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Estimation and orthogonal block structure

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Abstract

Estimators with good behaviors for estimable vectors and variance components are obtained for a class of models that contains the well known models with orthogonal block structure, OBS, see [15], [16] and [1], [2]. The study observations of these estimators uses commutative Jordan Algebras, CJA, and extends the one given for a more restricted class of models, the models with commutative orthogonal block structure, COBS, in which the orthogonal projection matrix on the space spanned by the means vector commute with all variance-covariance matrices, see [7].

Keywords: BLUE, LSE, OBS, UMVUE, Variance components.

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1. Introduction

Models with orthogonal block structure, OBS, are mixed models with the family $\nu = \left\{ \sum_{j=1}^{m} \gamma_j Q_j; \quad \boldsymbol{\gamma} \in \mathbb{R}_+^m \right\}$, of variance-covariance matrices where the $\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m$ are pairwise orthogonal orthogonal projection matrices, POOPM, summing to the identity matrix, I_n. These designs were introduced by [15] and [16], and continue to play an important part in the theory of randomized block designs, see for instance [1] and [2]. Refer to [9] and [18] for historical developments of the mixed model. The inference for these models is centered on the estimation of treatment contrasts, see [10]. These estimators are obtained from the orthogonal projections of the observation vector, $\boldsymbol{Y}$, on the strata which are the range spaces $\bigtriangledown_1, ..., \bigtriangledown_m$, of the $\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m$. Namely the problem of obtaining estimators from more than one strata has been dealt in detail. Then the weights to be given to each strata have to be estimated, see again [10].
We intend to follow a different approach using commutative Jordan algebras, CJA, to study the algebraic structure of these models. CJA are useful in discussing the algebraic structures of the models in a way that is convenient for deriving estimators both of variance components and estimable vectors through the introduction of sub-vectors. For our purpose it is convenient to write the mixed model as

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i, \tag{1.1}$$

where $\boldsymbol{\beta}_0$ is fixed and $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_w$ are random independent with null mean vectors and cross covariance matrices as well variance-covariance matrices $\theta_1 \boldsymbol{I}_{g_1}, ..., \theta_m \boldsymbol{I}_{g_w}$. This formulation enables an easy characterization of mixed models with OBS. Then when matrices $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^{'}, i = 1, ..., w$, commute they generate, as we will see, the CJA $\mathcal{A}(\underline{M})$. This is the smallest CJA of symmetric matrices that contains $\underline{M} = \{\boldsymbol{M}_1, ..., \boldsymbol{M}_w\}$. We recall, see [12], that these algebras are linear subspaces constituted by symmetric matrices and containing their squares. We will show that when matrices of $\underline{M}$ commute and constitute a basis for $\mathcal{A}(\underline{M})$ the models has OBS. Then we may use the sub-models $\boldsymbol{Y}_j = A_j \boldsymbol{Y}, j = 1, ..., m$, to obtain estimators for estimable vectors that are BLUE whatever the variance components. Following [21] we say that this estimators are uniformly BLUE, UBLUE. They are quite distinct from the ones for contrasts which are weighted means with estimated weights. Now no weight estimation is required and all estimable vectors may be treated as an unified approach. We point out that estimable contrasts are uni-dimensional estimable vectors so we have a widening of the class of estimable parameters and results that does not depend on weight and, as we shall see, have optimal properties.

Moreover we also obtain, using the sub-models, estimators for variance components which, when quasi-normality is assumed, also have optimal properties.

The role played by the CJA rests on the obtention of the sub-models which have variance-covariance matrices $\gamma_j \boldsymbol{I}_{g_j}$, with $g_j = rank(\boldsymbol{Q}_j), j = 1, ..., m$. The homoscedasticity of these sub-vectors leads to optimal estimators derived from each strata. Then the cross covariance matrices, $\Sigma(\boldsymbol{Y}_j; \boldsymbol{Y}_j^{'})$, are null which are the combinations of estimators derived from different sub-vectors. We will also consider a special class of models with OBS, the commutative orthogonal block structure, COBS, in which $\boldsymbol{T}$, the orthogonal projection matrix on the space Ω spanned by the mean vector commutes with the matrices in principal basis of a CJA $\mathcal{A}$, $pb(\mathcal{A})$. Then, whatever the $\gamma_1, ..., \gamma_m$, the matrix $\boldsymbol{T}$

will commute with

$$(1.2)\qquad \boldsymbol{V} = \sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j,$$

which, see [23], ensures that whatever the estimable vector $\boldsymbol{\psi}$ it's least square estimator, LSE, is the Best linear unbiased estimator, BLUE. We will say, see [21], that models with COBS have LSE that are UBLUE and show that, for theses models, the LSE are identical with the estimators we obtained for the general case of models with OBS.

2. Commutative Jordan Algebras

We already refer the importance of CJA in these models. We now point out that, see [17], the matrices of $\underline{M}$ commute if and only if they are diagonalized by the same orthogonal matrix $\boldsymbol{P}$. Then $\underline{M}$ will be contained in the CJA $\mathcal{A}(\boldsymbol{P})$ constituted by the matrices diagonalized by $\boldsymbol{P}$, thus $\underline{M}$ is contained in a CJA if and only if it's matrices commute. Since intersecting CJA gives a CJA, the intersection $\mathcal{A}(\underline{M})$ of all CJA containing $\underline{M}$ will be the least CJA containing $\underline{M}$, so we say that it is generated by $\underline{M}$.

[19] showed that any CJA, $\mathcal{A}$, has an unique basis, the $pb(\mathcal{A})$ of $\mathcal{A}$, constituted by POOPM. As stated by [5], Jordan algebras are used to present normal orthogonal models in a canonical form. Moreover:

(1) any family of POOPM is the principal basis of the CJA constituted by their linear combination;
(2) any orthogonal projection matrix, OPM, belonging to a CJA, $\mathcal{A}$, will be sum of matrices in $pb(\mathcal{A})$;
(3) if the matrices in $pb(\mathcal{A}_1)$ are some of matrices in $pb(\mathcal{A}_2)$ we have $\mathcal{A}_1 \subset \mathcal{A}_2$.

We recall that the product of two symmetric matrices is symmetric if they commute, then the product of two OPM that commute will be an OPM since it is symmetric and it is idempotent.

Given an OPM $\boldsymbol{K}$ that commutes with the matrices of $\underline{K} = \{\boldsymbol{K}_1, ..., \boldsymbol{K}_m\} = pb(\mathcal{A})$, the non null matrices $\boldsymbol{K}\boldsymbol{K}_j$ and $\boldsymbol{K}^c\boldsymbol{K}_j, j = 1, ..., m$, with $\boldsymbol{K}^c = \boldsymbol{I}_n - \boldsymbol{K}$, will be POOPM thus constituting the principal basis of a CJA, $\overline{\mathcal{A}}$. We can order the matrices in $pb(\overline{\mathcal{A}})$ so that the first are products by $\boldsymbol{K}$ of matrices in $pb(\mathcal{A})$ and the last $\overline{m} - z$ will be products by $\boldsymbol{K}^c$ also of matrices in $pb(\mathcal{A})$. Clearly we have $\mathcal{A} \subset \overline{\mathcal{A}}$. Those pairs of CJA appear in the theory of models with COBS. Models with this structure was also studied in [11], [5], [6] and [8]. $\mathcal{A}$ is now the CJA with principal basis $\boldsymbol{Q} = \{\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m\}$ when $\nu(\nabla)$ is the family of variance-covariance matrices and $\boldsymbol{T}$ playing the part of $\boldsymbol{K}$.

Let $\boldsymbol{\mu} = \boldsymbol{X}_0\boldsymbol{\beta}_0$ be the mean vector of the model. With $\boldsymbol{Q} = \{\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m\} = pb(\mathcal{A}(\underline{M}))$ let the row vectors of $\boldsymbol{A}_j$ constitute an orthonormal basis for the range space of $\boldsymbol{Q}_j$, $R(\boldsymbol{Q}_j)$, $j = 1, ..., m$, we have

$$(2.1)\qquad \begin{cases} \boldsymbol{A}_j\boldsymbol{A}_j^\top = \boldsymbol{I}_{g_j},\ j = 1, ..., m \\ \boldsymbol{A}_j^\top\boldsymbol{A}_j = \boldsymbol{Q}_j,\ j = 1, ..., m \end{cases},$$

with $g_j = rank(\boldsymbol{Q}_j)$, $j = 1, ..., m$. Let us take $\boldsymbol{X}_{0,j} = \boldsymbol{A}_j\boldsymbol{X}_0$ and represent by $\boldsymbol{P}_j$, $j = 1, ..., m$, and $\boldsymbol{P}_j^c$, $j = 1, ..., m$, the OPM on $\Omega_j = R(\boldsymbol{X}_{0,j})$ and it's orthogonal complement $\Omega_j^\perp$, $j = 1, ..., m$.

2.1. Lemma. *If the model has COBS we have* $\boldsymbol{T}\boldsymbol{Q}_j \neq \boldsymbol{0}_{n\times n}$ *if and only if* $\boldsymbol{X}_{0,j} \neq \boldsymbol{0}_{g_j\times k}$*, assuming* $\boldsymbol{X}_0$ *to be* $n \times k$*, j=1,...,m.*

Proof. We have $\boldsymbol{T}\boldsymbol{Q}_j = \boldsymbol{0}_{n\times n}$ if and only if $R(\boldsymbol{T}\boldsymbol{Q}_j) = \{\boldsymbol{0}_n\}$, so, if the model has COBS, $R(\boldsymbol{T}\boldsymbol{Q}_j) = R(\boldsymbol{Q}_j\boldsymbol{T}) = \boldsymbol{Q}_jR(\boldsymbol{T}) = \boldsymbol{Q}_jR(\boldsymbol{X}_0) = \boldsymbol{A}_j^\top\boldsymbol{A}_jR(\boldsymbol{X}) = \boldsymbol{A}_j^\top R(\boldsymbol{A}_j\boldsymbol{X}) = \boldsymbol{A}_j^\top R(\boldsymbol{X}_{0,j})$ and, since the column vectors of $\boldsymbol{A}_j^\top$ are linearly independent, $R(\boldsymbol{T}\boldsymbol{Q}_j) =$

$\boldsymbol{A}_j^\top R(\boldsymbol{X}_j) = \{\boldsymbol{0}_n\}$ if and only if $R(\boldsymbol{X}_{0,j}) = \{\boldsymbol{0}_n\}$ which is equivalent to $\boldsymbol{X}_{0,j} = \boldsymbol{0}_{g_j \times k}$, $j = 1, ..., m$.

2.2. Corollary. *If the model has COBS we have $\boldsymbol{TQ}_j \neq \boldsymbol{0}_{n\times n}$ if and only if $\boldsymbol{P}_j \neq \boldsymbol{0}_{g_j \times g_j}$, $j = 1, ..., m$.*

2.3. Corollary. *If the model has COBS we have $\boldsymbol{TQ}_j \neq \boldsymbol{0}_{n\times n}$ if and only if $\bar{\boldsymbol{Q}}_j = \boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j \neq \boldsymbol{0}_{n\times n}$, $j = 1, ..., m$.*

Proof. $\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j = (\boldsymbol{A}_j^\top \boldsymbol{P}_j)(\boldsymbol{A}_j^\top \boldsymbol{P}_j)^\top$ so, see [20],

$$rank(\boldsymbol{X}_j \boldsymbol{P}_j \boldsymbol{A}_j) = rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j), j = 1, ..., m.$$

Now the column vectors of $\boldsymbol{A}_j^\top$ are linearly independent so $\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j = \boldsymbol{0}_{n\times n}$. This is $rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j) = rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j) = 0$ if and only if $\boldsymbol{P}_j = \boldsymbol{0}_{g_j \times g_j}$, $j = 1, ..., m$. Thus, according to Corollary 2.2, $\boldsymbol{TQ}_j \neq \boldsymbol{0}_{n\times n}$ only when $\bar{\boldsymbol{Q}}_j \neq \boldsymbol{0}_{n\times n}$.

2.4. Corollary. *If the model has COBS we can order the $\boldsymbol{TQ}_1, ..., \boldsymbol{TQ}_m$ and the*

$$\bar{\boldsymbol{Q}}_1, ..., \bar{\boldsymbol{Q}}_m$$

to have $\boldsymbol{TQ}_j \neq \boldsymbol{0}_{n\times n}$ $[\bar{\boldsymbol{Q}}_j \neq \boldsymbol{0}_{n\times n}]$, if and only if $j \leq z$.

2.5. Proposition. *If the model has COBS we have $\boldsymbol{TQ}_j = \bar{\boldsymbol{Q}}_j$, $j = 1, ..., z$.*

Proof. Since $\boldsymbol{TQ}_j$ $[\bar{\boldsymbol{Q}}_j]$, $j = 1, ..., z$, are symmetric and idempotent matrices they are OPM. So we have only to show that $R(\boldsymbol{TQ}_j) = R(\bar{\boldsymbol{Q}}_j)$, $j = 1, ..., z$. Now

$$rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j) = rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{P}_j \boldsymbol{A}_j) = rank(\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j) = rank(\bar{\boldsymbol{Q}}_j), j = 1, ..., z,$$

so that

$$R(\bar{\boldsymbol{Q}}_j) = R(\boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j) = R(\boldsymbol{A}_j^\top \boldsymbol{P}_j), j = 1, ..., z,$$

since the first is a subspace of the last set with the same dimension.
Besides this

$$R(\boldsymbol{Q}_j \boldsymbol{TQ}_j) = R(\boldsymbol{Q}_j \boldsymbol{T}) = \boldsymbol{Q}_j R(\boldsymbol{T}) = \boldsymbol{Q} R(\boldsymbol{X}) =$$

$$= \boldsymbol{A}_j^\top \boldsymbol{A}_j R(\boldsymbol{X}) = \boldsymbol{A}_j^\top R(\boldsymbol{A}_j \boldsymbol{X}) = \boldsymbol{A}_j^\top R(\boldsymbol{X}_j) = \boldsymbol{A}_j^\top R(\boldsymbol{P}_j) = R(\boldsymbol{A}_j^\top \boldsymbol{P}_j) = R(\bar{\boldsymbol{Q}}_j),$$

$j = 1, ..., m$, which establish the thesis.

2.6. Corollary. *Putting $\boldsymbol{T}^c = \boldsymbol{I}_n - \boldsymbol{T}$ and $\bar{\boldsymbol{Q}}^\bullet{}_j = \boldsymbol{A}_j^\top \boldsymbol{P}_j^c \boldsymbol{A}_j$, $j = z+1, ..., m$, when the model has COBS we have $\boldsymbol{T}^c \boldsymbol{Q}_j = \bar{\boldsymbol{Q}}^\bullet{}_j$, $j = z+1, ..., m$.*

Proof. According to Corollary 2.4 we have $\boldsymbol{T}^c \boldsymbol{Q}_j = \boldsymbol{Q}_j - \boldsymbol{TQ}_j = \boldsymbol{Q}_j$, $j = z+1, ..., m$, as well as $\boldsymbol{A}_j^\top \boldsymbol{P}_j^c \boldsymbol{A}_j = \boldsymbol{A}_j^\top \boldsymbol{A}_j - \boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j = \boldsymbol{Q}_j - \bar{\boldsymbol{Q}}_j = \boldsymbol{Q}_j$, $j = z+1, ..., m$, so the thesis is established.

2.7. Corollary. *When the model has COBS the CJA with principal basis $\{\boldsymbol{TQ}_1, ..., \boldsymbol{TQ}_z, \boldsymbol{T}^c \boldsymbol{Q}_{z+1}, ..., \boldsymbol{T}^c \boldsymbol{Q}_m\}$ and $\{\bar{\boldsymbol{Q}}_1, ..., \bar{\boldsymbol{Q}}_z, \bar{\boldsymbol{Q}}^\bullet{}_{z+1}, ..., \bar{\boldsymbol{Q}}^\bullet{}_m\}$ are identical.*

Proof. The result follows from Corollary 2.6 and Proposition 2.5.

2.8. Corollary. *If the model has COBS we have $\boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j$.*

Proof. We have $\boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{TQ}_j$ so the thesis follows from Proposition 2.5.

3. Mixed Models

We now characterize mixed models with OBS and COBS. If the matrices of $\underline{M} = \{\boldsymbol{M}_1, ..., \boldsymbol{M}_w\}$ commute they will generate a CJA, $\mathcal{A}(\underline{M})$, as we saw in Section 2. With $\boldsymbol{Q} = \{\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m\} = pb(\mathcal{A}(\underline{M}))$ we have $\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j}\boldsymbol{Q}_j, i = 1, ..., w$, putting $\boldsymbol{B} = [b_{i,j}]$ and $\psi_i = \{j : b_{i,j} \neq 0\}, i = 1, .., w$, it is easy to see that the OPM on $R(\boldsymbol{M}_i) = R(\boldsymbol{X}_i)$ is $\sum_{j\in\psi_i} \boldsymbol{Q}_j$. Moreover the OPM on $R(\sum_{i=1}^{w} \boldsymbol{M}_i) = R([\boldsymbol{X}_1 ... \boldsymbol{X}_w])$ will be $\sum_{j=1}^{m} \boldsymbol{Q}_j$. Thus we have $R([\boldsymbol{X}_1 ... \boldsymbol{X}_w]) = \mathbb{R}^n$ if and only if $\boldsymbol{I}_n = \sum_{j=1}^{m} \boldsymbol{Q}_j$, which is, as we saw, one of the requirements on the POOPM that appear on the variance-covariance matrices of models with OBS. The mixed models will have variance-covariance matrices

$$\boldsymbol{V}(\boldsymbol{\theta}) = \sum_{i=1}^{w} \theta_i \boldsymbol{M}_i = \sum_{i=1}^{w} \theta_i (\sum_{j=1}^{m} b_{i,j}\boldsymbol{Q}_j) = \sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j, \tag{3.1}$$

where $\gamma_j = \sum_{i=1}^{w} b_{i,j}\theta_i, j = 1, ..., m$, so $\boldsymbol{\gamma} \in R(\boldsymbol{B}^\top)_+$, with ∇_+ the family of vectors of sub-space ∇ with non-negative components.

For the variance-covariance matrices of the model to be all the positive semi-definite matrices given by linear combination of $\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m$ we have to have

$$R(\boldsymbol{B}^\top) = \mathbb{R}^m,$$

this is matrix $\boldsymbol{B}$ must be invertible which occur when and only when $\underline{M}$ is a basis for $\mathcal{A}(\underline{M})$. Then, see [4], the family $\underline{M}$ will be perfect. We now establish

3.1. Proposition. *The mixed model enjoys OBS when $\underline{M}$ is a perfect family and*

$$R([\boldsymbol{X}_1 ... \boldsymbol{X}_w] = \mathbb{R}^w.$$

Proof. When $R([\boldsymbol{X}_1 ... \boldsymbol{X}_w]) = \mathbb{R}^w$ but $\underline{M}$ is not perfect we can always complete it adding some random effect terms to the model. We then restrict ourselves to perfect $\underline{M}$ families.

Going over to models with COBS we establish

3.2. Proposition. *$\boldsymbol{T}$ commutes with the matrices of $\underline{M}$ if and only if it commutes with matrices of $\boldsymbol{Q}$.*

Proof. If $\boldsymbol{T}$ commutes with the matrices of $\underline{M}$, the matrices of

$$\boldsymbol{M}^o = \{\boldsymbol{T}, \boldsymbol{M}_1, ..., \boldsymbol{M}_w\}$$

commute so they will generate a CJA $\mathcal{A}(\boldsymbol{M}^o)$ that contains $\boldsymbol{M}^o$, then containing $\boldsymbol{T}$, and the matrices of $\boldsymbol{Q}$ that will commute. Inversely if $\boldsymbol{T}$ commutes with the matrices of $\boldsymbol{Q}$ it commutes with matrices of $\underline{M}$ since $\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j}\boldsymbol{Q}_j, i = 1, ..., w$.

3.3. Corollary. *If a model has OBS and $\boldsymbol{T}$ commutes with the matrices of $\underline{M}$ it has COBS.*

4. Estimation

In this section we will use the sub-models $\boldsymbol{Y}_j = \boldsymbol{A}_j\boldsymbol{Y}, j = 1, ..., m$ to obtain estimators for estimable vectors. Taking $\boldsymbol{\mu}_j = \boldsymbol{A}_j\boldsymbol{\mu}, j = 1, ..., m$, where $\boldsymbol{\mu}_j = \boldsymbol{0}_{g_j}$, $j = z+1, ..., m$, a model with generalized OBS, GOBS, has the homoscedastic partition $\boldsymbol{Y} = \sum_{j=1}^{m} \boldsymbol{A}_j^\top \boldsymbol{Y}_j$

where the $\boldsymbol{Y}_1, ..., \boldsymbol{Y}_m$ have mean vectors $\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_m$, and variance-covariance matrices $\gamma_1 \boldsymbol{I}_{g_1}, ..., \gamma_m \boldsymbol{I}_{g_m}$.

Now $\boldsymbol{\psi} = \boldsymbol{G\beta}$ is estimable, see for instance [13], if and only if $\boldsymbol{G} = \boldsymbol{UX}_0$, so that $\boldsymbol{\psi} = \boldsymbol{U\mu} = \sum_{j=1}^{z} \boldsymbol{U}_j \boldsymbol{\mu}_j = \sum_{j=1}^{z} \boldsymbol{\psi}_j$ with $\boldsymbol{U}_j = \boldsymbol{UA}_j^\top$ and $\boldsymbol{\psi}_j = \boldsymbol{U}_j \boldsymbol{\mu}_j$, $j = 1, ..., z$. Now we establish

4.1. Proposition. *$\tilde{\boldsymbol{\psi}} = \sum_{j=1}^{z} \tilde{\boldsymbol{\psi}}_j$, with $\tilde{\boldsymbol{\psi}}_j = \boldsymbol{U}_j \boldsymbol{P}_j \boldsymbol{Y}_j$, $j = 1, ..., z$, is an unbiased estimator of $\boldsymbol{\psi}$, and if $\boldsymbol{\psi}^* = \sum_{j=1}^{z} \boldsymbol{\psi}_j^*$ with $\boldsymbol{\psi}_j^* = \boldsymbol{W}_j \boldsymbol{Y}_j$ is another unbiased estimator of $\boldsymbol{\psi}$, $j = 1, ..., z$, $\boldsymbol{\psi}^*$ is an unbiased estimator of $\boldsymbol{\psi}$, with $\Sigma(\tilde{\boldsymbol{\psi}}) \leq \Sigma(\boldsymbol{\psi}^*)$ where $\leq$ indicates that $\Sigma(\boldsymbol{\psi}^*) - \Sigma(\tilde{\boldsymbol{\psi}})$ is positive semi-definite.*

Proof. Since the mean vector of $\boldsymbol{P}_j \boldsymbol{Y}_j$ is $\boldsymbol{P}_j \boldsymbol{\mu}_j = \boldsymbol{\mu}_j$, $j = 1, ..., z$, $\boldsymbol{\psi}^*$ is an unbiased estimator of $\boldsymbol{\psi}$ and it is well known that $\Sigma(\tilde{\boldsymbol{\psi}}_j) \leq \Sigma(\boldsymbol{\psi}_j^*)$, $j = 1, ..., z$. Now the $\boldsymbol{Y}_1, ..., \boldsymbol{Y}_m$ have null variance-covariance matrices, so

$$(4.1) \qquad \begin{cases} \Sigma(\tilde{\boldsymbol{\psi}}) = \sum_{j=1}^{z} \boldsymbol{U}_j \Sigma(\tilde{\boldsymbol{\psi}}_j) \boldsymbol{U}_j^\top = \sum_{j=1}^{z} \Sigma(\tilde{\boldsymbol{\psi}}_j) \\ \Sigma(\boldsymbol{\psi}^*) = \sum_{j=1}^{z} \boldsymbol{U}_j \Sigma(\boldsymbol{\psi}_j^*) \boldsymbol{U}_j^\top = \sum_{j=1}^{z} \Sigma(\boldsymbol{\psi}_j^*) \end{cases}$$

and $\boldsymbol{U}_j \Sigma(\tilde{\boldsymbol{\psi}}_j) \boldsymbol{U}_j^\top \leq \boldsymbol{U}_j \Sigma(\boldsymbol{\psi}_j^*) \boldsymbol{U}_j^\top$, $j = 1, ..., z$.

4.2. Proposition. *When the model has COBS the $\tilde{\boldsymbol{\psi}}$ are LSE.*

Proof. Since the models enjoys COBS we have $\boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j$ and $\tilde{\boldsymbol{\psi}} = \sum_{j=1}^{z} \tilde{\boldsymbol{\psi}}_j = \sum_{j=1}^{z} \boldsymbol{U}_j \boldsymbol{P}_j \boldsymbol{Y}_j = \boldsymbol{U} \left(\sum_{j=1}^{z} \boldsymbol{A}_j^\top \boldsymbol{P}_j \boldsymbol{A}_j \right) \boldsymbol{Y} = \boldsymbol{UTY} = \boldsymbol{U}\tilde{\boldsymbol{\mu}}$, with $\tilde{\boldsymbol{\mu}}$ the LSE of $\boldsymbol{\mu}$, so the thesis is established.

This result is interesting since in COBS the LSE are UBLUE, being BLUE whatever $\boldsymbol{\theta}$, see [23]. Thus we validate the above proposition showing that our "sub-optimal estimator" is "optimal" when the model enjoys COBS. In the previous phrase "sub-optimal" must be taken in the sense of Proposition 4.1 and "optimal" in the sense of the LSE being UBLUE.

Let us put $q_j = rank(\boldsymbol{P}_j^c)$, $\;\; j = 1, ..., m$, as well as $\mathfrak{D} = \{j; \;\; q_{j>0}\}$, and

$$(4.2) \qquad \tilde{\gamma}_j = \frac{\boldsymbol{Y}_j^\top \boldsymbol{P}_j^c \boldsymbol{Y}_j}{q_j}, \quad j \in \mathfrak{D}.$$

It is also well known that, if $\boldsymbol{\gamma}_j^* = \boldsymbol{I}_j^\top \boldsymbol{W}_j \boldsymbol{Y}_j$, $\;\; j \in \mathfrak{D}$ is a quadratic unbiased estimator of $\boldsymbol{\gamma}_j$, $j \in \mathfrak{D}$, we have $var(\tilde{\boldsymbol{\gamma}}_j) \leq var(\boldsymbol{\gamma}_j^*)$, $\;\; j \in \mathfrak{D}$. Let us get the following Proposition. We leave out its proof which can be seen in [17], page 395.

4.3. Proposition. *If $\boldsymbol{Y}$ is quasi-normal we have*

$$(4.3) \qquad var\left(\sum_{j \in \mathfrak{D}} c_j \tilde{\boldsymbol{\gamma}}_j \right) \leq var\left(\sum_{j \in \mathfrak{D}} c_j \boldsymbol{\gamma}_j^* \right).$$

5. An application

The mixed model

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i,$$

where $\boldsymbol{\beta}_0$ is fixed and the $\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_w$ are random independent vectors with null mean vector and variance-covariance matrices $\sigma_1^2 \boldsymbol{I}_{g_1}, ..., \sigma_w^2 \boldsymbol{I}_{g_w}$ have GOBS, see for instance [14], when the matrices $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^\top, i = 1, ..., w$ commute.
Namely these matrices will belong to a CJA $\mathcal{A}$, with $pb(\mathcal{A}) = \{\boldsymbol{Q}_1, ..., \boldsymbol{Q}_m\}$, so that $\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j, i = 1, ..., w$. Note that to consider an extension of OBS we can replace

ν by $\nu(\nabla) = \left\{\sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j; \quad \boldsymbol{\gamma} \in \nabla_+\right\}$, where ∇_+ is the family of vectors belonging to subspace ∇ with non negative components. Then the model will have GOBS. This application is itself an extension of the one given, see [7], [3] and [14], for models with COBS, and the identity of the two algebras for models with COBS enables us to carry out an unified treatment for models with GOBS.

These models have variance-covariance matrices

$$\boldsymbol{V}(\sigma^2) = \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i = \sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j, \tag{5.1}$$

with $\gamma_j = \sum_{i=1}^{w} b_{i,j}\sigma_i^2, j = 1, ..., m$ so that now we have $\boldsymbol{\gamma} \in R(\boldsymbol{B}^\top)_+$, where $\boldsymbol{B} = [b_{i,j}]$.

We point out that for $\boldsymbol{V}(\sigma_1^2) = \boldsymbol{V}(\sigma_2^2)$ implying $\sigma_1^2 = \sigma_2^2$ the matrices $\boldsymbol{M}_1, ..., \boldsymbol{M}_w$ have to be linearly independent. Then the row vectors of $\boldsymbol{B}$ that are the column vectors of $\boldsymbol{B}^\top$, are linearly independent and we have $\boldsymbol{\sigma}^2 = \boldsymbol{B}^{\top+}\boldsymbol{\gamma}$, where $\boldsymbol{A}^+$ indicates MOORE-PENROSE inverse of matrix $\boldsymbol{A}$. Then, if $\boldsymbol{Y}$ is quasi-normal we may apply Proposition 4.3.

6. Final Remarks

Least squares estimators, LSE, have been widely used due to this algebraic structure and to having minimum variance.covariance matrices, under general conditions, whatever the variance components.

Following [21] we may say that, then, the LSE are UBLUE. Now these conditions rest on $\boldsymbol{T}$ commuting with the variance-covariance matrices of the model.

We showed that this commutativity condition was not necessary thus extending the class of models for which we have UBLUE for estimable vectors.We also showed that those UBLUE are LSE when the commutativity condition holds. Thus our results may be considered as an extension of the well known results on UBLUE that are LSE, for instance see [22] and [23].

Besides this we obtain an optimal result for estimators of linear combinations $\sum_{j=1}^{m} c_j\gamma_j$. We point out that in mixed models such as those considered in the application we have $\boldsymbol{\sigma}^2 = (\boldsymbol{B}^\top)^+\boldsymbol{\gamma}$ so we can apply that result to the components of $\widetilde{\boldsymbol{\sigma}}^2 = (\boldsymbol{B}^\top)^+\widetilde{\boldsymbol{\gamma}}$ whenever $\boldsymbol{Y}$ is quasi normal.

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