34.11	39.83	37.83	38.35	38.28	43.11	42.09	42.21
0.25	39.67	45.15	44.67	44.67	42.23	46.88	46.57	46.58
	41.39	46.83	46.62	46.63	43.01	47.51	47.36	47.36
	35.57	40.03	38.08	39.47	40.31	43.99	43.14	43.66
0.50	41.87	45.42	45.15	45.32	44.64	47.61	47.54	47.59
	43.98	47.17	47.14	47.14	45.64	48.10	47.99	48.00
	36.05	40.50	39.30	41.18	40.32	43.70	43.15	43.80
0.75	43.15	46.10	46.11	46.31	45.22	47.34	47.20	47.33
	45.30	47.41	47.40	47.47	46.67	48.21	48.16	48.19
	35.94	40.60	38.90	41.29	40.17	44.11	43.82	44.59
1.00	43.05	46.17	46.18	46.51	44.81	47.11	47.00	47.16
	44.87	47.09	47.08	47.21	46.24	47.89	47.90	47.94
	35.13	40.18	38.90	41.13	38.67	43.17	42.73	43.59
1.25	41.50	45.10	45.03	45.38	43.57	46.53	46.40	46.55
	45.11	47.91	47.81	47.98	46.13	48.47	48.41	48.46
	34.86	40.35	38.80	40.80	38.29	43.47	43.09	43.73
1.50	41.50	45.10	45.03	45.38	42.46	46.55	46.57	46.66
	43.66	47.15	47.14	47.21	44.54	47.79	47.79	47.80
	32.93	38.83	37.01	39.12	36.77	42.72	42.33	42.98
1.75	40.78	45.89	45.80	46.06	40.84	45.61	45.53	45.60
	42.76	47.15	47.06	47.13	43.34	47.87	47.82	47.85
	32.43	38.96	37.52	39.24	36.54	43.45	43.26	43.82
2.00	39.94	45.70	45.42	45.57	40.41	46.29	46.25	46.30
	41.20	46.73	46.72	46.77	42.46	48.01	48.02	48.04
	32.32	39.25	37.39	39.06	35.80	43.27	42.83	43.30
2.25	38.61	45.06	44.98	45.06	40.29	47.02	46.98	47.01
	40.91	46.86	46.69	46.73	41.55	48.12	48.15	48.16
	31.72	38.94	37.08	38.51	35.01	42.83	41.95	42.47
2.50	38.31	45.50	45.25	45.36	39.40	46.87	46.76	46.79
	40.09	46.68	46.59	46.60	40.80	48.27	48.31	48.32
	31.51	38.91	36.52	37.72	34.06	42.22	41.42	41.78
2.75	37.16	44.36	44.06	44.10	38.19	46.03	45.84	45.86
	39.62	47.24	47.22	47.23	40.20	48.16	48.11	48.11
	30.73	38.31	35.25	36.63	33.13	41.87	41.10	41.40
3.00	35.56	43.53	43.24	43.33	37.67	46.34	46.24	46.25
	38.97	46.56	46.28	46.29	39.28	47.99	48.01	48.01
	29.81	37.87	35.43	36.51	32.66	41.73	40.70	41.01
3.25	36.33	44.46	43.97	44.00	37.27	46.28	46.04	46.04
	37.90	46.33	46.25	46.25	38.66	47.66	47.58	47.58
	30.06	38.35	35.35	36.40	32.39	41.80	40.94	41.11
3.50	35.70	44.30	43.80	43.83	36.51	46.21	46.17	46.18
	37.02	45.54	45.39	45.39	37.52	47.06	47.03	47.03
	29.39	37.91	34.99	35.90	31.94	41.65	40.53	40.73
3.75	35.29	44.30	43.79	43.80	36.09	45.99	45.77	45.78
	37.56	46.46	46.20	46.20	37.51	47.30	47.19	47.19
	29.17	38.14	35.76	36.55	31.38	41.08	39.80	39.91
4.00	35.16	44.41	43.83	43.85	35.89	46.07	45.74	45.75
	36.46	45.71	45.53	45.53	37.51	47.38	47.18	47.18

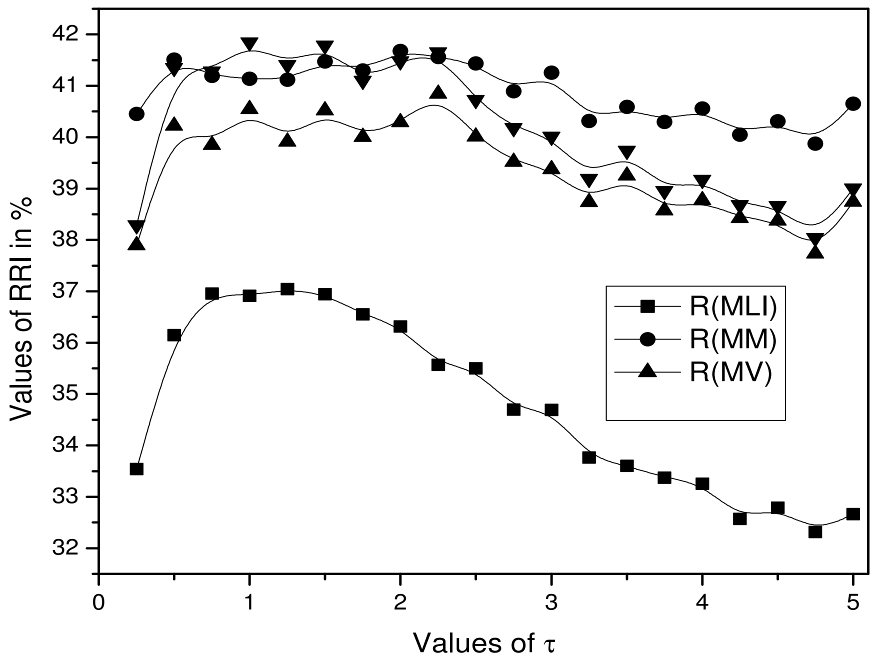

Figure 1. Comparison of RRI in % of improved estimators for μ when $m = n = 16$ and $r = s = 4$.

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References

[1] Balakrishnan, N. and Aggarwala, R. *Progrssive Censoring: Theory, Methods and Applications*, Birkhauser, Boston, 2000.

[2] Balakrishnan, N. and Cramer, E. *The Art of Progressive Censoring: Applications to Reliability and Quality*, Springer, New York., 2014.

[3] Brewster, J. F. and Zidek, J. V. *Improving on equivariant estimators*, The Annals of Statistics **2** (1), 21-38, 1974.

[4] Chandrasekar, B., Alexander, L. T. and Balakrishnan, N. *Equivariant estimation for parameters of exponential distributions based on type-II progressively censored samples*, Communications in Statistics- Theory ans Methods **31** (10), 1675-1686, 2002.

[5] Chiou, W. and Cohen, A. *Estimating the common location parameter of exponential distributions with censored samples*, Naval Research Logistics Quarterly **31**, 475-482, 1984.

[6] Elfessi, A. and Pal, N. *On location and scale parameters of exponential distributions with censored observations*, Communications in Statistics- Theory and Methods **20** (5-6), 1579-1592, 1991.

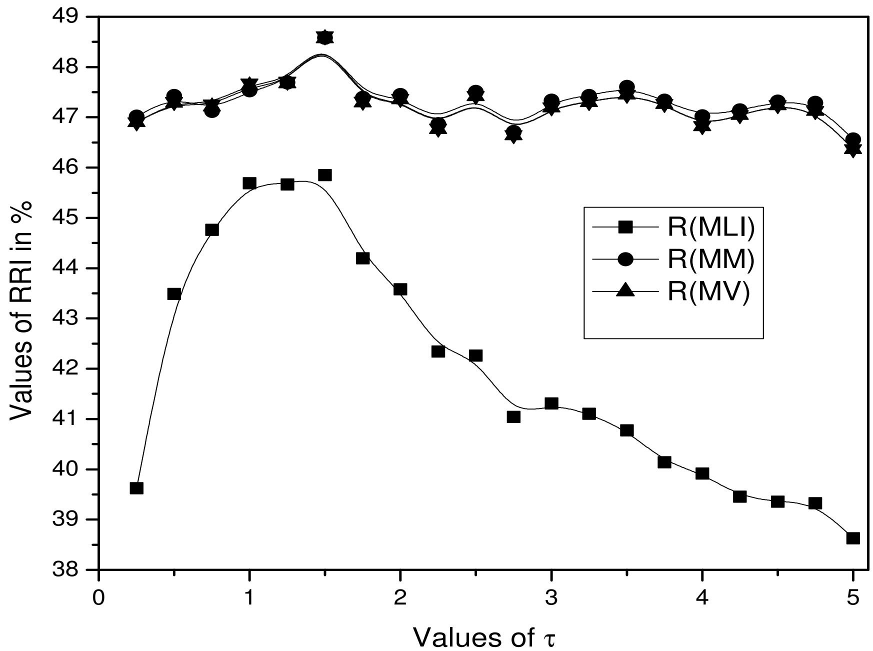

Figure 2. Comparison of RRI in % of improved estimators for μ when $m = n = 24$ and $r = s = 12$.

[7] Ghosh, M. and Razmpour, A. *Estimation of the common location parameter of several exponentials*, Sankhya: The Indian Journal of Statistics **46** Series A, Pt. 3, 383-394, 1984.

[8] Johnson, N. L., Kotz, S. and Balakrishnan, N. *Continuous Univariate Distributions, Vol-1,* John Wiley & Sons, INC, New York, 2004.

[9] Lawless, J. F. *Statistical Models and Methods for Lifetime Data*, Wiley, New York, 1982.

[10] Jin, C. and Crouse, H. *A note on the common location parameter of several exponential populations*, Communications in Statistics-Theory and Methods **27** (11), 2777-2789, 1998.

[11] Jin, C. and Pal, N. *On common location of several exponentials under a class of convex loss functions*, Calcutta Statistical Association Bulletin **42** (167-168), 191-200, 1992.

[12] Madi, M. T. *A note on the equivariant estimation of an exponential scale using progressively censored data*, Journal of Statistical Planning and Inference **140** 1437-1440, 2010.

[13] Suresh, R. P. *Estimation of location and scale parameters in a two-parameter exponential distribution from a censored sample*, Statistical Methods **6** (1), 82-89, 2004.

[14] Wang, B. X., Yu, K. and Jones, M. C. *Inference under progressively type-II right-censored sampling for certain lifetime distributions*, Technometrics **52** (4), 453-460, 2010.