Gamma-admissibility of generalized Bayes estimators under LINEX loss function in a non-regular family of distributions

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Abstract

Consider an estimation problem in a non-regular family of distributions under the LINEX loss function. Reviewing the admissibility of estimators under a vague prior information leads to the concept of gamma-admissibility. The purpose of this article is to give a sufficient conditions for a generalized Bayes estimator of a parametric function to be gamma-admissible. Some examples are given.

1. INTRODUCTION

Admissibility of estimator is an important problem in statistical decision theory; Consequently, this problem has been considered by many authors under various types of loss functions both in an exponential and in a non-regular family of distributions. For example under squared error loss function (Karlin [5], Ghosh & Meeden [3], Ralescu & Ralescu [10], Sinha & Gupta [13], Hoffmann [4], Pulskamp & Ralescu [9], Kim [6] and Kim & Meeden [7]), under entropy loss function (Sanjari Farsipour [11, 12]) and under LINEX loss function (Tanaka [14, 15, 16]) and squared-log error loss function (Zakerzadeh & Moradi Zahraie [18]).

In Bayesian statistical inference arbitrariness of a unique prior distribution is a permanent question. Robust Bayesian inference deals with the problem of expressing uncertainty of the prior information. A gamma-admissible approach is used which allows to take into account vague prior information on the distribution of the unknown parameter θ. The uncertainty about a prior is assumed by introducing a class Γ of priors. If prior information is scarce, the class Γ under consideration is large and a decision is close to a admissible decision. In the extreme case when no information is available the

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Γ-admissible setup is equivalent to the usual admissible setup. If, on the other hand, the statistician has an exactly prior information and the class Γ contains a single prior, then the Γ-admissible decision is an usual Bayes decision. So it is a middle ground between the subjective Bayes setup and full admissible. See Berger [1] for useful references on robust Bayesian analysis.

Eichenauer-Herrmann [2] gained a sufficient conditions for an estimator of the form $(aX + b)/(cX + d)$ to be Γ-admissible under the squared error loss in a one-parameter exponential family.

The most popular convex and symmetric loss function is the squared error loss function which is widely used in decision theory due to its simple mathematical properties. However in some cases, it does not represent the true loss structure. This loss function is symmetric in nature i.e. it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over-estimation may be more serious than under-estimation or vice versa. As an example, in construction an underestimate of the peak water level is usually much more serious than an overestimation.

The LINEX loss function was initially introduced by Varian [17] in the context of real estate assessment; estimation under this loss from the Bayesian perspective was studied by Zellner [19]. Subsequently, it became a workhorse in the literature on asymmetric loss. For an estimator δ of estimand $h(\theta)$, it is given by

$$(1.1) \qquad L(\delta, h(\theta)) = b\left\{e^{c(\delta - h(\theta))} - c(\delta - h(\theta)) - 1\right\},$$

where $c \neq 0$ and $b > 0$. If we define $\nabla := \delta - h(\theta)$, then $L(\nabla) = b\left\{e^{c\nabla} - c\nabla - 1\right\}$.

Some properties of the loss (1.1) are as follows:

  (i) The constant b serves to scale this loss and without loss of generality we can assume that it is equal 1.
 (ii) The constant c determines the shape of the loss; For $c > 0$ this loss function is quite asymmetric about 0 with overestimation being more costly than under-estimation. As $|\nabla| \to \infty$, $L(\nabla)$ increases almost exponentially when $\nabla > 0$ and almost linearly when $\nabla < 0$. For $c < 0$, the linearity-exponentially phenomenon is reversed.
(iii) For $|c| \to 0$, this loss is almost symmetric and not far from a squared error loss function; In fact since $e^{c\nabla} \approx 1 + c\nabla + c^2\nabla^2/2$, thus $L(\nabla) \approx c^2\nabla^2/2$.
 (iv) It is everywhere differentiable and its derivatives are continuous.

1.1. Remark. Linear-exponential where the name LINEX is justified by the fact that is this loss function rises approximately linearly on one side of zero and approximately exponentially on the other side.

A full discussion of the properties of this loss, may be found in Zellner [19] and Parsian & Kirmani [8].

In this paper we consider the Γ-admissibility of generalized Bayes estimators in a non-regular family of distributions under the loss (1.1) where class Γ consists of all distributions which are compatible with the vague prior information. To this end, in Section 2, we state some preliminary definitions and results. In Section 3, main theorem will obtain. Finally, in Section 4, we give an application of the Γ-admissibility in proof the Γ-minimaxity of estimators. Some examples are given.

2. Preliminaries

2.1. Definition of Γ-admissibility. In the present paper it is assumed that vague prior density on the distribution of the unknown parameter θ is available. Let Π denote the set of all priors, i.e. Borel probability measures on the parameter interval Θ and Γ

be a non-empty subset of Π. Suppose that the available vague prior information can be described by the set Γ, in the sense that Γ contains all prior which are compatible with the vague prior information.

Eichenauer-Herrmann [2] has defined the Γ-admissibility of an estimator as follows.

2.1. Definition. An estimator δ^* is called Γ-admissible, if

$$r(\pi, \delta) \leq r(\pi, \delta^*), \qquad \pi \in \Gamma,$$

for some estimator δ implies that

$$r(\pi, \delta) = r(\pi, \delta^*), \qquad \pi \in \Gamma,$$

where $r(\pi, \delta)$ is the Bayes risk of δ.

2.2. Remark. From Definition 2.1, it is obvious that
- A Π-admissible estimator is admissible.
- A $\{\pi\}$-admissible estimator is simply a Bayes strategy with respect to the prior π.
- In general neither Γ-admissibility implies admissibility nor admissibility implies Γ-admissibility.

Hence, the available results on admissibility cannot be applied in order to prove the Γ-admissibility of an estimator. Consequently, it is necessary to study the problem of Γ-admissibility of estimators.

2.2. A non-regular family of distributions. Let X be a random variable whose probability density function with respect to some σ-finite measure μ is given by

$$f_X(x; \theta) = \begin{cases} q(\theta)r(x), & \underline{\theta} < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \Theta =: (\underline{\theta}, \bar{\theta})$ and Θ is a nondegenerate interval (possibly infinite) on the real line. Also $r(x)$ is a positive μ-measurable function of x and

$$q^{-1}(\theta) = \int_{\underline{\theta}}^{\theta} r(x) d\mu(x) < \infty$$

for $\theta \in \Theta$. This family is known as *a non-regular family of distributions.*

Suppose $\pi(\theta)$ be a prior (possibly improper) by its Lebesgue density $p_\pi(\theta)$ over Θ which is positive and continuous. Let $h(\theta)$ be a continuous function to be estimated from Θ to $\mathbb{R}$ and the loss to be (1.1). The generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ is given by $\delta_\pi(X)$, where

$$(2.1) \qquad \delta_\pi(x) = -\frac{1}{c} \ln \left\{ \frac{\int_x^{\bar{\theta}} e^{-ch(\theta)} q(\theta) p_\pi(\theta) d\theta}{\int_x^{\bar{\theta}} q(\theta) p_\pi(\theta) d\theta} \right\}$$

for $\underline{\theta} < x < \bar{\theta}$, provided that the integrals in (2.1) exist and are finite.

3. Main results

In this section, main results will obtain.

For some real number λ_0 let $a, b : [\lambda_0, \infty) \mapsto \Theta$ be continuously differentiable functions with $a(\lambda_0) < b(\lambda_0)$, where a and b are supposed to be strictly decreasing and strictly increasing, respectively. For $\lambda \geq \lambda_0$ a prior π_λ is defined by its Lebesgue density p_{π_λ} of the form

$$p_{\pi_\lambda}(\theta) := \left(\int_{a(\lambda)}^{b(\lambda)} p_\pi(t) dt \right)^{-1} I_{[a(\lambda), b(\lambda)]}(\theta) p_\pi(\theta).$$

Throughout this paper, we restrict estimators to the class

$$\Delta := \{\delta | (A1) \text{ and } (A2) \text{ are satisfied}\},$$

where

(A1) $E_\theta[|\delta(X)|] < \infty$ and $E_\theta\left[e^{a\delta(X)}\right] < \infty$ for all $\theta \in \Theta$,

(A2) $\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[|\delta(X) - h(\theta)|\right] p_\pi(\theta)d\theta < \infty$ and $\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[e^{a(\delta(X)-h(\theta))}\right]p_\pi(\theta)d\theta$ for $\lambda \geq \lambda_0$ and all θ which $\underline{\theta} < a(\lambda) < \theta < b(\lambda) < \bar{\theta}$.

3.1. Remark. In the statistical game (Γ, Δ, r), a Γ-admissible estimator is an admissible strategy of the second player.

The next lemma is essential to obtain our results.

3.2. Lemma. *Let $S(\theta)$ be a continuous and non-negative function over $\Theta = (\underline{\theta}, \bar{\theta})$. Let $G(\lambda) := \int_{a(\lambda)}^{b(\lambda)} S(\theta)d\theta$. Suppose that there exists a positive function $R(\theta)$ such that*

$$G(\lambda) \leq 4\left(\min\left\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\right\}\right)^{-\frac{1}{2}}\left(G'(\lambda)\right)^{\frac{1}{2}}$$

for $\lambda \geq \lambda_0$. If

$$\int_{\lambda_0}^{\infty} \min\left\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\right\}d\lambda = \infty,$$

then $S(\theta) = 0$ for a.a. $\theta \in \Theta$.

Proof. See Eichenauer-Herrmann [2]. □

Now, the main result of the present paper can be stated.

3.3. Theorem. *Suppose that $\delta_\pi \in \Delta$ and put*

$$K(x, \theta) := \int_x^\theta \left\{e^{-c\delta_\pi(x)} - e^{-ch(t)}\right\}q(t)p_\pi(t)dt,$$

and

$$\gamma(\theta) := \frac{e^{ch(\theta)}}{p_\pi(\theta)q(\theta)}\int_{\underline{\theta}}^\theta r(x)e^{c\delta_\pi(x)}K^2(x, \theta)d\mu(x).$$

If $\pi_\lambda \in \Gamma$ for all $\lambda \geq \lambda_0$ and

(3.1) $$\int_{\lambda_0}^{\infty} \min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\}d\lambda = \infty,$$

then $\delta_\pi(X)$ is Γ-admissible under the loss (1.1).

Proof. Let $\delta \in \Delta$ be an estimator such that $r(\pi, \delta) \leq r(\pi, \delta_\pi)$ for every prior $\pi \in \Gamma$. Since $\pi_\lambda \in \Gamma$ for $\lambda \geq \lambda_0$, we must have

$$0 \leq \left(\int_{a(\lambda)}^{b(\lambda)} p_\pi(t)dt\right)\{r(\pi_\lambda, \delta_\pi) - r(\pi_\lambda, \delta)\}$$

$$= \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[L(\delta_\pi, h(\theta)) - L(\delta, h(\theta))\right]p_\pi(\theta)d\theta$$

for all $\theta \in \Theta$. From Condition (A1), we see that it is equivalent to

$$0 \leq \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[\left\{e^{\frac{c\delta(X)}{2}} - e^{\frac{c\delta_\pi(X)}{2}}\right\}^2\right]e^{-ch(\theta)}p_\pi(\theta)d\theta$$

$$\leq \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[c\{\delta(X) - \delta_\pi(X)\}\right]p_\pi(\theta)d\theta$$

$$- 2\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[e^{-ch(\theta)}e^{\frac{c\delta_\pi(X)}{2}}\left\{e^{\frac{c\delta(X)}{2}} - e^{\frac{c\delta_\pi(X)}{2}}\right\}\right]p_\pi(\theta)d\theta.$$

An application of the Fubini's theorem gives

$$
\begin{aligned}
0 \quad \leq \quad & \int_{a(\lambda)}^{b(\lambda)} \int_{\underline{\theta}}^{\theta} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\}^2 r(x)d\mu(x)e^{-ch(\theta)}p_\pi(\theta)q(\theta)d\theta \\
\leq \quad & \int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} \{c(\delta(x) - \delta_\pi(x))\}r(x)q(\theta)p_\pi(\theta)d\theta d\mu(x) \\
- \quad & 2\int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} e^{-ch(\theta)}e^{\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x)q(\theta)p_\pi(\theta)d\theta d\mu(x) \\
- \quad & \int_{\underline{\theta}}^{a(\lambda)} \int_{x}^{a(\lambda)} \{c(\delta(x) - \delta_\pi(x))\} r(x)q(\theta)p_\pi(\theta)d\theta d\mu(x) \\
+ \quad & 2\int_{\underline{\theta}}^{a(\lambda)} \int_{x}^{a(\lambda)} e^{-ch(\theta)}e^{\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x)q(\theta)p_\pi(\theta)d\theta d\mu(x)
\end{aligned}
$$

(3.2)

which is guaranteed by Condition (A2).

Using the inequality $x - y \leq e^{-y}(e^x - e^y)$ for all x and y, the first term of the right-hand side in (3.2) is less than

$$
2\int_{\underline{\theta}}^{b(\lambda)} \int_{x}^{b(\lambda)} e^{-\frac{c\delta_\pi(x)}{2}} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\} r(x)q(\theta)p_\pi(\theta)d\theta d\mu(x).
$$

By Schwartz inequality, sum of the first and the second terms of the right-hand side in (3.2) is less than

$$
2\left\{ \int_{\underline{\theta}}^{b(\lambda)} \left(e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right)^2 r(x)d\mu(x) \right\}^{\frac{1}{2}} \left\{ \int_{\underline{\theta}}^{b(\lambda)} e^{c\delta_\pi(x)}K^2(x, b(\lambda))r(x)d\mu(x) \right\}^{\frac{1}{2}}.
$$

Hence, if we define

$$
T(\theta) := \int_{\underline{\theta}}^{\theta} \left\{ e^{\frac{c\delta(x)}{2}} - e^{\frac{c\delta_\pi(x)}{2}} \right\}^2 r(x)d\mu(x),
$$

and

$$
M(\theta) := T(\theta)e^{-cb(\theta)}q(\theta)p_\pi(\theta),
$$

then Equation (3.2) implies

$$
\begin{aligned}
0 \quad \leq \quad & \int_{a(\lambda)}^{b(\lambda)} T(\theta)e^{-ch(\theta)}q(\theta)p_\pi(\theta)d\theta \\
\leq \quad & 2\left\{ T(b(\lambda))e^{-ch(b(\lambda))}q(b(\lambda))p_\pi(b(\lambda))b'(\lambda) \right\}^{\frac{1}{2}} \left\{ \gamma^{-1}(b(\lambda))b'(\lambda) \right\}^{-\frac{1}{2}} \\
+ \quad & 2\left\{ -T(a(\lambda))e^{-ch(a(\lambda))}q(a(\lambda))p_\pi(a(\lambda))a'(\lambda) \right\}^{\frac{1}{2}} \left\{ -\gamma^{-1}(a(\lambda))a'(\lambda) \right\}^{-\frac{1}{2}} \\
\leq \quad & 4\left(\min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\} \right)^{-\frac{1}{2}} \times \left(M(b(\lambda))b'(\lambda) - M(a(\lambda))a'(\lambda) \right)^{\frac{1}{2}}
\end{aligned}
$$

(3.3)

for $\lambda \geq \lambda_0$, where the definition of the function $\gamma(\theta)$ has been used. Now a continuous, differentiable and increasing function $H : [\lambda_0, \infty] \to \mathbb{R}$ is defined by

$$
H(\lambda) := \int_{a(\lambda)}^{b(\lambda)} T(\theta)e^{-ch(\theta)}q(\theta)p_\pi(\theta)d\theta.
$$

So (3.3) can be written in the form

$$H(\lambda) \leq 4 \left(\min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\} \right)^{-\frac{1}{2}} (H'(\lambda))^{\frac{1}{2}}$$

for $\lambda \geq \lambda_0$. Therefore, from Lemma 3.2 we obtain $T(\theta) = 0$ for $a.a.\theta \in \Theta$, and consequently from (A1), we have $\delta(x) = \delta_\pi(x)$ $a.e.\mu$. This completes the proof. $\qquad\square$

3.4. Remark. $K(x, \theta)$ can expressed as

$$K(x, \theta) = \frac{1}{\int_x^{\bar{\theta}} q(u)p_\pi(u)du} \int_x^{\theta} \int_{\theta}^{\bar{\theta}} \left\{ e^{-ah(s)} - e^{-ah(t)} \right\} q(s)p_\pi(s)q(t)p_\pi(t)dsdt,$$

by (2.1) and the symmetry of the integrand.

3.5. Example. As Example 1 in [18], suppose that $X_1, ..., X_n$ are i.i.d. random variables according to an exponential distribution whose probability distribution function is given by

$$f(x; \theta) = \begin{cases} e^{x-\theta}, & x < \theta \\ 0, & x > \theta \end{cases}$$

where $\theta(\in \mathbb{R})$ is unknown. $X = X_{(n)}$ is sufficient for θ and its probability distribution function is given by

$$f_X(x; \theta) = \begin{cases} ne^{n(x-\theta)}, & x < \theta \\ 0, & x > \theta \end{cases}$$

The generalized Bayes estimator of $h(\theta) = \theta$ with respect to the Lebesgue prior is given by

$$\delta_\pi(X) = X + \frac{1}{c} \ln \frac{n+c}{n},$$

if $n + c > 0$. A direct calculation gives

$$K(x, \theta) = \frac{1}{n+c} e^{-n\theta} \left(e^{-c\theta} - e^{-cx} \right),$$

and

$$\gamma(\theta) = \frac{2c^2}{n(n+c)^2(n-c)}.$$

Let class Γ_0 consists of all priors with mean 0, i.e., $\Gamma_0 := \{\pi \in \Pi| \int_\Theta \theta p_\pi(\theta)d\theta = 0\}$. Define functions a and b by $a(\lambda) = -\lambda$ and $b(\lambda) = \lambda$ for $\lambda \geq \lambda_0 > 0$, i.e., the prior π_λ is the uniform distribution on the interval $[-\lambda, \lambda]$. Hence, $\pi_\lambda \in \Gamma_0$ for all $\lambda \geq \lambda_0$. Since (3.1) is satisfied, Theorem 3.3 implies that $\delta_\pi(X)$ is Γ_0-admissible under the loss (1.1).

3.6. Remark. It is difficult to express $\gamma(\theta)$ explicitly and it can have a complicated form, so to apply Theorem 3.3, we have to seek the suitable upper bound of $\gamma(\theta)$. For the case when $h(\theta)$ is bounded, we can get the next corollary.

3.7. Corollary. *Suppose that $h(\theta)$ is bounded and $\delta_\pi \in \Delta$. Put*

$$\tilde{K}(x, \theta) := \frac{\int_\theta^{\bar{\theta}} q(s)p_\pi(s)ds \int_x^{\theta} q(t)p_\pi(t)dt}{\int_x^{\bar{\theta}} q(u)p_\pi(u)du},$$

and

$$\tilde{\gamma}(\theta) := \frac{1}{p_\pi(\theta)q(\theta)} \int_{\underline{\theta}}^{\theta} r(x)\tilde{K}^2(x, \theta)d\mu(x).$$

If $\pi_\lambda \in \Gamma$ for all $\lambda \geq \lambda_0$ and

$$\int_{\lambda_0}^{\infty} \min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda), -\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\}d\lambda = \infty,$$

then $\delta_\pi(X)$ is Γ-admissible under the loss (1.1).

Proof. It can be shown that there exist constants $\underline{C}$ and $\bar{C}$ such that $\underline{C} < e^{c\delta_\pi(x)} < \bar{C}$ for all $x \in (\underline{\theta}, \bar{\theta})$. Further, since $h(\theta)$ is bounded, there exists a constant C such that $|K(x, \theta)| \leq C\tilde{K}(x, \theta)$ for all $(x, \theta) \in \{(x, \theta)|\underline{\theta} < x < \theta < \bar{\theta}\}$. This completes the proof by Theorem 3.3. □

3.8. Example. As Example 2 in [18], suppose that $X_1, ..., X_n$ are i.i.d. random variables according to a uniform distribution over the interval $(0, \theta)$ where $\theta (\in \mathbb{R}^+)$ is unknown. Then the probability distribution function of the sufficient statistic $X = X_{(n)}$ is given by

$$f_X(x; \theta) = \begin{cases} \frac{n}{\theta^n} x^{n-1}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

Let $h(\theta) = P_\theta(X_1 \leq 1) = \frac{1}{\theta}I_{\{1 < \theta\}}(\theta) + I_{\{\theta < 1\}}(\theta)$, where $I_A(\theta)$ is the indicator function of the set A. Then the generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ by its density $p_\pi(\theta) = 1/\theta$ is given by $\delta_\pi(X)$, where

$$\delta_\pi(x) = \begin{cases} -\frac{1}{c} \ln \left\{ e^{-c}(1 - x^n) + n \int_0^x y^{n-1}e^{-c\frac{y}{x}} dy \right\}, & 0 < x < 1 \\ -\frac{1}{c} \ln \left\{ n \int_0^1 y^{n-1}e^{-c\frac{y}{x}} dy \right\}, & 1 < x \end{cases}$$

We can easily obtain

$$\tilde{K}(x, \theta) = \frac{1}{n\theta^n} \left\{ 1 - \left(\frac{x}{\theta}\right)^n \right\},$$

and

$$\tilde{\gamma}(\theta) = \frac{\theta}{3n^2}.$$

Let $\Gamma_m := \{\pi \in \Pi | \int_\Theta \theta p_\pi(\theta)d\theta = m\}$, i.e., Γ_m consists of all priors with mean m. Define functions a and b by $a(\lambda) = m \ln(\lambda)/(\lambda - 1)$ and $b(\lambda) = \lambda a(\lambda)$ for $\lambda \geq \lambda_0 > 1$. Since

$$\int_\Theta \theta p_{\pi_\lambda}(\theta)d\theta = \left(\int_{a(\lambda)}^{b(\lambda)} \frac{1}{t} dt \right)^{-1} (b(\lambda) - a(\lambda)) = m$$

for all $\lambda \geq \lambda_0$, so that $\pi_\lambda \in \Gamma_m$. A short calculation yields

$$a'(\lambda) = m\frac{\lambda - 1 - \lambda \ln(\lambda)}{\lambda(\lambda - 1)^2} < 0,$$

and

$$b'(\lambda) = m\frac{\lambda - 1 - \ln(\lambda)}{(\lambda - 1)^2} > 0,$$

for $\lambda \geq \lambda_0$. Because of $\lambda - 1 - \ln(\lambda) < \lambda \ln(\lambda) - \lambda + 1$ for $\lambda \geq \lambda_0$ and $\lim_{\lambda \to \infty} b(\lambda) = \infty$, one obtains

$$\int_{\lambda_0}^{\infty} \min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda), -\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\}d\lambda = (3n^2) \int_{\lambda_0}^{\infty} \min\left\{ \frac{b'(\lambda)}{b(\lambda)}, -\frac{a'(\lambda)}{a(\lambda)} \right\} d\lambda$$

$$= (3n^2) \int_{\lambda_0}^{\infty} \frac{b'(\lambda)}{b(\lambda)} d\lambda = \infty$$

which implies, according to Corollary 3.7 that $\delta_\pi(X)$ is Γ_m-admissible under the loss (1.1).

3.9. Remark. Typically all the result in this paper go through with some modifications for the density

$$f_X(x, \theta) = \begin{cases} q(\theta)r(x), & \underline{\theta} < x < \bar{\theta} \\ 0, & \text{otherwise} \end{cases}$$

where $\theta \in \Theta = (\underline{\theta}, \bar{\theta})$ is unknown.

4. An application

In the presence of vague prior information frequently the Γ-minimax approach is used as underlying principle. In this section, we provide the definition of the Γ-minimaxity of an estimator and then express the relation between this concept and the Γ-admissibility. Finally, we give an example.

4.1. Definition. A Γ-minimax estimator is a minimax strategy of the second player in the statistical game (Γ, Δ, r); δ^* is called a Γ-minimax estimator, if

$$\sup_{\pi \in \Gamma} r(\pi, \delta^*) = \inf_{\delta \in \Delta} \sup_{\pi \in \Gamma} r(\pi, \delta),$$

where $r(\pi, \delta)$ is the Bayes risk of δ.

4.2. Definition. A Γ-minimax estimator δ^* is said to be unique, if

$$r(\pi, \delta) = r(\pi, \delta^*), \qquad \pi \in \Gamma,$$

for any other Γ-minimax estimator δ.

4.3. Remark.
- From Definition 4.2, it is obvious that a unique Γ-minimax estimator is Γ-admissible.
- If a Γ-admissible estimator δ is an equalizer on Γ, i.e., $r(., \delta)$ is constant on Γ, then δ is a unique Γ-minimax estimator.

4.4. Example. In Example 3.5, we have $E_\theta[X] = \theta - (1/n)$ and $E_\theta[e^{cX}] = (n/(n+c))e^{c\theta}$. Thus, the risk function of δ_π is equal to

$$R(\delta_\pi, \theta) = bE_\theta \left[e^{c(\delta_\pi - \theta)} - c(\delta_\pi - \theta) - 1 \right] = b \left\{ \frac{c}{n} - \ln\left(\frac{n+c}{n}\right) \right\}.$$

So, δ_π is an equalizer on Γ_0, since its risk function is constant. Hence, $\delta_\pi(X)$ is the unique Γ_0-minimax estimator for θ.

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