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Approximations and adjoints for categories of complexes of Gorenstein projective modules

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Abstract

In the paper, it is proven that every object in C(R-GProj) has a special C(R-Proj)-preenvelope, and then some adjoints in homotopy categories related to Gorenstein projective modules are given, where C(R-Proj) is the subcategory of complexes of projective R-modules, and C(R-GProj) is the subcategory of complexes of Gorenstein projective R-modules.

Keywords: (special) preenvelopes, Gorenstein projective modules, DG-projective complexes, adjoints.

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1. Introduction

Let R be an associative ring, and let R-Proj, R-Flat, and R-GProj be the subcategory of projective, flat, and Gorenstein projective R-modules in R-Mod, the category of left R-modules. If $\mathcal{A}$ is one of the above categories then we use $\mathrm{C}(\mathcal{A})$ to denote the category of complexes of R-modules in $\mathcal{A}$. The category $\mathrm{K}(\mathcal{A})$ is the homotopy category which has the same objects as $\mathrm{C}(\mathcal{A})$, and the morphisms are homotopy equivalence classes of morphisms of complexes. It was shown in [15] and [16] that both the inclusions K(R-Proj) $\to$ K(R-Flat) and K(R-Flat) $\to$ K(R-Mod) have right adjoints. Recently, Diego Bravo, Edgar E. Enochs et. al in [5] showed that some adjoints to inclusion functors may exist if they were given complete cotorsion pairs in the category of complexes. The paper is motivated by the above work to show:

1.1. Theorem. Let R be any ring. Then every complex $G \in$ C(R-GProj) has a special C(R-Proj)-preenvelope.

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Now suppose that R is quasi-Frobenius. It is well known that the subcategory R-GProj is in fact R-Mod. Then the categories C(R-GProj) and C(R-Mod) are the same, and so Theorem 1.1 says that any complex admits a special C(R-Proj)-preenvelope. It is natural to ask whether every complex adimits a special DG-projective preenvelope since the class of DG-projective complexes is contained in C(R-Proj)? We find that the answer is negative in general. In fact, every complex adimits a special DG-projective preenvelope if and only if R has global dimension 0.

Note that the inclusion K(R-Proj) $\to$ K(R-Mod) always has a right adjoint ([5, Theorem 4.7]). We are inspired to consider whether there exists a left adjoint to it, and we show the following main result which is based on Theorem 1.1.

1.2. Theorem. Let R be any ring. Then the inclusion K(R-Proj) $\to$ K(R-GProj) has a left adjoint.

2. Preliminaries

Let Ω be a subcategory of an abelian category $\mathcal{A}$, and M is an object of $\mathcal{A}$. A morphism $f : M \to Q$ is called an Ω-preenvelope of M, if $Q \in \Omega$ and the sequence $\mathrm{Hom}(Q, Q') \to \mathrm{Hom}(M, Q') \to 0$ is exact for any $Q' \in \Omega$. If moreover, $g \circ f = f$ implies that g is an automorphism whenever $g \in \mathrm{End}(Q)$, then f is called an Ω-envelope. An Ω-preenvelope $f : M \to Q$ of M is said to be special, if f is injective and $\mathrm{Ext}^1(\mathrm{Coker}(f), Q') = 0$ for any $Q' \in \Omega$. An Ω-precover, an Ω-cover and a special Ω-precover $Q \to M$ are defined dually. See [9, 11] for detail.

Auslander and Reiten [2] and Auslander and Smalϕ [3] use the terminology left and right approximations and minimal left and right approximations for preenvolpes, precovers, envelopes and covers.

A complex X of R-modules is a sequence $\cdots \to X_{i+1} \xrightarrow{\delta_{i+1}^X} X_i \xrightarrow{\delta_i^X} X_{i-1} \to \cdots$ of R-modules and R-homomorphisms such that $\delta_i^X \delta_{i+1}^X = 0$ for all $i \in \mathbb{Z}$. A complex X is said to be acyclic (exact) if $\mathrm{Im}(\delta_{i+1}^X) = \mathrm{Ker}(\delta_i^X)$ for all $i \in \mathbb{Z}$. A complex X is said to be bounded above if $X_i = 0$ holds for $i \gg 0$, bounded below if $X_i = 0$ holds for $i \ll 0$, and bounded if it is bounded above and below, i.e. $X_i = 0$ holds for $|i| \gg 0$. Let X be a complex and let m be an integer. The m-fold shift of X is the complex $\Sigma^m X$ given by $(\Sigma^m X)_i = X_{i-m}$ and $\delta_i^{\Sigma^m X} = (-1)^m \delta_{i-m}^X$. Usually, $\Sigma^1 X$ is denoted simply by ΣX.

Let X and Y be two complexes. We will let $\mathrm{Hom}_R(X, Y)$ denote the complex of $\mathbb{Z}$-modules with mth component $\mathrm{Hom}_R(X, Y)_m = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+m})$ and differential $(\delta(g))_i = \delta_{i+m}^Y g_i - (-1)^m g_{i-1} \delta_i^X$ for $g = (g_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+m})$. By a morphism $f : X \to Y$ we mean a sequence $f_i : X_i \to Y_i$ such that $\delta_i^Y f_i = f_{i-1} \delta_i^X$ for all $i \in \mathbb{Z}$. The mapping cone $\mathrm{Cone}(f)$ of a morphism $f : X \to Y$ is defined as $\mathrm{Cone}(f)_i = Y_i \oplus X_{i-1}$ with $\delta_i^{\mathrm{Cone}(f)} = \begin{pmatrix} \delta_i^Y & f_{i-1} \\ 0 & -\delta_{i-1}^X \end{pmatrix}$.

If M is an R-module then we denote the complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M in the mth degree by $S^m(M)$, and denote the complex $\cdots \to 0 \to M \xrightarrow{Id} M \to 0 \to \cdots$ with M in the $m-1$ and mth degrees by $D^m(M)$. Usually, $S^0(M)$ is denoted simply by M. We use $\mathrm{Hom}(X, Y)$ to present the group of all morphisms from X to Y. Recall that a complex P is projective if the functor $\mathrm{Hom}(P, -)$ is exact. Equivalently, P is projective if and only if P is acyclic and $\mathrm{Im}(P_{i+1} \to P_i)$ is a projective R-module for each $i \in \mathbb{Z}$. For example, if M is a projective R-module then each complex $D^m(M)$ is projective. A injective complex is defined dually. Thus C(R-Mod), the category of complexes of

R-modules, has enough projectives and injectives, we can compute right derived functors $\mathrm{Ext}^i(X,Y)$ of $\mathrm{Hom}(-,-)$.

2.1. Definition. ([8]) We call an acyclic complex $P = \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$ with all P_i projective a complete projective resolution of an R-module M, if $M \cong \mathrm{Ker}(P_0 \to P_{-1})$, and $\mathrm{Hom}_R(P,N)$ is acyclic for any projective R-module N. An R-module M is called Gorenstein projective, if there exists a complete projective resolution of M.

The dual notions are those of a complete injective resolution and a Gorenstein injective R-module.

2.2. Remark. (1) The subcategory R-GProj is projectively resolving, that is, R-GProj contains R-Proj, and $F \in R$-GProj if and only if $H \in R$-GProj for any exact sequence $0 \to F \to H \to G \to 0$ with $G \in R$-GProj ([12, Theorem 2.5]).

(2) An R-module $M \in R$-GProj with finite projective dimension is projective ([12, Proposition 2.7]).

3. The existence of $\mathrm{C}(R$-Proj$)$-preenvelopes

In this section, we focus on $\mathrm{C}(R$-Proj$)$-preenvelopes of special complexes over general associative rings. We begin with the following

3.1. Lemma. Assume that the following diagram of complexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mu"] & B \arrow[r, "g"] \arrow[d, "\nu"] & C \arrow[r] \arrow[d, "\omega"] & 0 \\ 0 \arrow[r] & X \arrow[r, "p"] & Y \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

is commutative. Then the sequence

$$0 \longrightarrow Cone(\mu) \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & \Sigma f \end{pmatrix}} Cone(\nu) \xrightarrow{\begin{pmatrix} q & 0 \\ 0 & \Sigma g \end{pmatrix}} Cone(\omega) \longrightarrow 0$$

is exact.

Proof. It can be checked by standard computation. □

3.2. Lemma. Let $G \in \mathrm{C}(R$-GProj$)$ be acyclic and bounded above. If $\mathrm{Hom}_R(G,A)$ is acyclic for any projective R-module A then $\mathrm{Ext}^1(G,P) = 0$ for any $P \in \mathrm{C}(R$-Proj$)$.

Proof. See [13, Lemma 3.1]. □

3.3. Definition. Let X be a complex and let m be an integer. The hard truncation above of X at m, denoted $X_{\leqslant m}$, is the complex

$$X_{\leqslant m} = \ 0 \to X_m \xrightarrow{\delta^X_m} X_{m-1} \xrightarrow{\delta^X_{m-1}} X_{m-2} \to \cdots .$$

Similarly, the hard truncation below of X at m, denoted $X_{\geqslant m}$, is the complex

$$X_{\geqslant m} = \ \cdots \to X_{m+2} \xrightarrow{\delta^X_{m+2}} X_{m+1} \xrightarrow{\delta^X_{m+1}} X_m \to 0.$$

3.4. Lemma. Let $G \in \mathrm{C}(R$-GProj$)$ be bounded above. Then there is an exact sequence $0 \to G \to P \to C \to 0$ such that $P \in \mathrm{C}(R$-Proj$)$ and $C \in \mathrm{C}(R$-GProj$)$ are both bounded above, C is acyclic, and $\mathrm{Hom}_R(C,A)$ is acyclic for any projective R-module A.

Proof. Assume without loss of generality that $G =: \ 0 \to G_0 \to G_{-1} \to G_{-2} \to \cdots$ is a complex of Gorenstein projective R-modules with G_0 in the 0th degree. If for each $n \geqslant 0$, we let $G(n) = G_{\geqslant -n}$, the hard truncation below of G at $-n$, then $\{(G(n), \alpha_{mn}) | m \geqslant n \geqslant 0\}$ forms a inverse system in $C(R\text{-Mod})$ and $G = \varprojlim G(n)$, where $\alpha_{mn} : G(m) \to G(n)$ is a natural projection for any $m \geqslant n$.

We will show by induction on n. For $n = 0$, since G_0 is Gorenstein projective, there exists an exact sequence $0 \to G_0 \xrightarrow{f} P_0 \to P_{-1} \to P_{-2} \to \cdots$ with each P_i projective and it remains exact after applying the functor $\mathrm{Hom}_R(-, A)$ for any projective R-module A. Let $P(0) =: \ 0 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$, and consider the following monomorphism of complexes $\phi(0) : G(0) \to P(0)$.

$$\begin{tikzcd}
G(0) \arrow[d] & 0 \arrow[r] & G_0 \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
P(0) & 0 \arrow[r] & P_0 \arrow[r] & P_{-1} \arrow[r] & P_{-2} \arrow[r] & \cdots
\end{tikzcd}$$

Let $C(0) = \mathrm{Coker}(\phi(0))$, that is $C(0) =: \ 0 \to G'_0 \to P_{-1} \to P_{-2} \to \cdots$ with $G'_0 = \mathrm{Coker}(f)$. Clearly, $P(0) \in \mathrm{C}(R\text{-Proj})$ and $C(0) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, $C(0)$ is acyclic, and $\mathrm{Hom}_R(C(0), A)$ is acyclic for any projective R-module A.

Now for $n \geqslant 0$, suppose that there is a monomorphism $\phi(n) : G(n) \to P(n)$ as follows.

$$\begin{tikzcd}
G(n) \arrow[d] & 0 \arrow[r] & G_0 \arrow[r] \arrow[d, "f_0"'] & G_{-1} \arrow[r] \arrow[d, "f_{-1}"'] & \cdots \arrow[r] & G_{-n} \arrow[r] \arrow[d, "f_{-n}"'] & 0 \arrow[r] \arrow[d] & \cdots \\
P(n) & 0 \arrow[r] & P_0 \arrow[r, "\delta_0"] & P_{-1} \arrow[r, "\delta_{-1}"] & \cdots \arrow[r] & P_{-n} \arrow[r, "\delta_{-n}"] & P_{-n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Where $P(n) \in \mathrm{C}(R\text{-Proj})$ and $C(n) = \mathrm{Coker}(\phi(n)) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, $C(n)$ is acyclic, and $\mathrm{Hom}_R(C(n), A)$ is acyclic for any projective R-module A. Let $G(n+1) =: \ 0 \to G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} \cdots \to G_{-n} \xrightarrow{d_{-n}} G_{-n-1} \to 0 \to \cdots$, and let $0 \to G_{-n-1} \xrightarrow{g} Q_{-n-1} \to Q_{-n-2} \to Q_{-n-3} \to \cdots$ be an exact sequence with each Q_i projective and it remains exact after applying the functor $\mathrm{Hom}_R(-, A)$ for any projective R-module A. We denoted by Q the complex $0 \to Q_{-n-1} \to Q_{-n-2} \to Q_{-n-3} \to \cdots$ with Q_{-n-1} in the $(-n-1)$th degree. By the above proof, we have a monomorphism $\iota : S^{-n-1}(G_{-n-1}) \to Q$ such that $\mathrm{Coker}(\iota) \in \mathrm{C}(R\text{-GProj})$ is acyclic and bounded above, and also $\mathrm{Hom}_R(\mathrm{Coker}(\iota), A)$ is acyclic for any projective R-module A.

Let $\mu : \Sigma^{-1} G(n) \to S^{-n-1}(G_{-n-1})$ be the following morphism

$$\begin{tikzcd}
\Sigma^{-1} G(n) \arrow[d] & 0 \arrow[r] & G_0 \arrow[r, "-d_0"] \arrow[d] & \cdots \arrow[r] & G_{-n+1} \arrow[r] \arrow[d] & G_{-n} \arrow[r] \arrow[d, "d_{-n}"'] & 0 \arrow[r] \arrow[d] & \cdots \\
S^{-n-1}(G_{-n-1}) & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & G_{-n-1} \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Note that the sequence $0 \to \Sigma^{-1} G(n) \to \Sigma^{-1} P(n) \to \Sigma^{-1} C(n) \to 0$ is exact. Then it follows from Lemma 3.2 that the sequence

$$0 \to \mathrm{Hom}(\Sigma^{-1} C(n), Q) \to \mathrm{Hom}(\Sigma^{-1} P(n), Q) \to \mathrm{Hom}(\Sigma^{-1} G(n), Q) \to \mathrm{Ext}^1(\Sigma^{-1} C(n), Q) = 0$$

is exact, and so there exits a morphism $\nu : \Sigma^{-1}P(n) \longrightarrow Q$ such that the following diagram commutes.

$$\begin{tikzcd}
\Sigma^{-1}G(n) \arrow[r, "\Sigma^{-1}\phi(n)"] \arrow[d, "\mu"'] & \Sigma^{-1}P(n) \arrow[d, "\nu"] \\
S^{-n-1}(G_{-n-1}) \arrow[r, "\iota"] & Q
\end{tikzcd}$$

Thus there exists a morphism $\omega : \Sigma^{-1}C(n) \to \mathrm{Coker}(\iota)$ such that the following diagram with exact rows commutes.

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{-1}G(n) \arrow[r, "\Sigma^{-1}\phi(n)"] \arrow[d, "\mu"'] & \Sigma^{-1}P(n) \arrow[r] \arrow[d, "\nu"] & \Sigma^{-1}C(n) \arrow[r] \arrow[d, "\omega"] & 0 \\
0 \arrow[r] & S^{-n-1}(G_{-n-1}) \arrow[r, "\iota"] & Q \arrow[r] & \mathrm{Coker}(\iota) \arrow[r] & 0
\end{tikzcd}$$

By lemma 3.1, the sequence

$$0 \longrightarrow Cone(\mu) \longrightarrow Cone(\nu) \longrightarrow Cone(\omega) \longrightarrow 0$$

is exact. Note that $G(n+1) = Cone(\mu)$. If we put $P(n+1) = Cone(\nu)$ and $C(n+1) = Cone(\omega)$ then we have an exact sequence

$$0 \longrightarrow G(n+1) \xrightarrow{\phi(n+1)} P(n+1) \longrightarrow C(n+1) \longrightarrow 0 .$$

On one hand, exactness of the sequence

$$0 \longrightarrow Q \longrightarrow P(n+1) \longrightarrow P(n) \longrightarrow 0$$

implies that $P(n+1) \in \mathrm{C}(R\text{-Proj})$ is bounded above since $P(n) \in \mathrm{C}(R\text{-Proj})$ and $Q \in \mathrm{C}(R\text{-Proj})$ are so, and $P(n+1)_{-k} = P(n)_{-k}$ for $0 \leqslant k \leqslant n$. On the other hand, exactness of the sequence

$$0 \longrightarrow \mathrm{Coker}(\iota) \longrightarrow C(n+1) \longrightarrow C(n) \longrightarrow 0$$

implies that $C(n+1) \in \mathrm{C}(R\text{-GProj})$ is bounded above and acyclic with $\mathrm{Hom}_R(C(n+1), A)$ acyclic for any projective R-module A since $\mathrm{Coker}(\iota)$ and $C(n)$ are so. Clearly, one has $C(n+1)_{-k} = C(n)_{-k}$ for $0 \leqslant k \leqslant n$.

Note that every morphism $G(n+1) \to G(n)$ is surjective. By [9, Theorem 1.5.13], the sequence

$$0 \longrightarrow G = \varprojlim G(n) \xrightarrow{\varprojlim \phi(n)} \varprojlim P(n) \longrightarrow \varprojlim C(n) \longrightarrow 0$$

is exact. Let $P = \varprojlim P(n)$, and $C = \varprojlim C(n)$. Then $P_{-k} = \varprojlim P(n)_{-k} = P(k)_{-k}$ for any $k \geqslant 0$ and $P_{-k} = 0$ for any $k \leqslant -1$, $C_{-k} = \varprojlim C(n)_{-k} = C(k)_{-k}$ for any $k \geqslant 0$ and $C_{-k} = 0$ for any $k \leqslant -1$. Thus one can check easily that $P \in \mathrm{C}(R\text{-Proj})$ and $C \in \mathrm{C}(R\text{-GProj})$ are bounded above, C is acyclic, and also $\mathrm{Hom}_R(C, A)$ is acyclic for any projective R-module A. □

Now we give the following main result which contains Theorem 1.1.

3.5. Theorem. Every complex $G \in \mathrm{C}(R\text{-GProj})$ has a special $\mathrm{C}(R\text{-Proj})$-preenvelope $\eta : G \to P$ with $\mathrm{Coker}(\eta) \in \mathrm{C}(R\text{-GProj})$ acyclic.

Proof. If we write $G(n) = G_{\leqslant n}$ for each $n \geqslant 0$ then we get that $((G(n)), (\alpha_{mn}))_{n\geqslant 0}$ is a direct system in $C(R\text{-Mod})$ and $\varinjlim G(n) = G$, where $\alpha_{mn} : G(m) \to G(n)$ is a natural injection for any $m \leqslant n$.

By Lemma 3.4, there exists an exact sequence $0 \to G(0) \xrightarrow{\eta_0} P(0) \to C(0) \to 0$ such that $P(0) \in \mathrm{C}(R\text{-Proj})$ and $C(0) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, $C(0)$ is acyclic, and $\mathrm{Hom}_R(C(0), A)$ is acyclic for any projective R-module A. It follows from lemma 3.2 that $\mathrm{Ext}^1(C(0), Q) = 0$ for any $Q \in \mathrm{C}(R\text{-Proj})$. Thus the monomorphism $\eta_0 : G(0) \to P(0)$ is a special $\mathrm{C}(R\text{-Proj})$-preenvelope of $G(0)$. Consider the push-out diagram of morphisms $\eta_0 : G(0) \to P(0)$ and $\alpha_{01} : G(0) \to G(1)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G(0) \arrow[r, "\eta_0"] \arrow[d, "\alpha_{01}"'] & P(0) \arrow[r] \arrow[d, "\lambda_0"] & C(0) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G(1) \arrow[r, "\mu_0"] \arrow[d] & U \arrow[r] \arrow[d] & C(0) \arrow[r] & 0 \\
 & S^1(G_1) \arrow[r, equal] \arrow[d] & S^1(G_1) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly, $U \in \mathrm{C}(R\text{-GProj})$ is bounded above since $P(0)$ and $S^1(G_1)$ are so. By Lemma 3.4 again, we get that there exists an exact sequence $0 \to U \xrightarrow{\nu} P(1) \to L(1) \to 0$ such that $P(1) \in \mathrm{C}(R\text{-Proj})$ and $L(1) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, and $L(1)$ and $\mathrm{Hom}_R(L(1), A)$ are acyclic for any projective R-module A. Consider the push-out diagram of morphisms $U \to C(0)$ and $\nu : U \to P(1)$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G(1) \arrow[r, "\mu_0"] \arrow[d, equal] & U \arrow[r] \arrow[d, "\nu"] & C(0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G(1) \arrow[r] & P(1) \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & & L(1) \arrow[r, equal] \arrow[d] & L(1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The exactness of the rightmost column implies that $V \in \mathrm{C}(R\text{-GProj})$ is bounded above, V is acyclic, and $\mathrm{Hom}_R(V, A)$ is acyclic for any projective R-module A. It follows from Lemma 3.2 that the monomorphism $\eta_1 = \nu\mu_0 : G(1) \to P(1)$ is a special $\mathrm{C}(R\text{-Proj})$-preenvelope of $G(1)$. Let $C(1) = V$, and $\beta_{01} = \nu\lambda_0$. Therefore we get, by the construction above, a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G(0) \arrow[r, "\eta_0"] \arrow[d, "\alpha_{01}"'] & P(0) \arrow[r] \arrow[d, "\beta_{01}"'] & C(0) \arrow[r] \arrow[d, "\gamma_{01}"'] & 0 \\
0 \arrow[r] & G(1) \arrow[r, "\eta_1"] \arrow[d] & P(1) \arrow[r] \arrow[d] & C(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^1(G_1) \arrow[r] \arrow[d] & N(1) \arrow[r] \arrow[d] & L(1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since it is easily seen from the lower row of the above diagram that $N(1) \in \mathrm{C}(R\text{-GProj})$, and the middle column that $N(1)_i$ has finitely projective dimension for each $i \in \mathbb{Z}$, we get by Remark 2.2 that $N(1) \in \mathrm{C}(R\text{-Proj})$.

If we continue this process, then we get a commutative diagram with exact rows as follows

$$\begin{tikzcd}
0 \arrow[r] & G(0) \arrow[r, "\eta_0"] \arrow[d, "\alpha_{01}"'] & P(0) \arrow[r] \arrow[d, "\beta_{01}"'] & C(0) \arrow[r] \arrow[d, "\gamma_{01}"'] & 0 \\
0 \arrow[r] & G(1) \arrow[r, "\eta_1"] \arrow[d, "\alpha_{12}"'] & P(1) \arrow[r] \arrow[d, "\beta_{12}"'] & C(1) \arrow[r] \arrow[d, "\gamma_{12}"'] & 0 \\
0 \arrow[r] & G(2) \arrow[r, "\eta_2"] \arrow[d, "\alpha_{23}"'] & P(2) \arrow[r] \arrow[d, "\beta_{23}"'] & C(2) \arrow[r] \arrow[d, "\gamma_{23}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where each row $0 \longrightarrow G(n) \xrightarrow{\eta_n} P(n) \longrightarrow C(n) \longrightarrow 0$ satisfies that $P(n) \in \mathrm{C}(R\text{-Proj})$ and $C(n) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, $C(n)$ is acyclic, and $\mathrm{Hom}_R(C(n), A)$ is acyclic for any projective R-module A. In particular, by Lemma 3.2, the monomorphism $\eta_n : G(n) \to P(n)$ is a special $\mathrm{C}(R\text{-Proj})$-preenvelope of $G(n)$ for each $n \geqslant 0$. Also each row $0 \longrightarrow G(n) \xrightarrow{\eta_n} P(n) \longrightarrow C(n) \longrightarrow 0$ has the property that the following diagram with exact rows and columns is commutative.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G(n) \arrow[r, "\eta_n"] \arrow[d, "\alpha_{n,n+1}"'] & P(n) \arrow[r] \arrow[d, "\beta_{n,n+1}"'] & C(n) \arrow[r] \arrow[d, "\gamma_{n,n+1}"'] & 0 \\
0 \arrow[r] & G(n+1) \arrow[r, "\eta_{n+1}"] \arrow[d] & P(n+1) \arrow[r] \arrow[d] & C(n+1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^{n+1}(G_{n+1}) \arrow[r] \arrow[d] & N(n+1) \arrow[r] \arrow[d] & L(n+1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Where $N(n+1) \in \mathrm{C}(R\text{-Proj})$ and $L(n+1) \in \mathrm{C}(R\text{-GProj})$ are both bounded above, $L(n+1)$ is acyclic, $\mathrm{Hom}_R(L(n+1), A)$ is acyclic for any projective R-module A and for each $n \geqslant 0$. By Lemma 3.2, one has $\mathrm{Ext}^1(L(n+1), Q) = 0$ for any $Q \in \mathrm{C}(R\text{-Proj})$ and for each $n \geqslant 0$. Clearly, $((P(n)), (\beta_{mn}))_{n\geqslant 0}$ forms a continuous direct systems of monomorphisms in $\mathrm{C}(R\text{-Proj})$ such that $\mathrm{Coker}(\beta_{n,n+1}) = N(n+1) \in \mathrm{C}(R\text{-Proj})$, and we have $\varinjlim P(n) \in \mathrm{C}(R\text{-Proj})$ since $\mathrm{C}(R\text{-Proj})$ is closed under direct transfinite extension. Again since $((C(n)), (\gamma_{mn}))_{n\geqslant 0}$ forms a continuous direct systems of monomorphisms in $\mathrm{C}(R\text{-GProj})$ such that $\mathrm{Coker}(\gamma_{n,n+1}) = L(n+1) \in \mathrm{C}(R\text{-GProj})$, we get that $\varinjlim C(n) \in \mathrm{C}(R\text{-GProj})$ since R-GProj is closed under direct transfinite extension [7, Theorem 3.2]. Note that each $C(n)$ is acyclic and the class of acyclic complexes is a left side of a cotorsion pair [10], we get that $\varinjlim C(n)$ is acyclic by [6, Theorem 1.2]. In fact, the monomorphism $\eta : \varinjlim G(n) \to \varinjlim P(n)$, $\eta = \varinjlim \eta_n$, is a special $\mathrm{C}(R\text{-Proj})$-preenvelope of $\varinjlim G(n) = G$. To show this we need only to prove $\mathrm{Ext}^1(\varinjlim C(n), Q) = 0$ for any $Q \in \mathrm{C}(R\text{-Proj})$, but the latter is easily seen by [6, Theorem 1.5] and by the above construction. This completes the proof. □

3.6. Remark. The above special $\mathrm{C}(R\text{-Proj})$-preenvelope $\eta : G \to P$ of G is a homology isomorphism since η is monomorphic and $\mathrm{Coker}(\eta)$ is acyclic.

Recall from [4] that a complex P is called DG-projective if each P_i is projective and if $\mathrm{Hom}_R(P, E)$ is an acyclic complex of abelian groups for any acyclic complex E. Let R be a quasi-Frobenius ring, that is, An R-module M is projective if and only if it is injective. Then it is easily seen by Theorem 3.5 that every complex of left R-modules has a special $\mathrm{C}(R\text{-Proj})$ preenvelope since every left R-module is Gorenstein projective, so it is natural to ask whether every complex of left R-modules has a special DG-projective preenvelope, and we have the following result.

3.7. Proposition. Let R be a quasi-Frobenius ring. Then every complex of R-modules has a special DG-projective preenvelope of and only if $l.gl.dim(R) = 0$.

Proof. For the necessity. Suppose $l.gl.dim(R) > 0$ and let M be a non-projective R-module. If $S^0(M) \to P$ is a special DG-projective preenvelope (which is injective), then there is an induced morphism $S^0(M) \to P_{\leqslant 0}$. Since the sequence $0 \to P_{\leqslant 0} \to P \to P_{\geqslant 1} \to 0$ is exact and $P_{\geqslant 1}$ and P are DG-projective, it follows that the subcomplex $P_{\leqslant 0}$ is DG-projective. Thus one can check easily that $S^0(M) \to P_{\leqslant 0}$ is a DG-projective preenvelope of $S^0(M)$. In fact, let $K_0 = \mathrm{Coker}(M \to P_0)$. Then $\mathrm{Ext}^1(X, T) = 0$ for any DG-projective complex T since $S^0(M) \to P$ is a special DG-projective preenvelope, where $X = \mathrm{Coker}(S^0(M) \to P) =: \cdots \to P_2 \to P_1 \xrightarrow{0} K_0 \to P_{-1} \to P_{-2} \to \cdots$. But it is easily seen that $K =: 0 \to K_0 \to P_{-1} \to P_{-2} \to \cdots$ is a direct summand of X, and so $\mathrm{Ext}^1(K, T) = 0$ for any DG-projective complex T. This shows that $S^0(M) \to P_{\leqslant 0}$ is a special DG-projective preenvelope of $S^0(M)$.

Let $0 \to M \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots$ be a right minimal projective (injective) resolution of M, that is to say, $M \to Q_0$ and each $L_{-i+1} \to Q_{-i}$ are projective envelopes of M and L_{-i+1} for $i > 0$, respectively, where $L_0 = \mathrm{Coker}(M \to Q_0)$, and $L_{-i} = \mathrm{Coker}(L_{-i+1} \to Q_{-i})$. Denote the complex $0 \to L_0 \to Q_{-1} \to Q_{-2} \to \cdots$ by L with L_0 in the 0th degree. Then we have a morphism $S^0(M) \to S^0(Q_0)$ with $S^0(Q_0)$ DG-projective. Thus there is a commutative diagram

$$\begin{tikzcd}
S^0(M) \arrow[r] \arrow[d, equal] & P_{\leqslant 0} \arrow[d, dotted] \\
S^0(M) \arrow[r] & S^0(Q_0)
\end{tikzcd}$$

In particular, its commutative square frame in the 0th degree implies that there exists a morphism of R-modules $K_0 \to L_0$ such that the following diagram with the bottom row exact is commutative.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & P_0 \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & Q_0 \arrow[r] & L_0 \arrow[r] & 0
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
K \arrow[d] & 0 \arrow[r] & K_0 \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & P_{-2} \arrow[r] \arrow[d] & \cdots \\
S^0(L_0) & 0 \arrow[r] & L_0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
L_{\geqslant -1} \arrow[u] & 0 \arrow[r] & L_0 \arrow[r] \arrow[u, equal] & Q_{-1} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Since the subcomplex $0 \to Q_{-1} \to 0$ of $0 \to L_0 \to Q_{-1} \to 0$ is DG-projective and since $\mathrm{Ext}^1(K, T) = 0$ for any DG-projective complex T, we can lift the morphism $K \to S^0(L_0)$ to a morphism $K \to L_{\geqslant -1}$. Then consider the morphism $K \to L_{\geqslant -1}$ and the exact sequence $0 \to S^{-2}(Q_{-2}) \to L_{\geqslant -2} \to L_{\geqslant -1} \to 0$, for the same reason, we can lift the morphism $K \to L_{\geqslant -1}$ to a morphism $K \to L_{\geqslant -2}$. Repeating the procedure, we see that there is a commutative diagram

$$\begin{tikzcd}
K \arrow[d] & 0 \arrow[r] & K_0 \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] & P_{-2} \arrow[r] \arrow[d] & P_{-3} \arrow[r] \arrow[d] & \cdots \\
L & 0 \arrow[r] & L_0 \arrow[r] & Q_{-1} \arrow[r] & Q_{-2} \arrow[r] & Q_{-3} \arrow[r] & \cdots
\end{tikzcd}$$

On the other hand, since $0 \to M \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots$ is a right minimal projective (injective) resolution of M, there exist morphisms $Q_i \to P_i$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & Q_0 \arrow[r] \arrow[d] & Q_{-1} \arrow[r] \arrow[d] & Q_{-2} \arrow[r] \arrow[d] & Q_{-3} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & M \arrow[r] & P_0 \arrow[r] & P_{-1} \arrow[r] & P_{-2} \arrow[r] & P_{-3} \arrow[r] & \cdots
\end{tikzcd}$$

is commutative, this induces a commutative diagram

$$\begin{tikzcd}
L \arrow[d] & 0 \arrow[r] & L_0 \arrow[r] \arrow[d] & Q_{-1} \arrow[r] \arrow[d] & Q_{-2} \arrow[r] \arrow[d] & Q_{-3} \arrow[r] \arrow[d] & \cdots . \\
K & 0 \arrow[r] & K_0 \arrow[r] & P_{-1} \arrow[r] & P_{-2} \arrow[r] & P_{-3} \arrow[r] & \cdots
\end{tikzcd}$$

But $0 \to L_0 \to Q_{-1} \to Q_{-2} \to \cdots$ is a minimal projective resolution of L_0, so one can check easily that L is isomorphic to a direct summand of K, and so $0 \to Q_{-1} \to Q_{-2} \to Q_{-3} \to \cdots$ is a direct summand of $P_{\leqslant -1}$. It follows that $P_{\leqslant -1}$ is DG-projective since $S^0(P_0)$ and $P_{\leqslant 0}$ in the exact sequence $0 \to P_{\leqslant -1} \to P_{\leqslant 0} \to S^0(P_0) \to 0$ are so, hence $0 \to Q_{-1} \to Q_{-2} \to Q_{-3} \to \cdots$ is DG-projective and of course then $0 \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots$ is DG-projective. Now assembling the (left) projective resolution $\cdots \to Q_2 \to Q_1 \to M \to 0$ and the complex $0 \to M \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots$, one gets an exact sequence $0 \to Q_{\leqslant 0} \to Q \to Q_{\geqslant 1} \to 0$ with $Q_{\leqslant 0}$ and $Q_{\geqslant 1}$ DG-projective, where $Q =: \cdots \to Q_2 \to Q_1 \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots$. Thus Q is clearly DG-projective. But this complex is acyclic, and so it is a projective complex by [10, Proposition 3.7]. This contradicts to the fact that M is a non-projective R-module. Hence $l.gl.dim(R) = 0$.

The sufficiency is trivial. $\square$

4. Adjoints to inclusion functors

We have mentioned in the introduction that the inclusion $\mathrm{K}(R\text{-Proj}) \to \mathrm{K}(R\text{-Mod})$ always has a right adjoint ([5, Theorem 4.7]). We are inspired to consider whether there exists a left adjoint to it in this section.

4.1. Definition. Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$. The subcategory is said to be thick if it is a triangulated subcategory, and if every direct summand of any object of $\mathcal{C}$ is in $\mathcal{C}$.

The following result is dual to [16, Proposition 1.4], we give its proof for completeness.

4.2. Proposition. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{S}$ a thick subcategory of $\mathcal{T}$. Assume further that

(1) Every object $T \in \mathcal{T}$ admits an $\mathcal{S}$-preenvelope.
(2) Every idempotent in $\mathcal{T}$ splits.

Then the inclusion $\rho : \mathcal{S} \to \mathcal{T}$ has a left adjoint.

Proof. Let T be an object in $\mathcal{T}$. In the following we will show that there exists a morphism $g : T \to S$ with $S \in \mathcal{S}$ such that every other morphism $T \to \overline{S}$ with $\overline{S} \in \mathcal{S}$ must factor uniquely through g. Firstly, we choose an $\mathcal{S}$-preenvelope $f : T \to \widetilde{S}$ which must exist by hypothesis, every morphism $T \to \overline{S}$, $\overline{S} \in \mathcal{S}$ clearly factors through f, but not necessarily uniquely. We will show next that we can choose a direct summand S of $\widetilde{S}$ for which the factorization is unique.

Complete $f : T \to \widetilde{S}$ to a triangle $T \xrightarrow{f} \widetilde{S} \xrightarrow{a} X \longrightarrow \Sigma T$ and then choose an $\mathcal{S}$-preenvelope $b : X \to S'$. Again complete $ba : \widetilde{S} \to S'$ to a triangle

$S'' \xrightarrow{c} \widetilde{S} \xrightarrow{ba} S' \longrightarrow \Sigma S''$ and then we get a morphism of triangles:

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "d"'] & \widetilde{S} \arrow[r, "a"] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "b"'] & \Sigma T \arrow[d, "\Sigma d"] \\ S'' \arrow[r, "c"] & \widetilde{S} \arrow[r, "ba"] & S' \arrow[r] & \Sigma S'' \end{tikzcd} \tag{$*$}$$

We get that $S'' \in \mathcal{S}$ since $\widetilde{S}$ and S' are in the thick subcategory $\mathcal{S}$. Since f is an $\mathcal{S}$-preenvelope, the morphism $d : T \to S''$ can be factored as $d = \widetilde{c}f$ with $\widetilde{c}$ a morphism $\widetilde{S} \to S''$. Now let $e = c\widetilde{c} : \widetilde{S} \to \widetilde{S}$ be the composite $\widetilde{S} \xrightarrow{\widetilde{c}} S'' \xrightarrow{c} \widetilde{S}$. Then the diagram $(*)$ implies that $f = cd = c\widetilde{c}f = ef$. To obtain the desired summand S of $\widetilde{S}$, we need the following more steps.

- If the composite $T \xrightarrow{f} \widetilde{S} \xrightarrow{\rho} \overline{S}$ vanishes for some morphism $\rho : \widetilde{S} \to \overline{S}$ with $\overline{S} \in \mathcal{S}$, then so does the composite $\widetilde{S} \xrightarrow{e} \widetilde{S} \xrightarrow{\rho} \overline{S}$.

Let ρ satisfy $\rho f = 0$ as above. Then we have the following morphism of triangles:

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "0"'] & \widetilde{S} \arrow[r, "a"] \arrow[d, "\rho"'] & X \arrow[r] \arrow[d, "a'"'] & \Sigma T \arrow[d, "0"] \\ 0 \arrow[r] & \overline{S} \arrow[r, "="] & \overline{S} \arrow[r] & \Sigma 0 \end{tikzcd}$$

This shows $\rho = a'a$. Since $b : X \to S'$ is an $\mathcal{S}$-preenvelope of X, there exists a morphism $b' : S' \to \overline{S}$ such that $a' = b'b$. Thus by the diagram $(*)$ we get that $\rho e = a'ae = b'bae = b'bac\widetilde{c} = b'(bac)\widetilde{c} = 0$.

- Note that $f = ef$, i.e., $(1 - e)f = 0$, it follows from above that the morphism $e : \widetilde{S} \to \widetilde{S}$ is an idempotent, that is, $e^2 = e$.

By the hypothesis that any idempotent in $\mathcal{T}$ splits, the morphism $e : \widetilde{S} \to \widetilde{S}$ has a factorization $\widetilde{S} \xrightarrow{u} S \xrightarrow{v} \widetilde{S}$ with uv being the identity $1_S : S \to S$. We get that S must belong to $\mathcal{S}$ since it is a direct summand of $\widetilde{S}$ and the subcategory $\mathcal{S}$ is thick. Now we assert:

- Let $e = vu$ be a splitting as above, and $g : T \to S$ be the composite $T \xrightarrow{f} \widetilde{S} \xrightarrow{u} S$. Then g has the property that any morphism $T \to \overline{S}$, $\overline{S} \in \mathcal{S}$ factors uniquely through g.

It remains to show the last assertion. Suppose that we are given a morphism $h : T \to \overline{S}$ with $\overline{S} \in \mathcal{S}$. Because $f : T \to \widetilde{S}$ is an $\mathcal{S}$-preenvelope the map h must factor as $T \xrightarrow{f} \widetilde{S} \xrightarrow{\sigma} \overline{S}$. Now observe

$$h = \sigma f = \sigma e f = \sigma v u f = (\sigma v)(uf) = (\sigma v)g,$$

and we have factored h through g. It remains to show the uniqueness. Suppose $\tau : S \to \overline{S}$ is such that the composite $\tau g = \tau u f$ vanishes. By above proof we have $\tau u e = 0$. Note that $e = vu$, we have $\tau u v u = 0$, and of cause $\tau u v u v = 0$. But $uv = 1$, we conclude that $\tau = 0$, as desired. □

The categories K(R-Proj), K(R-GProj) and K(R-Mod) have coproducts, hence idempotents split by [14, Proposition 1.6.8]. It is clear that K(R-Proj) is a thick subcategory of either K(R-GProj) or K(R-Mod). Now we give the main result in this section.

4.3. Theorem. Let R be any ring. Then the inclusion K(R-Proj) $\to$ K(R-GProj) has a left adjoint.

Proof. It follows from Proposition 4.2 and Theorem 3.5. $\square$

4.4. Corollary. Let R be any ring. Then the composition functor $J\widetilde{I} : \mathrm{K}(R\text{-GProj}) \to \mathrm{K}(R\text{-Mod})$ has a right adjoint, where $I : \mathrm{K}(R\text{-Proj}) \to \mathrm{K}(R\text{-GProj})$ and $J : \mathrm{K}(R\text{-Proj}) \to \mathrm{K}(R\text{-Mod})$ are the inclusions, and $\widetilde{I}$ is a left adjoint to I.

Proof. By [5, Theorem 4.7], the inclusion $J : \mathrm{K}(R\text{-Proj}) \to \mathrm{K}(R\text{-Mod})$ has a right adjoint $\widehat{J}$. Since we have isomorphisms for any $G \in \mathrm{K}(R\text{-GProj})$ and $M \in \mathrm{K}(R\text{-Mod})$

$$\mathrm{Hom}_{K(R\text{-Mod})}(J\widetilde{I}G, M) \cong \mathrm{Hom}_{K(R\text{-Proj})}(\widetilde{I}G, \widehat{J}M) \cong \mathrm{Hom}_{K(R\text{-GProj})}(G, I\widehat{J}M),$$

it follows that $I\widehat{J}$ is a right adjoint to $J\widetilde{I} : \mathrm{K}(R\text{-GProj}) \to \mathrm{K}(R\text{-Mod})$. $\square$

At the end of this section we give adjoints to inclusion functors over special rings.

4.5. Proposition. If R is left perfect and right coherent, then the inclusions of $\mathrm{K}(R\text{-Proj})$, into either of the categories $\mathrm{K}(R\text{-GProj})$ and $\mathrm{K}(R\text{-Mod})$, have left adjoints.

Proof. By [1, Proposition 3.5], a ring R is left perfect and right coherent if and only if every left R-module has a projective preenvelope. Thus it follows from [17, Theorem 4.2] that every complex in $\mathrm{K}(R\text{-GProj})$ or $\mathrm{K}(R\text{-Mod})$ admits a $\mathrm{K}(R\text{-Proj})$-preenvelope since every flat R-module is projective under the hypothesis. Now the result follows from Proposition 4.2. $\square$

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