Single server queueing model with Gumbel distribution using Bayesian approach

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Abstract

Bayesian methodology is an important technique in statistics, and especially in mathematical statistics. It consists of the sample information along with the prior information available about the parameter before the sample has been observed. This paper exhibits the estimation of the parameters of queueing model with inter-arrival time and service time which follows Gumbel distribution. Bayesian procedure is applied to obtain the estimation of the model parameters and the traffic intensity of queueing model based on the informative and the non-informative prior knowledges. In this paper, the Bayesian estimates are carried out by numerically and graphically with the help of Markov Chain Monte Carlo (MCMC) simulation technique, particularly in Gibbs sampling algorithm.

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1. Introduction

Statistical inference in queueing theory has drawn the attention of researcher in the past few decades. The problem of estimation is concerned with the parameters of the queueing process such as arrival rate, service rate and traffic intensity. It is the most important thing in the queueing systems [6]. The pioneer investigators have derived the Maximum Likelihood Estimates (MLE) for the arrival and service parameters of an M/M/1 queueing model [9] and an infinite server queueing model [4]. The hypothesis testing for the point and the interval estimations of the M/M/1 queueing model using Bayesin approaches [7] and the non-zero waiting time of the model has been disscussed by using the convential and the Bayesian approaches along with the risk factors [8]. The five different approaches has been applied for the constructed 100(1-α) % of the Confidence Interval (CI) of the intensity of the queuing system [22]. Examining the MLE and Moment Estimate (ME) of the parameters of the inter-arrival and the service time distributions of GI/G/1 queueing model are discussed [3]. Consequently, the inferential procedures are concerned with the traffic intensity of $M/E_k/1$ queueing model which discussed [16]. The stationary solution of MLE of Markovian two server queueing model parameters have been obtained in the case of the non-identical servers [11]. Later, the stationary solution of the MLE of the generalized form of the multi-server queueing model in the presence of the non-identical servers and some of the CI of these model parameters are obtained [28]. Meanwhile, the MLE and the Bayesian estimates of the M/M/1/1 queueing model parameters are explained and the large sample test for the model parameters are also discussed [17]. The inferential process for the parameters of the bulk service queues is derived by using the Bayesian hierarchical model approaches [1]. Recently, the single server queueing model with working vacations has considered based on MLE approaches and simulation studies are carried out by the performance measures of the model [21].

The service times and the inter-arrival times of queueing model are not followed by the exponential distribution because of the high variability is observed in the inter-arrival time and the service time, most of the times are smaller than the minor proportion of the time and this leads to the characterisation of the heavy tails not only by the exponentially distributed [25]. In this regard, serveral authors have been devoted by queueing models based on the heavy tailed distribution [13], [15], [26]. Weibull, Parato, lognormal, Burr type III, Burr type XII and Gumbel distributions are some heavy tailed behaviour distributions [19]. The Bayesian estimation for the double Pareto lognormal (dPlN) distribution which has been proposed by the model in the queueing system for the heavy-tailed phenomena [10]. The evaluation of M/G/1 queueing model with the service time as assumed to Gumbel distribution, which has been explained by numerically and graphically based on the various combinations of the arbitrary values [20]. The extended queueing model when service time distributed according to Gumbel distribution under multiple working vacations scenario and the model parameters has been estimated based on Bayesian approaches with Gibbs sampling algorithm through Markov Chain Monte Carlo (MCMC) technique [18].

This article introduces tele-traffic and insurance data and some of the unusual characteristics of these types of data which motivate some of the inter-arrival and service time model that are analyzed through heavy tailed nature, particularly in Gumbel distribution. In insurance context, the claim sizes can take on extremely large values so they can be well modeled by heavy-tailed distribution. However, one difference between the insurance data and the internet traffic data is that in the insurance context, high

autocorrelations are not observed to such an extent as with the tele-traffic data and that the insurance claims processes do not exhibit burstiness so much as the tele-traffic data, which suggests that heavy-tailed, but independent distributions may be reasonable for modeling insurance claims data in many contexts [12].

This paper proposes the new queueing model when exponential times of the inter-arrival time and service time are disappeared due to unusual characteristics. Therefore, the inter-arrival times of two successive arrival of customers and service times becomes a heavy tailed. For this reason, here the inter-arrival times and service times of the system follows Gumbel distribution. No attempts are found in the literature on evaluating the queueing models under Gumbel distribution based on Bayesian approaches. Determination of Gumbel/Gumbel/1 queueing model using Bayesian approach is discussed. The posterior distribution of the queueing model is derived incorporating the natural conjugate prior and non-informative prior to the parameters of the Gumbel distribution. The objective of this paper is to analyse the traffic congestion of the Gumbel/Gumbel/1 queueing model satisfying the stability condition of the system.

The probability generating function and cumulative distribution function of the Gumbel distribution are based on the location parameter, α and the scale parameter, β, respectively,

$$(1.1) \qquad f(x:\alpha,\beta) = \frac{1}{\beta} e^{-\frac{(x-\alpha)}{\beta}} e^{-e^{-\frac{(x-\alpha)}{\beta}}} \quad for \quad x \in \Re, \alpha \in \Re; \beta > 0 \quad and$$

$$(1.2) \qquad F(x) = e^{-e^{-\frac{(x-\alpha)}{\beta}}}$$

with the mean $\alpha + \beta\gamma$ where $\gamma = 0.5722...$ is the Euler's constant.

This paper is organized into the five sections, this is being the first. Section 2 contains model descriptions. The frame work of Bayesian estimation of model parameters is presented in section 3. The computational studies for the empirical Bayesian estimates by using Gibbs sampling algorithm in MCMC technique of the model are discussed in section 4 and section 5 provides the summary and conclusion of this work.

2. Model descriptions

Consider an Gumbel/Gumbel/1 queueing model,

- The inter-arrival time of two consecutive arrival of the customers which follows Gumbel distribution (α, β) with mean inter-arrival time, $1/\lambda = 1/[\alpha + \beta\zeta]$ where Euler's constant, $\zeta = 0.5277...$.
- The service time of the system is distributed according to Gumbel distribution (γ, δ) with mean service time, $1/\mu = 1/[\gamma + \delta\zeta]$ where Euler's constant, $\zeta = 0.5277...$.
- The inter-arrival times and the service times are mutually independent of each other.
- The server gives the services the single stage service with First In First Out (FIFO) discipline.
- In order to learn about the congestion of the system, the inference about the parameters governing the whole system $\theta = \{\alpha, \beta, \gamma, \delta\}$ is considered.

- The queueing system is consolidated and operated for the long time which indicates that it is working at equilibrium and satisfies the ergodic condition.
- Note that, the ergodic assumption implies that the parameters can only move freely in the reduced parametric space $\Theta_e = \{\theta : \lambda < \mu, \lambda, \mu > 0\}$. Hence, the traffic intensity of the model is $\rho = \frac{\gamma + \delta\zeta}{\alpha + \beta\zeta} < 1$.

3. Estimation on Gumbel/Gumbel/1 model

The Bayesian methodology consists of the sample information along with the prior information available about the parameter before the sample has been observed. The Bayesian approach treats that the model parameters are the random variables. The suitable probability distribution is determined for the models parameters for the queueing system say $\tau(\theta)$ with reference to the prior information. The information about the parameter given by the sample $\underline{x}$ is obtained from the likelihood function, $L(\theta|\underline{x})$. A prior probability distribution that represents perfect ignorance or indifference would produce the posterior probability distribution that represents that one should need about the parameter on the basis of the evidence alone. The prior distributions can be classified into two main categories like the informative prior and non-informative prior (vague, objective, and diffuse). The informative prior expresses the previous knowledge about parameter and the non-informative prior provides the formal way of expressing ignorance of the value of the parameter over the permitted range. The efforts to construct the priors may be represented by the absence of the knowledge. They have failed because no probability distribution to represent the pure ignorance. Combining these two information, the updated information about the parameter is obtained as the posterior distribution, $\tau(\theta|\underline{x})$. The inference about the parameter, θ is drawn from this posterior distribution. More details about the Bayesian methods can be found in [2], [27].

In the Gumbel/Gumbel/1 queueing model, the n_a inter-arrival times $x_a = (x_{1_a}, x_{2_a}, ..., x_{n_a})$ are a random samples distributed according to Gumbel (α, β) and the n_s recorded service time $x_s = (x_{1_s}, x_{2_s}, ..., x_{n_s})$ constitute a random sample from Gumbel (γ, δ). The joint probability generating function of this model is

$$(3.1) \qquad f(x|\theta) = \frac{1}{\beta} e^{\frac{(x_a - \alpha)}{\beta}} e^{e^{\frac{(x_a - \alpha)}{\beta}}} \frac{1}{\delta} e^{\frac{(x_s - \gamma)}{\delta}} e^{e^{\frac{(x_s - \gamma)}{\delta}}} \qquad \forall \quad x_a, \ x_s > 0$$

where $\theta = \{\lambda < \mu; \ \alpha, \beta, \gamma \ and \ \delta > 0\}$.

From Eqn. 3.1, the corresponding likelihood equation are as follows,

$$(3.2) \qquad L(\theta|x) = \Pi_{i=1}^{n} \frac{1}{\beta^{n_a}} e^{\frac{(x_{i_a} - \alpha)}{\beta}} e^{e^{\frac{(x_{i_a} - \alpha)}{\beta}}} \Pi_{j=1}^{n} \frac{1}{\delta^{n_s}} e^{\frac{(x_{j_s} - \gamma)}{\delta}} e^{e^{\frac{(x_{j_s} - \gamma)}{\delta}}}$$

where, $x_a = \sum_{i=1}^{n} x_{i_a}$ is the total time until the arrival of n_a customer considered in the queue and $x_s = \sum_{j=1}^{n} x_{j_s}$ is the total time taken by the server to compelete the service under consideration. Note that, the restriction in the domain of the likelihood in Eqn. 3.2 corresponding to the ergodic condition of the queueing model.

Suppose that, the inverted Gamma distribution is employed as a probability model for the inter-arrival and service parameters based on the information obtained from the history of previous process of the queues respectively. The inverted Gamma distribution is a natural conjugate prior for sampling from the gumbel distribution for the inter-arrival

and service parameters. The probability density function of inverted Gamma distribution is given in Eqn. 3.3.

$$(3.3) \qquad \tau(c,d) = \begin{cases} \frac{d}{\Gamma(c)} x^{-(c+1)} e^{-c/x} & \text{c,d>0; x>0} \\ 0 & \text{otherwise} \end{cases}$$

In certain situations, especially in the investigation of new problems of a pioneering nature, useful prior information may not be available. In such situations, the statistician will be forced to select a prior distribution which will reflect a situation of no prior information. This led to the notion of vague or diffused or non-informative prior distributions. The parameters is continuous and can take any value in a finite interval, then one can use a continuous uniform distribution as the prior distribution for the parameter. Such prior distributions are called non-informative priors and sometimes as vague priors (see more [5], [2], [27]). Furthermore, it may be considered that the uniform distribution is a non-informative prior knowledge about the model parameters α, β, γ and δ. The probability density function of uniform distribution is

$$(3.4) \qquad \tau_3(\phi) = \frac{1}{q-p}; 0 < p \le \phi \le q$$

The updated inforamtions of posterior distribution is obtained for the model parameters is given by

$$
\begin{aligned}
\tau_I\left(\alpha,\beta,\gamma,\delta|data\right) \quad &\propto \quad \frac{d_1 d_2 d_3 d_4}{\beta^{n_a} \delta^{n_s} \Gamma\left(c_1\right) \Gamma\left(c_2\right) \Gamma\left(c_3\right) \Gamma\left(c_4\right)} \alpha^{-(c_1+1)} \beta^{-(c_2+1)} \times \\
&\gamma^{-(c_3+1)} \delta^{-(c_4+1)} e^{-(c_1/\alpha + c_2/\beta + c_3/\gamma + c_4/\delta)} \times
\end{aligned}
$$

$$(3.5) \qquad \Pi_{i=1}^n e^{\frac{(x_{i_a}-\alpha)}{\beta}} e^{e^{\frac{(x_{i_a}-\alpha)}{\beta}}} \Pi_{j=1}^n e^{\frac{(x_{j_s}-\gamma)}{\delta}} e^{e^{\frac{(x_{j_s}-\gamma)}{\delta}}}$$

$$(3.6) \qquad \tau_{NI}\left(\alpha,\beta,\gamma,\delta|data\right) \propto \frac{1}{\beta^{n_a} \delta^{n_s}} \Pi_{i=1}^n e^{\frac{(t_{i_a}-\alpha)}{\beta}} e^{e^{\frac{(x_{i_a}-\alpha)}{\beta}}} \Pi_{j=1}^n e^{\frac{(x_{j_s}-\gamma)}{\delta}} e^{e^{\frac{(x_{j_s}-\gamma)}{\delta}}}$$

Since, the posterior distributions of the informative and the non-informative prior knowledges are not attained in the closed form expression. Hence, MCMC simulation technique is more appropriate to deal with the empirical estimates of the model parameters. The empirical Bayesian estimates are computed particularly through Gibbs sampling algorithm [24] using OpenBugs software.

4. Gibbs sampling algorithm in MCMC technique

The Markov chains have significant role in Bayesian statistics because it is generally possible to construct the Markov chain in such a way that the target distribution is the joint posterior distribution of all the unknown parameters in the Bayesian model. Thus, the Markov chain Monte Carlo methods provide a way of generating samples from the joint posterior distribution in the realistic and high-dimensional Bayesian models. The Gibbs sampling algorithm is a special case of Metropolis-Hastings sampling algorithm which one particular way of constructing the transition kernel to produce the Markov chain with the desired target distribution. The Gibbs sequence converges to the stationary(equilibrium) distribution that is independent of the initial values, and by the attaining this stationary distribution is the target distribution. The step-by-step procedure in Gibbs sampling alogrithm for the proposed queueing model as follows:

(1) Set initial values $\alpha^{(0)}$, $\beta^{(0)}$, $\gamma^{(0)}$, $\delta^{(0)}$

(2) For t=1,...,T

    (a) For i=1,2,...,n

        (i) Generate $x_i^{(t)}$ from $f\left(x|\alpha^{(t-1)}, \beta^{(t-1)}, \gamma^{(t-1)}, \delta^{(t-1)}\right)$

    (b) Generate $\alpha^{(t)} \sim \tau\left(\alpha|x^{(t)}\right)$

    (c) Generate $\beta^{(t)} \sim \tau\left(\beta|x^{(t)}\right)$

    (d) Generate $\gamma^{(t)} \sim \tau\left(\gamma|x^{(t)}\right)$

    (e) Generate $\delta^{(t)} \sim \tau\left(\delta|x^{(t)}\right)$

The MCMC samples are generated through Gibbs sampling algorithm from the posterior distribution of model parameters for the given set of the informative priors Eqn. 3.3 and non-informative prior Eqn. 3.4 for obtaining the Bayes estimates of the model. The markov chain is run in OpenBugs for 10,000 number of iterations for various arbitrary values and samples.

4.1. Convergence diagnostics of MCMC. From the outputs of OpenBugs, the diagnostic checking plots for each model parameters are presented in Appendix. The Markov chain has converged in both informative and non-informative priors since it likely to be sampling from the stationary distribution and horizontal band, with no long upward or downward trends as shown in Figure [15, 16, 17, 18]. Moreover, the autocorrelation is almost negligible for all the model parameters (see Figure[19, 20, 21, 22]). Therefore, the generated samples, in each iteration from posterior densities under informative and non-informative priors are independent to each other. Further, the kernal densities of model parameters α, β, γ, δ for various samples 50, 100, 150, 200, 250 and $\alpha = 0.2$, 0.3, 0.4, 0.5, $\beta = 0.3$, 0.4, 0.5, 0.6, $\gamma = 0.1$, 0.2, 0.3, 0.4, & $\delta = 0.2$, 0.3, 0.4, 0.5 under informative and non-informative priors are displayed (see Figure[23, 24, 25, 26]) for checking the convergence of the algorithm. Also, the Monte Carlo Error (MC.E) of Gumbel/Gumbel/1 queueing model is presented Table.1 & Table.3. It is to be observed that, MC error is minimum for each estimates in model parameters.

4.2. Numerical results of Bayesian estimation. The posterior mean and 95 % credible region of Gumbel/Gumbel/1 queueing model parameters are presented in Table.1 - Table.4. Meanwhile, the empirical Bayesian estimates of traffic intensity of the model are computed from the posterior mean of corresponding parameter and it is to be observed that in Figure [1, 2] , the stable intensity level has been maintained when sample observations and values of model parameters increase in both prior informations. The congenstion level of the each model belongs to the interval of 0.5 - 0.95.

5. Summary and conclusion

In this paper, the Bayesian estimates of an Gumbel/Gumbel/1 queueing model under the informative and non- informative prior knowledges is considered. The empirical posterior mean, 95 % credible region and the diagonostic checking plots are carried out for various size of the sample observations and different sets of arbitrary values based on Gibbs sampler through the MCMC simulation technique using the OpenBugs software. From those results, the traffic intensity of the model has been increased when the model parameters of inter-arrival time and service time are increased but not in the increasing size sample observations. Meanwhile, the stable intensity level has been maintained when the sample observations and the values of model parameters are increased in both

Table 1. Empirical Bayesian estimates of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples based on informative priors

Arbitrary	Samples	$\hat{\alpha}$	MC.E	$\hat{\beta}$	MC.E	$\hat{\gamma}$	MC.E	$\hat{\delta}$	MC.E
$\alpha = 0.2$,	50	0.257	0.00040	0.233	0.00038	0.195	0.00026	0.212	0.00029
$\beta = 0.3$,	100	0.331	0.00030	0.268	0.00027	0.178	0.00017	0.154	0.00015
$\gamma = 0.1$,	150	0.331	0.00024	0.258	0.00018	0.218	0.00016	0.189	0.00014
$\delta = 0.2$	200	0.294	0.00019	0.250	0.00017	0.193	0.00014	0.156	0.00011
	250	0.299	0.00020	0.264	0.00014	0.183	0.00011	0.154	0.00093
$\alpha = 0.3$,	50	0.359	0.00050	0.290	0.00041	0.297	0.00045	0.338	0.00050
$\beta = 0.4$,	100	0.331	0.00029	0.268	0.00025	0.279	0.00025	0.238	0.00021
$\gamma = 0.2$,	150	0.402	0.00035	0.333	0.00025	0.318	0.00022	0.259	0.00019
$\delta = 0.3$	200	0.407	0.00024	0.345	0.00022	0.314	0.00019	0.247	0.00017
	250	0.428	0.00020	0.311	0.00018	0.298	0.00017	0.253	0.00013
$\alpha = 0.4$,	50	0.574	0.00057	0.411	0.00050	0.435	0.00054	0.587	0.00062
$\beta = 0.5$,	100	0.464	0.00038	0.379	0.00032	0.347	0.00036	0.342	0.00027
$\gamma = 0.3$,	150	0.508	0.00043	0.473	0.00033	0.386	0.00026	0.316	0.00021
$\delta = 0.4$	200	0.515	0.00038	0.446	0.00028	0.428	0.00026	0.353	0.00025
	250	0.484	0.00030	0.419	0.00025	0.428	0.00023	0.331	0.00019
$\alpha = 0.5$,	50	0.687	0.00073	0.585	0.00061	0.468	0.00069	0.440	0.00055
$\beta = 0.6$,	100	0.464	0.00035	0.379	0.00030	0.460	0.00050	0.394	0.00039
$\gamma = 0.4$,	150	0.582	0.00045	0.487	0.00038	0.560	0.00041	0.473	0.00032
$\delta = 0.5$	200	0.670	0.00039	0.551	0.00037	0.571	0.00037	0.473	0.00028
	250	0.608	0.00035	0.513	0.00023	0.541	0.00031	0.415	0.00023

Table 2. 95% Credible region of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples based on informative priors

Arbitrary Values	Samples	α		β		γ		δ	
		LB	UB	LB	UB	LB	UB	LB	UB
$\alpha = 0.2$,	50	0.1908	0.3266	0.1823	0.3002	0.1578	0.2697	0.1541	0.2491
$\beta = 0.3$,	100	0.2778	0.3880	0.2268	0.3164	0.1471	0.2099	0.1310	0.1811
$\gamma = 0.1$,	150	0.2887	0.3762	0.2270	0.2956	0.1872	0.2500	0.1654	0.2158
$\delta = 0.2$	200	0.2584	0.3322	0.2235	0.2802	0.1709	0.2156	0.1397	0.1749
	250	0.2653	0.3337	0.2385	0.2934	0.1632	0.2035	0.1388	0.1707
$\alpha = 0.3$,	50	0.2777	0.4459	0.2297	0.3685	0.2373	0.3763	0.2539	0.4272
$\beta = 0.4$,	100	0.2774	0.3865	0.2267	0.3166	0.2313	0.3280	0.204	0.2797
$\gamma = 0.2$,	150	0.3458	0.459	0.2932	0.381	0.2765	0.3619	0.2279	0.2958
$\delta = 0.3$	200	0.3589	0.4580	0.3071	0.3873	0.2792	0.3507	0.2217	0.2769
	250	0.3888	0.4688	0.4688	0.3444	0.2658	0.3309	0.2286	0.2800
$\alpha = 0.4$,	50	0.4551	0.6915	0.3293	0.5134	0.3503	0.5456	0.4655	0.7124
$\beta = 0.5$,	100	0.3879	0.5410	0.3242	0.4439	0.2767	0.4197	0.2900	0.4040
$\gamma = 0.3$,	150	0.4322	0.5860	0.4152	0.5422	0.3352	0.4396	0.2782	0.3627
$\delta = 0.4$	200	0.4516	0.5802	0.3999	0.5000	0.3766	0.4809	0.3140	0.3958
	250	0.4297	0.5382	0.3790	0.4637	0.3865	0.4716	0.2995	0.3665
$\alpha = 0.5$,	50	0.5258	0.8328	0.4738	0.7197	0.3503	0.5582	0.3422	0.5937
$\beta = 0.6$,	100	0.3877	0.5430	0.3238	0.4437	0.3823	0.5415	0.3358	0.4632
$\gamma = 0.4$,	150	0.5016	0.6628	0.4265	0.5559	0.4819	0.6391	0.4163	0.5381
$\delta = 0.5$	200	0.5925	0.7496	0.4936	0.6155	0.5031	0.6422	0.4224	0.5303
	250	0.5417	0.6749	0.4647	0.5683	0.4880	0.5964	0.3756	0.4589

prior informations. Hence, this model is appropriate for studying the queueing system when the inter-arrival time and service time follows heavy tailed particularly in Gumbel distribution.

References

[1] Armero, C. and Conesa, D. *Bayesian hierarchical models in manufacturing bulk service queues*, Journal of Statistical Planning and Inference, 136, 335-354, 2006.

Table 3. Empirical Bayesian estimates of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples based on non-informative priors

Arbitrary	Samples	$\hat{\alpha}$	M.C.E	$\hat{\beta}$	M.C.E	$\hat{\gamma}$	M.C.E	$\hat{\delta}$	M.C.E
$\alpha = 0.2$,	50	0.251	0.00035	0.227	0.00030	0.210	0.00029	0.193	0.00027
$\beta = 0.3$,	100	0.329	0.00028	0.266	0.00024	0.177	0.00018	0.153	0.00014
$\gamma = 0.1$,	150	0.351	0.00030	0.261	0.00023	0.215	0.00018	0.186	0.00018
$\delta = 0.2$	200	0.293	0.00019	0.248	0.00016	0.192	0.00013	0.155	0.00012
	250	0.298	0.00018	0.264	0.00013	0.182	0.00010	0.153	0.00093
$\alpha = 0.3$,	50	0.357	0.00044	0.287	0.00041	0.339	0.00053	0.297	0.00043
$\beta = 0.4$,	100	0.410	0.00036	0.333	0.00032	0.278	0.00029	0.238	0.00022
$\gamma = 0.2$,	150	0.419	0.00044	0.339	0.00035	0.313	0.00028	0.262	0.00027
$\delta = 0.3$	200	0.407	0.00030	0.345	0.00022	0.313	0.00020	0.247	0.00016
	250	0.428	0.00023	0.310	0.00015	0.297	0.00021	0.252	0.00018
$\alpha = 0.4$,	50	0.590	0.00078	0.415	0.00065	0.602	0.00006	0.441	0.00057
$\beta = 0.5$,	100	0.466	0.00045	0.379	0.00037	0.347	0.00039	0.342	0.00029
$\gamma = 0.3$,	150	0.512	0.00043	0.476	0.00037	0.387	0.00027	0.317	0.00024
$\delta = 0.4$	200	0.518	0.00045	0.448	0.00030	0.428	0.00026	0.353	0.00022
	250	0.485	0.00031	0.419	0.00025	0.430	0.00022	0.332	0.00020
$\alpha = 0.5$,	50	0.750	0.00102	0.616	0.00091	0.475	0.00007	0.446	0.00069
$\beta = 0.6$,	100	0.770	0.00065	0.589	0.00057	0.462	0.00049	0.396	0.00036
$\gamma = 0.4$,	150	0.588	0.00043	0.490	0.00036	0.566	0.00051	0.475	0.00036
$\delta = 0.5$	200	0.682	0.00047	0.557	0.00038	0.575	0.00035	0.476	0.00030
	250	0.613	0.00037	0.516	0.00029	0.543	0.00030	0.415	0.00023

Table 4. 95% Credible region of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples based on non-informative priors
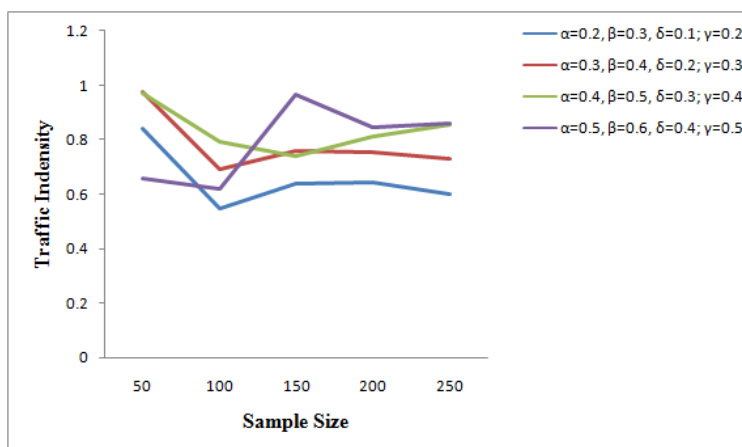
Arbitrary Values	Samples	α		β		γ		δ	
		LB	UB	LB	UB	LB	UB	LB	UB
$\alpha = 0.2$,	50	0.1846	0.3178	0.1781	0.2925	0.1546	0.2681	0.1525	0.2452
$\beta = 0.3$,	100	0.2759	0.3854	0.2268	0.3127	0.1460	0.2090	0.1307	0.1800
$\gamma = 0.1$,	150	0.2971	0.4060	0.2230	0.3063	0.1784	0.2537	0.1587	0.2212
$\delta = 0.2$	200	0.2573	0.3295	0.2216	0.2795	0.1701	0.2152	0.1391	0.1748
	250	0.2634	0.3325	0.2384	0.2925	0.1627	0.2027	0.1386	0.1702
$\alpha = 0.3$,	50	0.2725	0.4415	0.2282	0.3622	0.2516	0.4292	0.2366	0.3766
$\beta = 0.4$,	100	0.3424	0.4788	0.2822	0.3941	0.2296	0.3274	0.2033	0.2788
$\gamma = 0.2$,	150	0.3495	0.4894	0.2890	0.3989	0.2607	0.3675	0.2235	0.3093
$\delta = 0.3$	200	0.3576	0.4583	0.30751	0.3883	0.2781	0.3506	0.2217	0.2764
	250	0.3877	0.4692	0.2809	0.3430	0.2643	0.3307	0.2280	0.2795
$\alpha = 0.4$,	50	0.4634	0.7169	0.3279	0.5256	0.4744	0.7313	0.3517	0.554
$\beta = 0.5$,	100	0.3878	0.5430	0.3235	0.4451	0.2778	0.4198	0.2904	0.4056
$\gamma = 0.3$,	150	0.4332	0.5950	0.4164	0.5470	0.3343	0.4407	0.2775	0.3626
$\delta = 0.4$	200	0.4551	0.5855	0.3999	0.5042	0.3767	0.4815	0.3137	0.3966
	250	0.4314	0.5408	0.3793	0.4638	0.3864	0.4731	0.2999	0.3686
$\alpha = 0.5$,	50	0.5712	0.9297	0.4893	0.7790	0.3451	0.6087	0.3527	0.5693
$\beta = 0.6$,	100	0.6481	0.8922	0.5001	0.6953	0.3816	0.548	0.3376	0.4654
$\gamma = 0.4$,	150	0.5064	0.6710	0.4306	0.5620	0.4862	0.6497	0.4183	0.5428
$\delta = 0.5$	200	0.6010	0.7654	0.4979	0.6260	0.5065	0.6471	0.4257	0.5340
	250	0.5474	0.6807	0.4661	0.5713	0.4902	0.598	0.3761	0.4600

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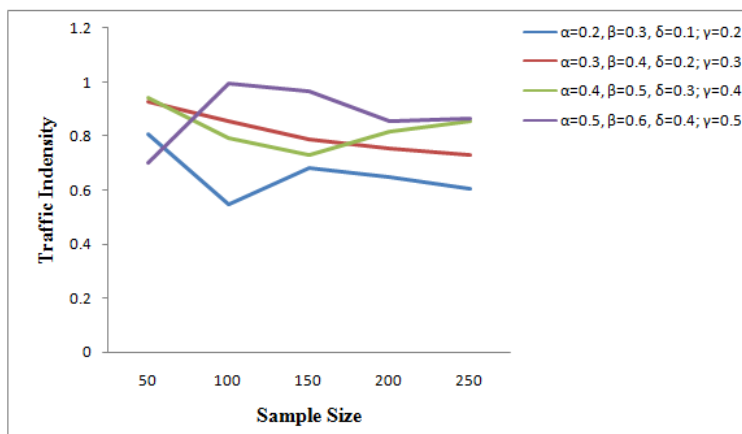
Figure 1. Traffic intensity of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples under informative prior


Figure 2. Traffic intensity of Gumbel/Gumbel/1 queueing model for various arbitrary values and samples under non-informative prior

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Appendices

Diagnostics checking plots

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
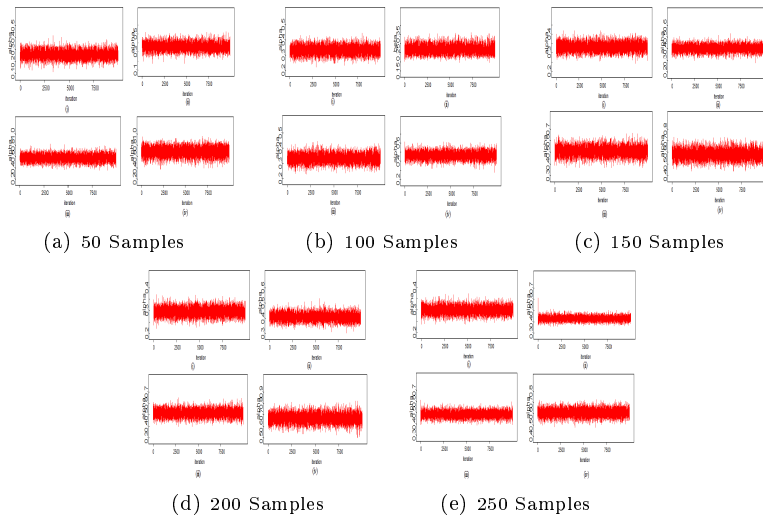
Figure 3. History plots of model parameter α under informative prior for various samples and arbitrary values

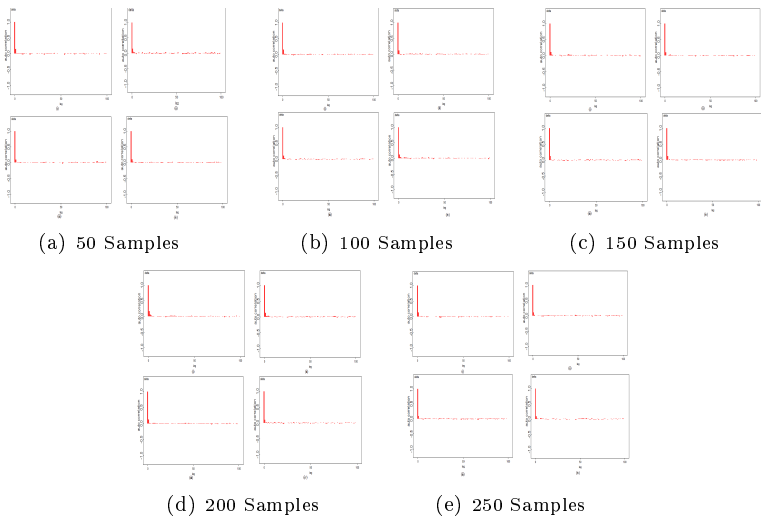(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
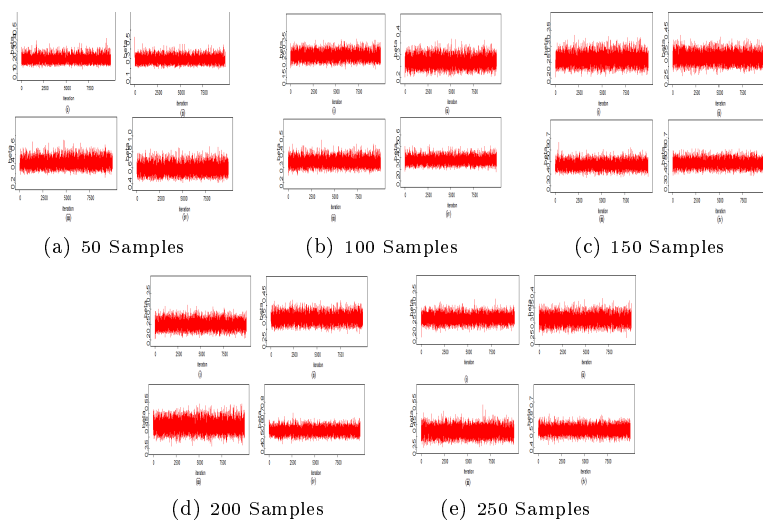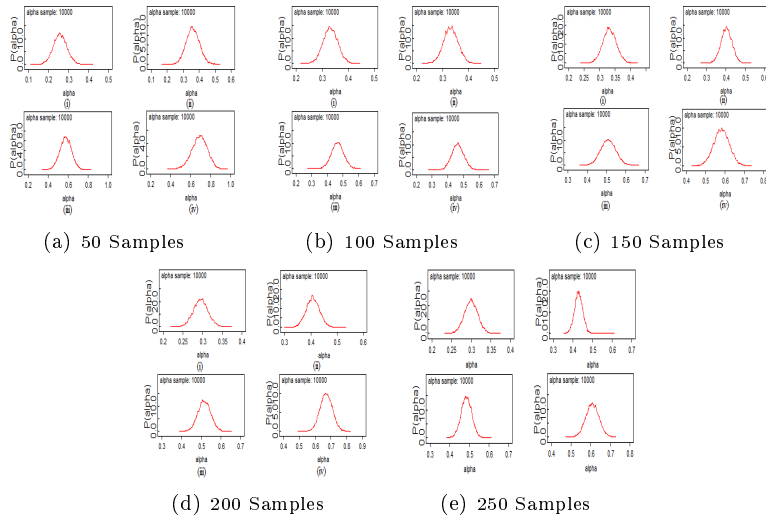
Figure 4. History plots of model parameter β under informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
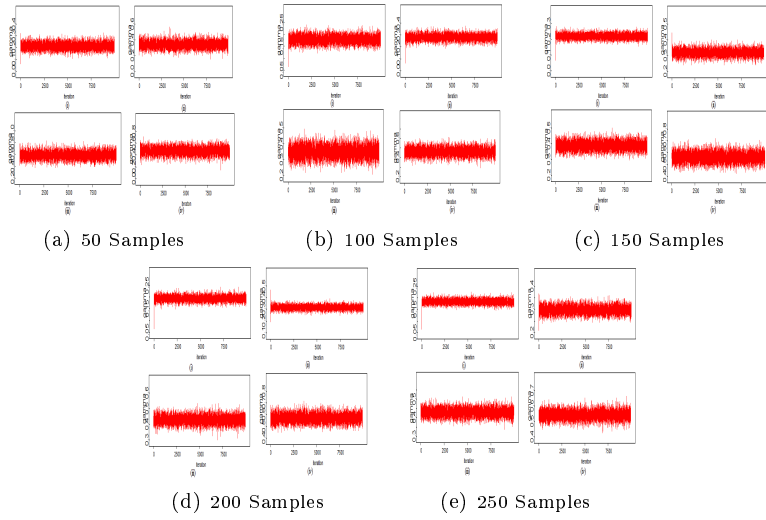
Figure 5. History plots of model parameter γ under informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
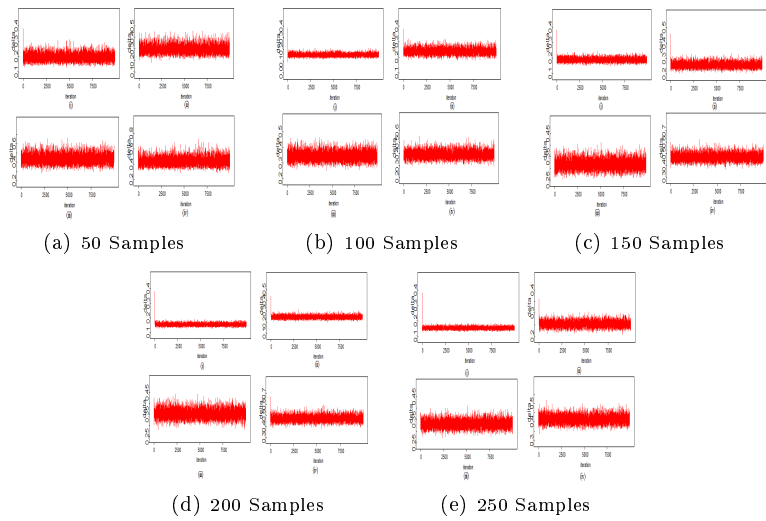
Figure 6. History plots of model parameter δ under informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples
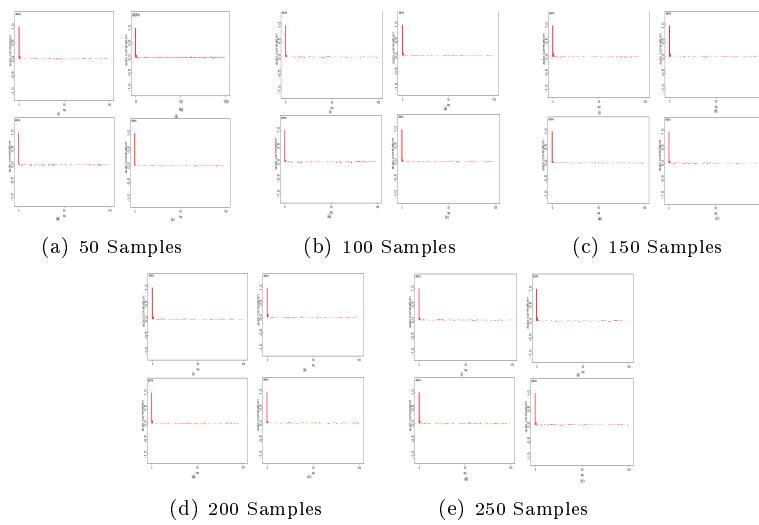
Figure 7. Auto correlation plots of model parameter α under informative prior for various samples and arbitrary values
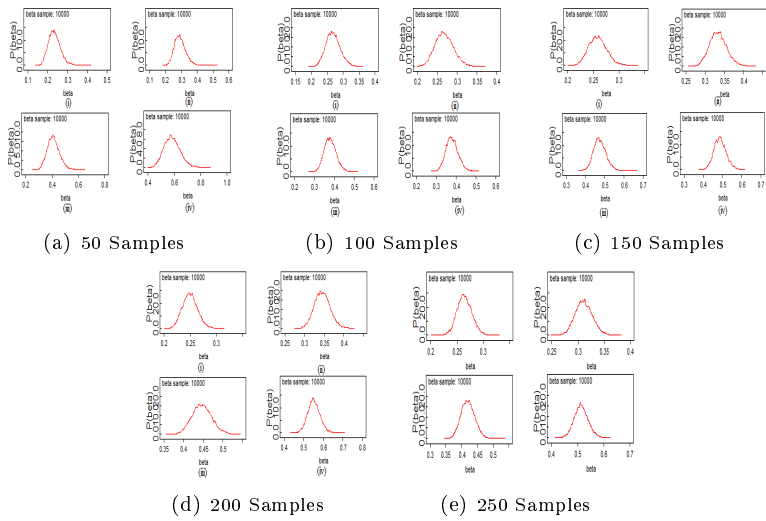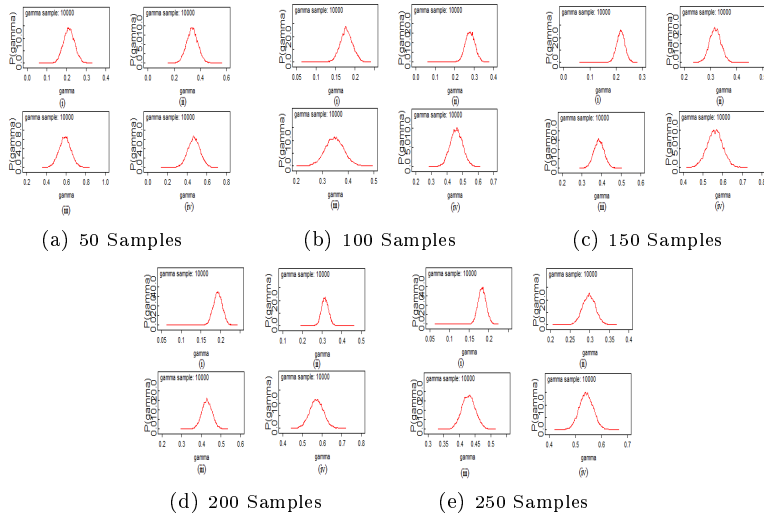

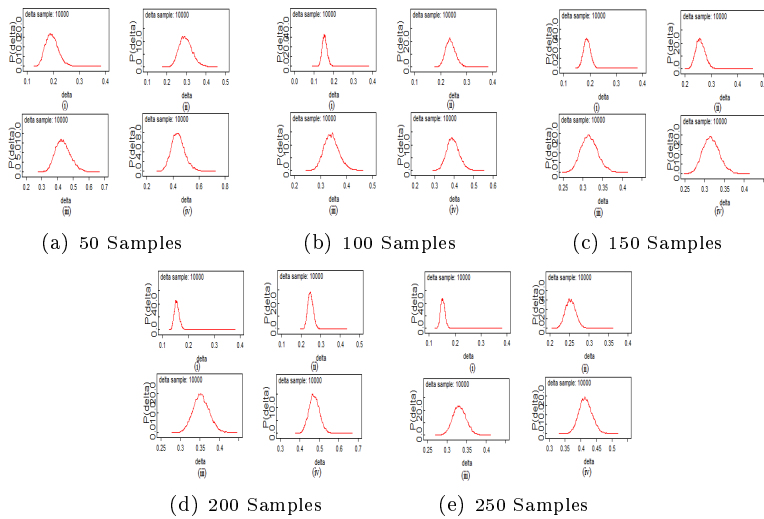(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples
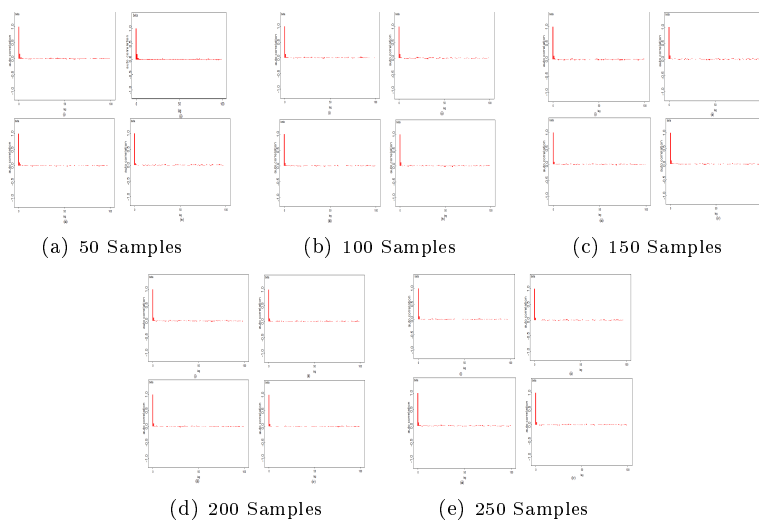
Figure 8. Auto correlation plots of model parameter β under informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples
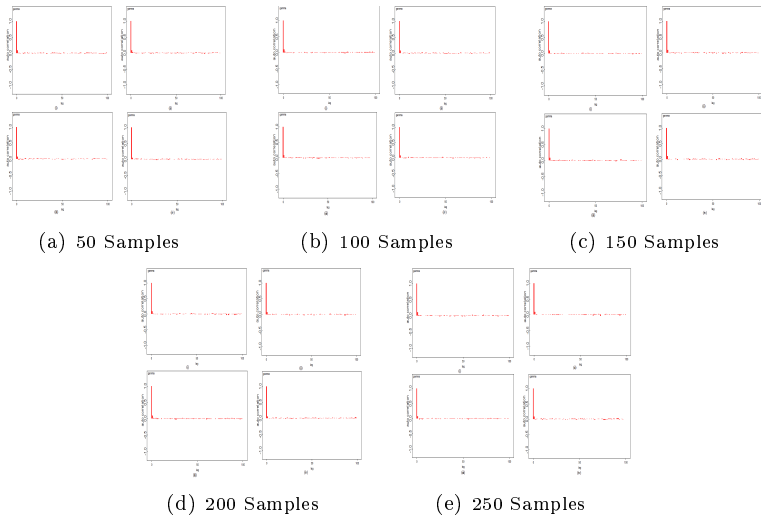
Figure 9. Auto correlation plots of model parameter γ under informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 10. Auto correlation plots of model parameter δ under informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 11. Kernal densities of model parameter α under informative prior for various samples and arbitrary values


(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 12. Kernal densities of model parameter β under informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples

Figure 13. Kernal densities of model parameter γ under informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples

Figure 14. Kernal densities of model parameter δ under informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
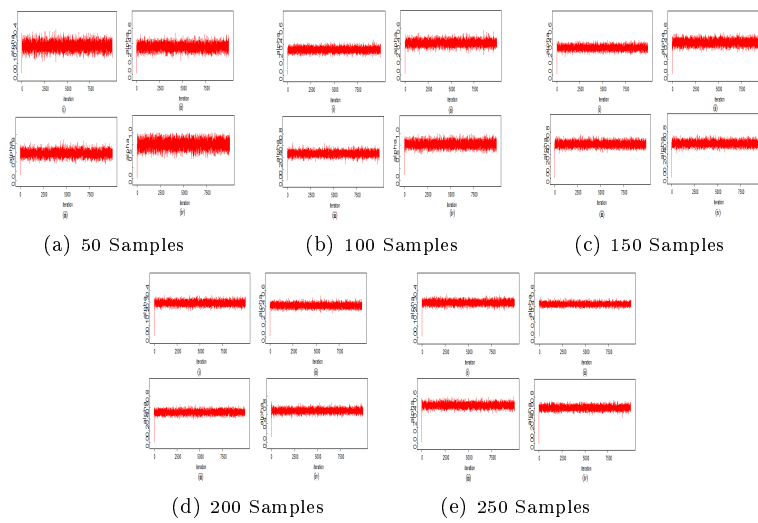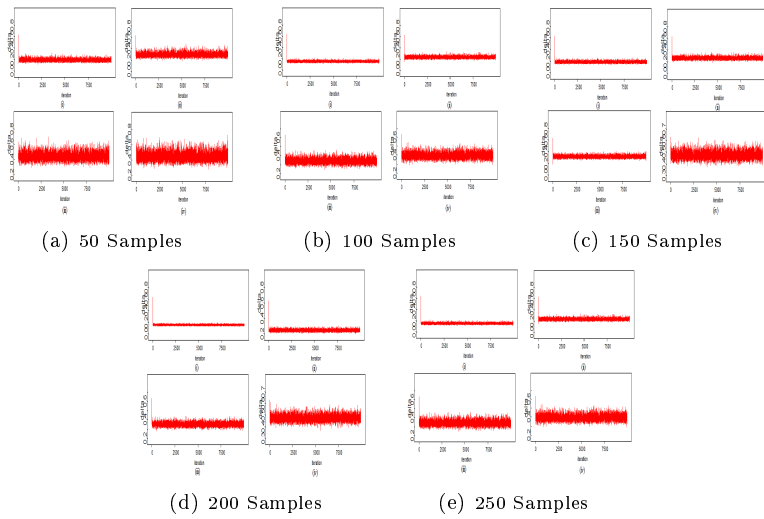
Figure 15. History plots of model parameter α under non-informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
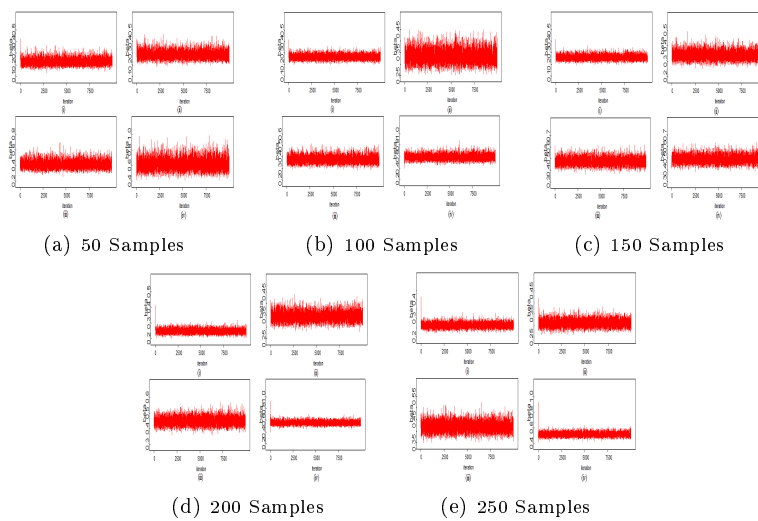
Figure 16. History plots of model parameter β under non-informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
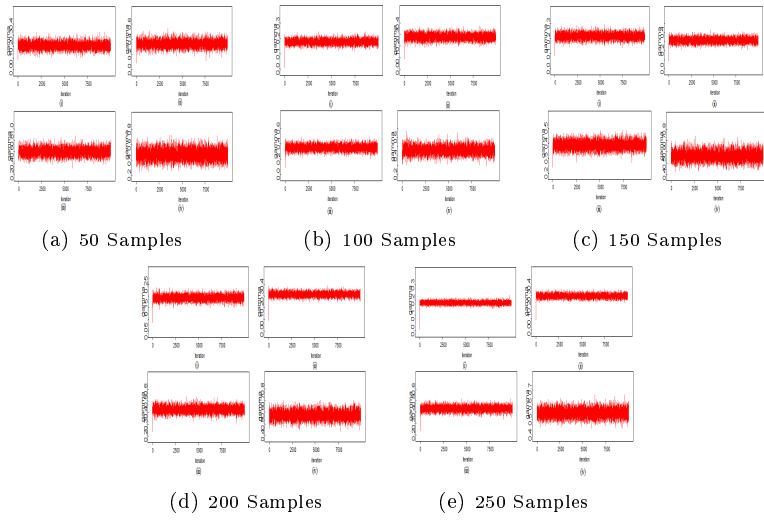
Figure 17. History plots of model parameter γ under non-informative prior for various samples and arbitrary values

(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples

Figure 18. History plots of model parameter δ under non-informative prior for various samples and arbitrary values

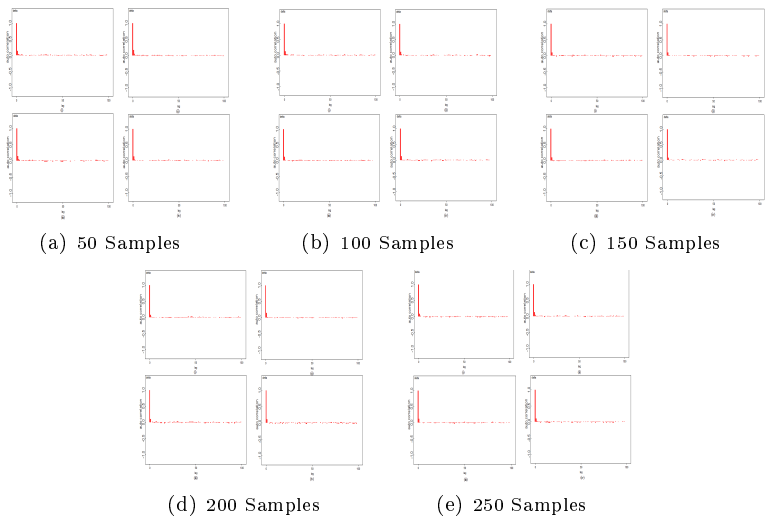(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples
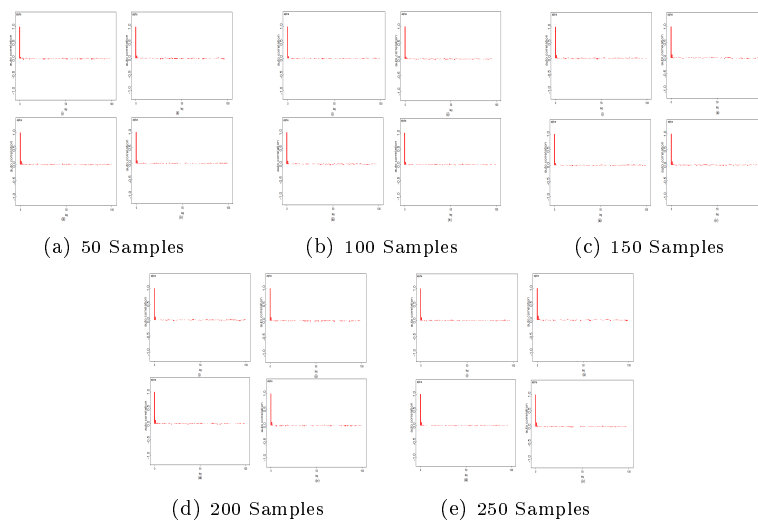
Figure 19. Auto correlation plots of model parameter α under non-informative prior for various samples and arbitrary values

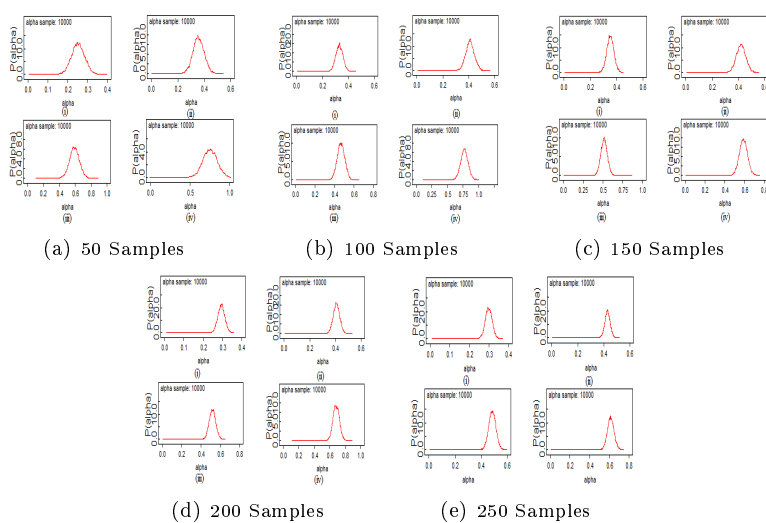(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples
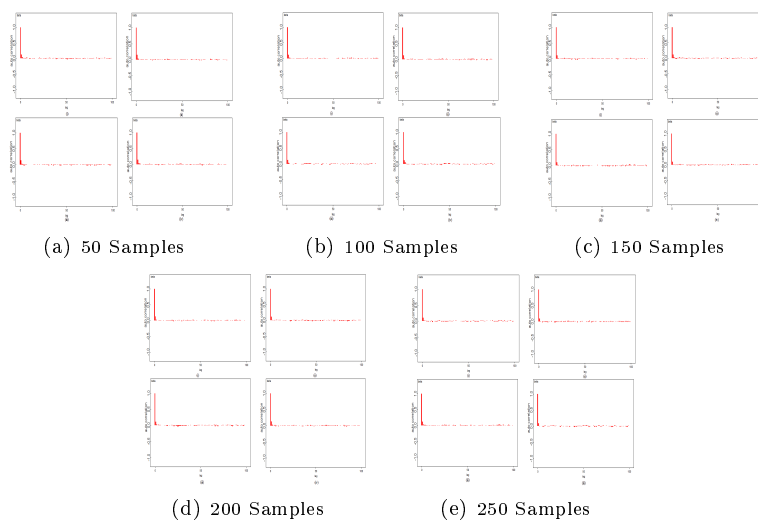
Figure 20. Auto correlation plots of model parameter β under non-informative prior for various samples and arbitrary values

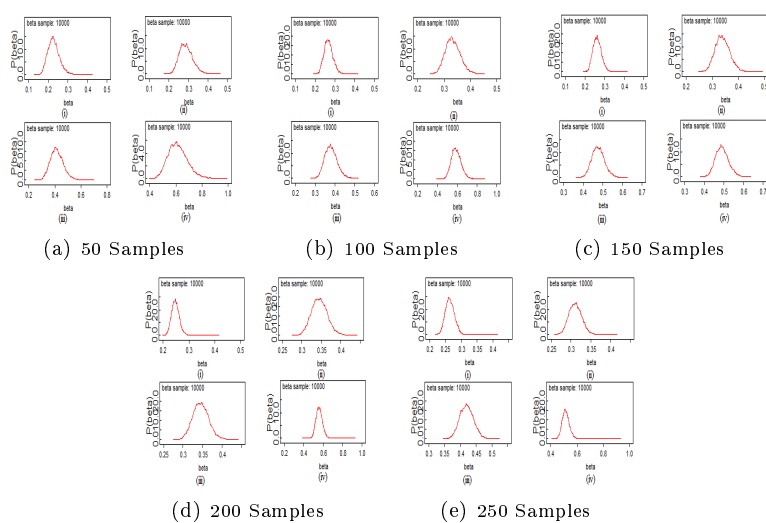(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples
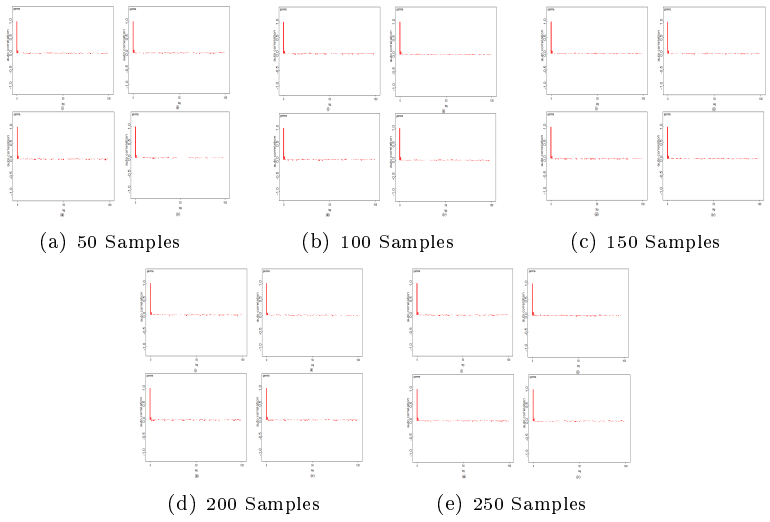
Figure 21. Auto correlation plots of model parameter γ under non-informative prior for various samples and arbitrary values
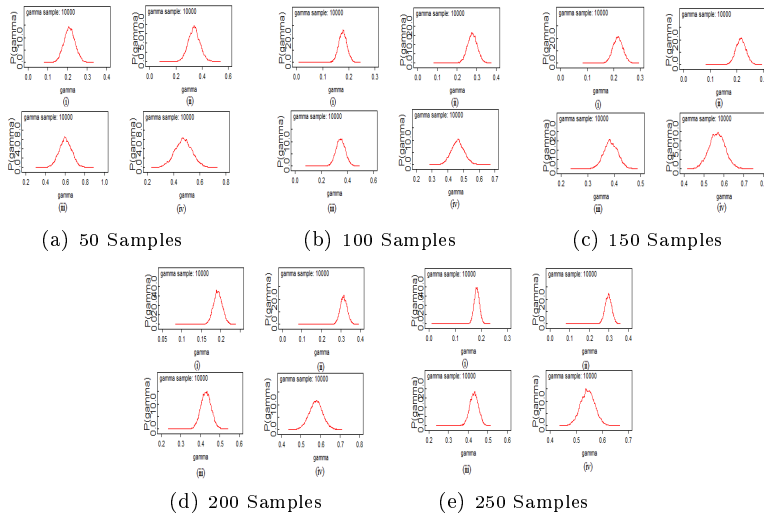
(a) 50 Samples   (b) 100 Samples   (c) 150 Samples

(d) 200 Samples   (e) 250 Samples

Figure 22. Auto correlation plots of model parameter δ under non-informative prior for various samples and arbitrary values
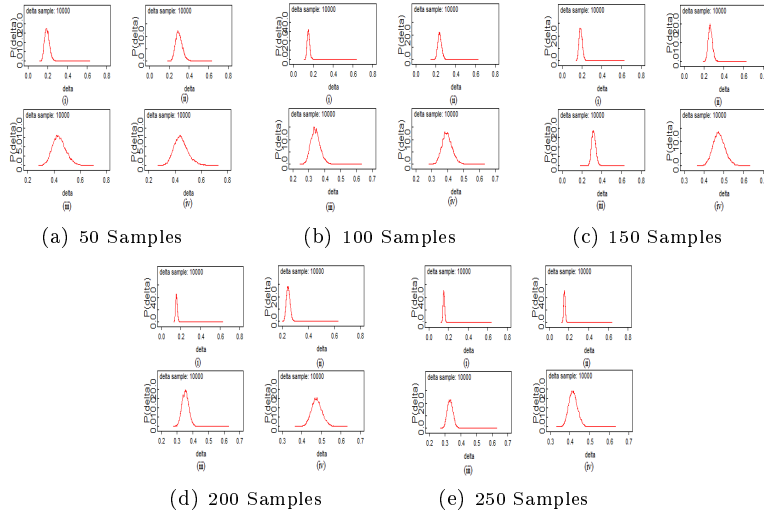
(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 23. Kernal densities of model parameter α under non-informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 24. Kernal densities of model parameter β under non-informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 25. Kernal densities of model parameter γ under non-informative prior for various samples and arbitrary values

(a) 50 Samples (b) 100 Samples (c) 150 Samples

(d) 200 Samples (e) 250 Samples

Figure 26. Kernal densities of model parameter δ under non-informative prior for various samples and arbitrary values