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Two different shrinkage estimator classes for the shape parameter of classical Pareto distribution

Meral Ebegil*† and Senay Ozdemir‡

Abstract

In this study, biased estimators for the shape parameter of a classical Pareto distribution are proposed using two different shrinkage techniques which give a smaller mean square error than an unbiased estimator. Then these obtained biased estimators are compared with the unbiased estimator by the means of their mean square error.

Keywords: Mean Square Error, Pareto Distribution, Shape Parameter, Shrinkage Estimation.

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1. Introduction

Primarily descriptive parameters of the population are used to make a statistical inference about any population. Unbiased estimators are widely used for this purpose. It can be mentioned that using biased estimators have a smaller mean square error (MSE) if the unbiased estimator has a high MSE.

There have been some studies on biased but smaller MSE estimators of an unknown population parameter. Thompson [1, 2] considered a technique of shrinking best linear unbiased estimator (BLUE) by multiplying it by a shrinking factor to obtain an estimator which has a smaller MSE than that of BLUE. Other important studies about this issue are made by Metha and Srinivasan [3], Govindarajulu and Sahai [4], Das [5], Srivastava et. al. [6], Rao and Singh [7], Bhatnagar [8], Singh and Katyar [9], Singh [10], Jani [11], Kourouklis [12], Singh et. al. [13], Singh and Singh [14], Singh and Shukla [15], Singh and Saxena [16], Prakash et. al. [17], Prakash [18].

*Department of Statistics Gazi University Ankara Turkiye, Email: mdemirel@gazi.edu.tr
†Corresponding Author.
‡Department of Statistics Afyon Kocatepe University Afyonkarahisar Turkiye,
Email: senayozdemir@aku.edu.tr

Pareto distribution was first used by Pareto [19] to describe an income distribution. Rytgaard [20] studied on the maximum likelihood estimator (MLE) and the moment estimator for the shape parameter of the Pareto distribution. Furthermore, he found a minimum variance unbiased (MVU) estimator for the shape parameter of the Pareto distribution using the obtained MLE. Sing et. al. [13] proposed new shrinkage estimators for scale parameter of the Pareto distribution using the MLE and the unbiased estimator. Then they compared the proposed estimators with the MLE and the unbiased estimator by the means of their MSE. Prakash et. al. [17] obtained that some test estimators for the scale parameter of a classical Pareto distribution are considered when a prior point guess value of the shape parameter is available. Then they showed that their proposed biased test estimators were better than other estimators through a squared error loss function. Prakash [18] derived some shrinkage test estimators and the Bayes estimators for the shape parameter of the Pareto distribution under the general entropy loss function.

In this study, two different estimator classes are obtained for the shape parameter of the Pareto distribution. These estimator classes are compared with the unbiased estimator by the means of their MSE. After that, it is tried to find out in which case obtained estimator classes are better than the unbiased estimator.

2. Shrinkage Estimators Classes

Jani [11] and Singh and Singh [14] proposed two different shrinkage estimator classes for scale parameters of exponential and normal distribution.

First, Jani [11] proposed a shrinkage estimator class for the scale parameter of the exponential distribution is given as

$$T_{(p)} = \theta_0[1 + k(\theta_0/\widehat{\theta})^p] \tag{2.1}$$

where θ_0 is a priori value of θ parameter, k is a shrinking factor minimizing MSE value, p is a nonzero real number and $\widehat{\theta}$ is the unbiased estimator of θ parameter.

Second, Singh and Singh [14] studied on the estimation problem of population variance σ^2 by adapting the estimation class defined equation (2.1) to a normal population. This estimation class is given as the following:

$$\widetilde{\sigma}^2_{(p)} = \sigma_0^2\left[1 + w\left(\frac{s^2}{\sigma_0^2}\right)^p\right] \tag{2.2}$$

where σ_0^2 is a prior value of σ^2 parameter, w is a shrinking factor minimizing MSE value, p is a nonzero real number and s^2 is the unbiased estimator of σ^2 parameter.

The biased estimators, which have a smaller MSE than the unbiased estimator for the shape parameter of Pareto distribution, are obtained using the estimator classes defined in equation (2.1) and equation (2.2).

3. The Obtained Estimators for the Shape Parameter of the Pareto Distribution and Their Properties

In this section, the shape parameter of the Pareto distribution's MVU estimator, which is proposed by Rytgaard [20], is introduced. Then the biased estimator classes, which have smaller MSE than the unbiased estimator, is obtained using various shrinking factors and MSE values of these estimators are calculated.

Let's consider, X is a Pareto distributed random variable. The probability density function (pdf) is as in equation (3.1)

$$(3.1) \qquad f_X(x) = \begin{cases} (\beta\alpha^{\beta})/x^{(\beta+1)} & ; x > \alpha \\ 0 & ; x \leq \alpha \end{cases}$$

where α is the scale parameter, β is the shape parameter.

If X random variable has the pdf defined in equation (3.1), the MLE for the shape parameter of the Pareto distribution is

$$(3.2) \qquad \widehat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln \frac{x_i}{\alpha}}.$$

Using equation (3.2) estimator, Rytgaard [20] obtained an unbiased estimator which is defined as

$$(3.3) \qquad \tilde{\beta} = \frac{n-1}{n}\widehat{\beta}.$$

It can be found that the expected value of this estimator is

$E[\tilde{\beta}] = \beta.$

Variance of $\tilde{\beta}$ estimator is

$Var(\tilde{\beta}) = \frac{1}{(n-2)}\beta^2$ where $E[\tilde{\beta}^2] = \frac{(n-1)}{(n-2)}\beta^2.$

3.1. Corollary. *The shrinkage estimator class for the shape parameter of Pareto distribution, which is obtained by help of equation(2.1), given as*

$$(3.4) \qquad \beta^{\star}_{(p)} = \beta_0 + (\tilde{\beta} - \beta_0)k_{(p)}$$

where

$$(3.5) \qquad k_{(p)} = (n-1)^p \frac{(n+p-1)!}{(n+2p-1)!}$$

and p is a nonzero real number. MSE of $\beta^{\star}_{(p)}$ estimator class is

$$(3.6) \qquad MSE(\beta^{\star}_{(p)}) = \beta^2 \left[\frac{k^2_{(p)}}{(n-2)} + (k_{(p)} - 1)^2(1-\lambda)^2 \right]$$

where $\lambda = \beta_0/\beta$. Furthermore bias of $\beta^{\star}_{(p)}$ estimators class given by

$$(3.7) \qquad Bias(\beta^{\star}_{(p)}) = (1 - k_{(p)})(\beta_0 - \beta).$$

Proof. The shrinkage estimator class for the shape parameter of Pareto distribution is described as

$$(3.8) \qquad \beta^{\star}_{(p)} = \beta_0 \left[1 + k \left(\frac{\beta_0}{\widetilde{\beta}} \right)^p \right]$$

which is obtained by means of equation (2.1).

$$E\left[\left(\frac{1}{\beta}\right)^{jp}\right] = K_{1(jp)}(1/\beta)^{jp} \ ,(j=1,2)$$

and

$$K_{1(jp)} = (n-1)^{-jp}\frac{(n+jp-1)!}{(n-1)!}$$

functions are used to calculate the MSE of $\beta^{\star}_{(p)}$ estimator class. As it is known the value of $MSE\left(\beta^{\star}_{(p)}\right)$ is

$$MSE\left(\beta^{\star}_{(p)}\right) = E\big[\beta^{\star}_{(p)} - \beta\big]^2. \tag{3.9}$$

If required information is written in equation (3.9) where $\lambda = \beta_0/\beta$, the MSE value is obtained as

$$MSE(\beta^{\star}_{(p)}) = \beta^2\left[k^2(\lambda)^{2(1+p)}K_{1(2p)} + 2k(\lambda)^{(1+p)}(\lambda-1)\,K_{1(p)} + (\lambda-1)^2\right].$$

Differentiating this equation with respect to k and setting the derivative equal to zero, we find

$$k = (\lambda)^{-p}\left(\frac{1}{\lambda}-1\right)K_{1(p)}/K_{1(2p)} \tag{3.10}$$

which minimizes the MSE value. If the required values are inserted into equation (3.10), the k value is obtained as given in equation (3.11);

$$k = \left(\frac{\beta-\beta_0}{\beta_0}\right)\left(\frac{\beta}{\beta_0}\right)^p(n-1)^p\frac{(n+p-1)!}{(n+2p-1)!} \tag{3.11}$$

The shrinking parameter k is obtained as a function of β parameter. In practice, it is impossible to attain parameter β. Therefore the unknown parameters in equation (3.11) are replaced by their unbiased estimators. So an estimator for k is obtained as

$$\hat{k} = \left(\frac{\widetilde{\beta}-\beta_0}{\beta_0}\right)\left(\frac{\beta}{\beta_0}\right)^p(n-1)^p\frac{(n+p-1)!}{(n+2p-1)!}.$$

On conclusion, when necessary adjustment is made, the estimator class for the shape parameter of Pareto distribution is obtained as

$$\beta^{\star}_{(p)} = \beta_0 + \left(\widetilde{\beta}-\beta_0\right)k_{(p)}$$

where $k_{(p)} = (n-1)^p\dfrac{(n+p-1)!}{(n+2p-1)!}$.Thus, the MSE value of $\beta^{\star}_{(p)}$ is obtained as

$$MSE\left(\beta^{\star}_{(p)}\right) = \frac{k^2_{(p)}\beta^2}{(n-2)} + \left(k_{(p)}-1\right)^2(\beta-\beta_0)^2 \tag{3.12}$$

by making necessary adjustment in equation (3.9). If $\lambda = \beta_0/\beta$ is written on its place, equation (3.12) is written as

$$MSE\left(\beta^{\star}_{(p)}\right) = \beta^2\left[\frac{k^2_{(p)}}{(n-2)} + \left(k_{(p)}-1\right)^2(1-\lambda)^2\right]. \tag{3.13}$$

The bias of $\beta^{\star}_{(p)}$ estimators class is obtained as

$$Bias\left(\beta^{\star}_{(p)}\right) = E\left(\beta^{\star}_{(p)}\right) - \beta = \left(1 - k_{(p)}\right)\left(\beta_0 - \beta\right) \ .$$

Thus the proof is completed. □

The relative efficiency of $\beta^{\star}_{(p)}$ estimator class with respect to $\widetilde{\beta}$ estimator is calculated by means of

$$\text{(3.14)} \quad \frac{MSE\left(\beta^{\star}_{(p)}\right)}{Var\left(\widetilde{\beta}\right)} = k^2_{(p)} + (n-2)\left(k_{(p)} - 1\right)^2 (1-\lambda)^2 \ .$$

If equation (3.14) is smaller than 1, it is clear that $MSE(\beta^{\star}_{(p)}) < Var(\widetilde{\beta})$.

Case Study 1: Consider that p=1. By using equation(3.4) and equation(3.5) an estimator is obtain as

$$\beta^{\star}_{(1)} = \beta_0 + \left(\widetilde{\beta} - \beta_0\right)\frac{(n-1)}{(n+1)} \ .$$

The MSE value of this estimator is

$$MSE\left(\beta^{\star}_{(1)}\right) = \beta^2\left[\frac{(n-1)^2}{(n-2)(n+1)^2} + \left(\frac{(n-1)}{(n+1)} - 1\right)^2 (1-\lambda)^2\right].$$

The relative efficiency of $\beta^{\star}_{(1)}$ estimator with respect to $\widetilde{\beta}$ estimator is

$$\begin{aligned} \frac{MSE(\beta^{\star}_{(1)})}{Var(\widetilde{\beta})} &= \frac{(n-1)^2}{(n+1)^2} + (n-2)\left(\frac{(n-1)}{(n+1)} - 1\right)^2 (1-\lambda)^2 \\ &= \frac{(n-1)^2}{(n+1)^2} + \frac{4(n-2)}{(n+1)^2}(1-\lambda)^2. \end{aligned}$$

It is clear that $\beta^{\star}_{(1)}$ estimator is better than $\widetilde{\beta}$ estimator if $\frac{MSE\left(\beta^{\star}_{(1)}\right)}{Var\left(\widetilde{\beta}\right)} < 1$ inequality is true. Thus

$$\text{(3.15)} \quad \frac{(n-1)^2}{(n+1)^2} + \frac{4(n-2)}{(n+1)^2}(1-\lambda)^2 < 1$$

inequality can be written. If the necessary adjustment is made in equation (3.15), it is obtained that

$$0 < \lambda < 1 + \left(\frac{n}{n-2}\right)^{1/2}.$$

In case this inequality is true, it can be said that $\beta^{\star}_{(1)}$ estimator is better than $\widetilde{\beta}$ estimator. Further, when n is very large (i.e. $n \to \infty$)

$$(1-\lambda)^2 < \frac{n}{n-2}$$

inequality reduces to $0 < \lambda < 2$.

Case Study 2: Consider that $p = 2$. By using equation(3.4) and equation(3.5) an estimator is obtained as

$$\beta^{\star}_{(2)} = \beta_0 + \left(\widetilde{\beta} - \beta_0\right) \frac{(n-1)^2}{(n+1)(n+2)}.$$

The MSE value of this estimator is

$$MSE\left(\beta^{\star}_{(2)}\right) = \beta^2 \left[\tfrac{(n-1)^4}{(n+1)^2(n+2)^2(n-2)} + \left(\tfrac{(n-1)^2}{(n+1)(n+2)} - 1\right)^2 (1-\lambda)^2\right].$$

The relative efficiency of $\beta^{\star}_{(2)}$ estimator with respect to $\widetilde{\beta}$ estimator is

$$\begin{aligned}\frac{MSE(\beta^{\star}_{(2)})}{Var(\widetilde{\beta})} &= \left[\frac{(n-1)^2}{(n+1)(n+2)}\right]^2 + (n-2)\left[\frac{(n-1)^2}{(n+1)(n+2)} - 1\right]^2 (1-\lambda)^2 \\ &= \frac{(n-1)^4}{(n+1)^2(n+2)^2} + \frac{(n-2)(5n+1)^2}{(n+1)^2(n+2)^2}(1-\lambda)^2.\end{aligned}$$

It is clear that $\beta^{\star}_{(2)}$ estimator is better than $\widetilde{\beta}$ estimator if $\frac{MSE\left(\beta^{\star}_{(2)}\right)}{Var\left(\widetilde{\beta}\right)} < 1$ inequality is true. Thus

$$\frac{(n-1)^4}{(n+1)^2(n+2)^2} + \frac{(n-2)(5n+1)^2}{(n+1)^2(n+2)^2}(1-\lambda)^2 < 1 \tag{3.16}$$

inequality can be written. If the necessary adjustment is made in equation (3.16), it is obtained that

$$1 - \left(\frac{2n^2+n+3}{(n-2)(5n+1)}\right)^{1/2} < \lambda < 1 + \left(\frac{2n^2+n+3}{(n-2)(5n+1)}\right)^{1/2}.$$

In case this inequality is true, it can be said that $\beta^{\star}_{(2)}$ estimator is better than $\widetilde{\beta}$ estimator. Furthermore, when n is very large (i.e. $n \to \infty$)

$$(1-\lambda)^2 < \frac{2n^2+n+3}{(n-2)(5n+1)}$$

inequality reduces to $0.37 < \lambda < 1.63$.

3.2. Corollary. *The shrinkage estimator class for the shape parameter of Pareto distribution, which is obtained by means of equation(2.2), is given as*

$$\beta^{*}_{(p)} = \beta_0 + \left(\widetilde{\beta} - \beta_0\right) w_{(p)} \tag{3.17}$$

where

$$w_{(p)} = (n-1)^{-p} \frac{(n-p-1)!}{(n-2p-1)!}. \tag{3.18}$$

of $\beta^{*}_{(p)}$ estimator class is

$$MSE\left(\beta^{*}_{(p)}\right) = \beta^2 \left[\tfrac{w^2_{(p)}}{(n-2)} + \left(w_{(p)} - 1\right)^2 (1-\lambda)^2\right]$$

where $\lambda = \beta_0/\beta$.Furthermore bias of $\beta^{*}_{(p)}$estimator class is given by

$$Bias\left(\beta^{*}_{(p)}\right) = \left(1 - w_{(p)}\right)(\beta_0 - \beta) \ . \tag{3.19}$$

Proof. The shrinkage estimator class for the shape parameter of Pareto distribution is described as

$$\beta^*_{(p)} = \beta_0 \left[1 + w \left(\frac{\widetilde{\beta}}{\beta_0} \right)^p \right]$$

which is obtained by means of equation(2.2).

$$E\left(\widetilde{\beta}^{jp}\right) = K_{2(jp)}(\beta)^{jp} \ , (j = 1, 2)$$

and

$$K_{2(jp)} = (n-1)^{jp} \frac{(n-jp-1)!}{(n-1)!}$$

functions are used to calculate the MSE of $\beta^*_{(p)}$ estimator class. The value of $MSE\left(\beta^*_{(p)}\right)$ is

$$\text{(3.20)} \quad MSE(\beta^*_{(p)} = E\left[\beta^*_{(p)} - \beta\right]^2$$

If necessary information is written in equation (3.20) where $\lambda = \frac{\beta_0}{\beta}$, the MSE value is obtained as

$$MSE\left(\beta^*_{(p)}\right) = \beta^2 \left[w^2 (\lambda)^{2(1-p)} K_{2(2p)} + 2w(\lambda)^{1-p} (\lambda - 1) K_{2(p)} + (\lambda - 1)^2 \right].$$

Differentiating this equation with respect to w and setting the derivate equal to zero, we find

$$\text{(3.21)} \quad w = \left(\frac{1}{\lambda} - 1 \right) (\lambda)^p \left(\frac{K_{2(p)}}{K_{2(2p)}} \right)$$

which is a constant minimizing the MSE value. If necessary information is written in equality which is introduced equation (3.21) w is obtained as follows:

$$\text{(3.22)} \quad w = \left(\frac{\beta - \beta_0}{\beta_0} \right) \left(\frac{\beta_0}{\beta} \right)^p (n-1)^{-p} \frac{(n-p-1)!}{(n-2p-1)!}.$$

The shrinking parameter k is obtained as a function of βparameter. In practice, it is impossible to attain parameter β. Therefore the unknown parameter in equation (3.22) is replaced by its unbiased estimator. So an estimator for w is obtained as

$$\hat{w} = \left(\frac{\widetilde{\beta} - \beta_0}{\beta_0} \right) \left(\frac{\beta_0}{\widetilde{\beta}} \right)^p (n-1)^{-p} \frac{(n-p-1)!}{(n-2p-1)!}.$$

On conclusion, when necessary adjustment is made, the estimator class for the shape parameter of Pareto distribution is obtained as

$$\beta^*_{(p)} \ = \beta_0 + \left(\widetilde{\beta} - \beta_0 \right) w_{(p)}$$

where $w_{(p)} = (n-1)^{-p} \frac{(n-p-1)!}{(n-2p-1)!}$ .Thus, the MSE value of $\beta^*_{(p)}$ is obtained as

$$\text{(3.23)} \quad MSE\left(\beta^*_{(p)}\right) = \frac{w^2_{(p)} \beta^2}{(n-2)} + \left(w_{(p)} - 1\right)^2 (\beta - \beta_0)^2$$

by making necessary adjustment in equation (3.20). If $\lambda = \beta_0/\beta$ is written on its place, equation (3.23) is written as

$$MSE\left(\beta^*_{(p)}\right) = \beta^2 \left[\frac{w^2_{(p)}}{(n-2)} + \left(w_{(p)} - 1\right)^2 (1-\lambda)^2\right].$$

Furthermore the bias of $\beta^\star_{(p)}$estimator class can be obtained as

$$Bias\left(\beta^*_{(p)}\right) = E\left(\beta^*_{(p)}\right) - \beta = \left(1 - w_{(p)}\right)(\beta_0 - \beta) \;.$$

Thus the proof is completed. □

The relative efficiency of $\beta^*_{(p)}$ estimator class with respect to $\widetilde{\beta}$ estimator is calculated by means of

$$\frac{MSE\left(\beta^*_{(p)}\right)}{Var\left(\widetilde{\beta}\right)} = w^2_{(p)} + (n-2)\left(w_{(p)} - 1\right)^2 (1-\lambda)^2. \tag{3.24}$$

If equation (3.24) is smaller than 1, it is clear that $MSE\left(\beta^*_{(p)}\right) < Var\left(\widetilde{\beta}\right)$.

Case Study 3: Consider that p=1. An estimator is obtained as

$$\beta^*_{(1)} = \beta_0 + \left(\widetilde{\beta} - \beta_0\right)\frac{(n-2)}{(n-1)}$$

by using equation (3.17) and equation (3.18). The MSE value of this estimator is

$$MSE\left(\beta^*_{(1)}\right) = \beta^2\left[\frac{(n-2)^2}{(n-2)(n-1)^2} + \left(\frac{(n-2)}{(n-1)} - 1\right)^2 (1-\lambda)^2\right].$$

The relative efficiency of $\beta^*_{(1)}$ estimator with respect to $\widetilde{\beta}$ estimator is

$$\begin{aligned} \frac{MSE(\beta^*_{(1)})}{Var(\widetilde{\beta})} &= \frac{(n-2)^2}{(n-1)^2} + (n-2)\left[\frac{(n-2)}{(n-1)} - 1\right]^2 (1-\lambda)^2 \\ &= \frac{(n-2)^2}{(n-1)^2} + \frac{(n-2)}{(n-1)^2}(1-\lambda)^2. \end{aligned}$$

It is clear that $\beta^*_{(1)}$ estimator is better than $\widetilde{\beta}$ estimator if$\frac{MSE\left(\beta^*_{(1)}\right)}{Var\left(\widetilde{\beta}\right)} < 1$ inequality is true. Thus

$$\frac{(n-2)^2}{(n-1)^2} + \frac{(n-2)}{(n-1)^2}(1-\lambda)^2 < 1$$

inequality can be written. If the necessary adjustment is made in above inequality, it is obtained that

$$0 < \lambda < 1 + \left(\frac{2n-3}{n-2}\right)^{1/2}.$$

In case this inequality is true, it can be said that $\beta^*_{(1)}$estimator is better than $\widetilde{\beta}$ estimator. Further, when n is very large (i.e. $n \to \infty$)

$$(1-\lambda)^2 < \frac{2n-3}{n-2}$$

inequality reduces to $0 < \lambda < 2.41$.

Case Study 4: Consider that $p = 2$. By using equation (3.17) and equation (3.18) an estimator is obtained as

$$\beta^*_{(2)} = \beta_0 + \left(\widetilde{\beta} - \beta_0\right) \frac{(n-3)(n-4)}{(n-1)^2}.$$

The MSE value of this estimator is

$$MSE\left(\beta^*_{(2)}\right) = \beta^2 \left[\tfrac{(n-3)^2(n-4)^2}{(n-2)(n-1)^4} + \left(\tfrac{(n-3)(n-4)}{(n-1)^2} - 1\right)^2 (1-\lambda)^2\right].$$

The relative efficiency of $\beta^*_{(2)}$ estimator with respect to $\widetilde{\beta}$ estimator is

$$\begin{aligned} \frac{MSE(\beta^*_{(2)})}{Var(\widetilde{\beta})} &= \frac{(n-3)^2(n-4)^2}{(n-1)^4} + (n-2)\left(\frac{(n-3)(n-4)}{(n-1)^2} - 1\right)^2 (1-\lambda)^2 \\ &= \frac{(n-3)^2(n-4)^2}{(n-1)^4} + \frac{(n-2)(5n-11)^2}{(n-1)^2}. \end{aligned}$$

It is clear that $\beta^*_{(2)}$ estimator is better than $\widetilde{\beta}$ estimator if $\frac{MSE\left(\beta^*_{(2)}\right)}{Var\left(\widetilde{\beta}\right)} < 1$ inequality is true. Thus

$$\frac{(n-3)^2(n-4)^2}{(n-1)^4} + \frac{(n-2)(5n-11)^2}{(n-1)^2}(1-\lambda)^2 < 1$$

inequality can be written. If the necessary adjustment is made in above inequality, it is obtained that

$$1 - \left(\frac{2n^2 - 9n + 13}{(n-2\ )(5n-11)}\right)^{\frac{1}{2}} < \lambda < 1 + \left(\frac{2n^2 - 9n + 13}{(n-2\ )(5n-11)}\right)^{\frac{1}{2}}.$$

In case this inequality is true, it can be said that $\beta^*_{(2)}$estimator is better than $\widetilde{\beta}$ estimator. Further, when n is very large (i.e. $n \to \infty$)

$$(1-\lambda)^2 < \frac{2n^2 - 9n + 13}{(n-2\ )(5n-11)}$$

inequality reduces to $0.37 < \lambda < 1.63$.

Note: It can be seen that the estimator class proposed by Jani [11] is directly related with that of Singh and Singh [14] for the shape parameter of the Pareto distribution. This relationship is expressed as $k_{(p)} = w_{(-p)}$.

4. Comparisons of the estimators

Here, the relative efficiency of the obtained estimator classes with respect to the unbiased estimator for the shape parameter of the Pareto distribution is calculated using different values of n, p and λ. The handled λ values are selected by considering the efficiency range for large n values in case studies.

The relative efficiency of the estimator class introduced in equation (3.4) with respect to the estimator given in equation (3.3) is calculated for the different value of n, p and λ by the help of equation (3.14). These calculated values are summarized in Table 1.

Table 1. The relative efficiency of the estimator class proposed equation (3.4) with respect to estimator given by equation (3.3)

		Sample Size n				
λ	**Estimator**	**5**	**10**	**15**	**25**	**50**
0.125	$\beta^{\star}_{(-1)}$	0.8657	0.9130	0.9357	0.9490	0.9749
	$\beta^{\star}_{(-1/2)}$	0.9738	0.9831	0.9875	0.9901	0.9951
	$\beta^{\star}_{(1/2)}$	0.9022	0.9354	0.9518	0.9615	0.9809
	$\beta^{\star}_{(1)}$	0.8719	0.9211	0.9436	0.9563	0.9796
	$\beta^{\star}_{(3/2)}$	1.1139	1.1369	1.1304	1.1190	1.0761
	$\beta^{\star}_{(2)}$	1.6853	1.6965	1.6407	1.5788	1.3687
	$\beta^{\star}_{(5/2)}$	2.4890	2.5955	2.5228	2.4111	1.9533
0.50	$\beta^{\star}_{(-1)}$	0.8148	0.8788	0.9100	0.9284	0.9646
	$\beta^{\star}_{(-1/2)}$	0.9730	0.9826	0.9871	0.9898	0.9950
	$\beta^{\star}_{(1/2)}$	0.8858	0.9236	0.9426	0.9541	0.9770
	$\beta^{\star}_{(1)}$	0.7355	0.8164	0.8594	0.8861	0.9416
	$\beta^{\star}_{(3/2)}$	0.6667	0.7647	0.8182	0.8519	0.9231
	$\beta^{\star}_{(2)}$	0.7319	0.8302	0.8785	0.9065	0.9581
	$\beta^{\star}_{(5/2)}$	0.9107	1.0270	1.0697	1.0858	1.0823
1.00	$\beta^{\star}_{(-1)}$	0.7901	0.8622	0.8975	0.9184	0.9596
	$\beta^{\star}_{(-1/2)}$	0.9726	0.9823	0.9869	0.9896	0.9949
	$\beta^{\star}_{(1/2)}$	0.8778	0.9179	0.9382	0.9504	0.9751
	$\beta^{\star}_{(1)}$	0.6694	0.7656	0.8186	0.8521	0.9231
	$\beta^{\star}_{(3/2)}$	0.4499	0.5842	0.6668	0.7224	0.8490
	$\beta^{\star}_{(2)}$	0.2696	0.4103	0.5090	0.5805	0.7590
	$\beta^{\star}_{(5/2)}$	0.1455	0.2665	0.3652	0.4432	0.6600
1.50	$\beta^{\star}_{(-1)}$	0.8148	0.8788	0.9100	0.9284	0.9646
	$\beta^{\star}_{(-1/2)}$	0.9730	0.9826	0.9871	0.9898	0.9950
	$\beta^{\star}_{(1/2)}$	0.8858	0.9236	0.9426	0.9541	0.9770
	$\beta^{\star}_{(1)}$	0.7355	0.8164	0.8594	0.8861	0.9416
	$\beta^{\star}_{(3/2)}$	0.6667	0.7647	0.8182	0.8519	0.9231
	$\beta^{\star}_{(2)}$	0.7319	0.8302	0.8785	0.9065	0.9581
	$\beta^{\star}_{(5/2)}$	0.9107	1.0270	1.0697	1.0858	1.0823
2.50	$\beta^{\star}_{(-1)}$	1.0123	1.0115	1.0097	1.0082	1.0046
	$\beta^{\star}_{(-1/2)}$	0.9760	0.9846	0.9887	0.9910	0.9956
	$\beta^{\star}_{(1/2)}$	0.9494	0.9693	0.9781	0.9830	0.9920
	$\beta^{\star}_{(1)}$	1.2645	1.2227	1.1859	1.1583	1.0892
	$\beta^{\star}_{(3/2)}$	2.4013	2.2085	2.0290	1.8880	1.5164
	$\beta^{\star}_{(2)}$	4.4301	4.1901	3.8347	3.5142	2.5509
	$\beta^{\star}_{(5/2)}$	7.0327	7.1109	6.7060	6.2265	4.4606

Table 1 shows that $\beta^{\star}_{(-1/2)}$ and $\beta^{\star}_{(1/2)}$ estimators are better than the unbiased estimators for all values of λ. Further when $0.50 \leq \lambda \leq 1.50$, the all proposed biased estimators are better than the unbiased estimators. Hence the efficiency of the proposed biased estimator class with respect to the unbiased estimator decreases as λ values differ from 1. Besides increased p values cause a decrease in efficiency of the proposed biased estimator class with respect to the unbiased estimator.

Table 2. The relative efficiency of the estimator class proposed equation (3.17) with respect to estimator given by equation (3.3)

		Sample Size n				
λ	**Estimator**	**5**	**10**	**15**	**25**	**50**
0.125	$\beta^*_{(-1)}$	0.8719	0.9211	0.9436	0.9563	0.9796
	$\beta^*_{(-1/2)}$	0.9022	0.9354	0.9518	0.9615	0.9809
	$\beta^*_{(1/2)}$	0.9738	0.9831	0.9875	0.9901	0.9951
	$\beta^*_{(1)}$	0.8657	0.9130	0.9357	0.9490	0.9749
	$\beta^*_{(3/2)}$	0.9948	0.9987	0.9997	1.0000	1.0003
	$\beta^*_{(2)}$	1.6888	1.5148	1.4053	1.3331	1.1750
	$\beta^*_{(5/2)}$	2.9280	2.6133	2.3354	2.1289	1.6252
0.50	$\beta^*_{(-1)}$	0.7355	0.8164	0.8594	0.8861	0.9416
	$\beta^*_{(-1/2)}$	0.8858	0.9236	0.9426	0.9541	0.9770
	$\beta^*_{(1/2)}$	0.9730	0.9826	0.9871	0.9898	0.9950
	$\beta^*_{(1)}$	0.8148	0.8788	0.9100	0.9284	0.9646
	$\beta^*_{(3/2)}$	0.6758	0.7768	0.8302	0.8630	0.9304
	$\beta^*_{(2)}$	0.7325	0.8001	0.8412	0.8686	0.9297
	$\beta^*_{(5/2)}$	1.0254	1.0314	1.0250	1.0197	1.0083
1.00	$\beta^*_{(-1)}$	0.6694	0.7656	0.8186	0.8521	0.9231
	$\beta^*_{(-1/2)}$	0.8778	0.9179	0.9382	0.9504	0.9751
	$\beta^*_{(1/2)}$	0.9726	0.9823	0.9869	0.9896	0.9949
	$\beta^*_{(1)}$	0.7901	0.8622	0.8975	0.9184	0.9596
	$\beta^*_{(3/2)}$	0.5212	0.6692	0.7480	0.7966	0.8966
	$\beta^*_{(2)}$	0.2689	0.4536	0.5677	0.6433	0.8108
	$\beta^*_{(5/2)}$	0.1030	0.2644	0.3897	0.4819	0.7092
1.50	$\beta^*_{(-1)}$	0.7355	0.8164	0.8594	0.8861	0.9416
	$\beta^*_{(-1/2)}$	0.8858	0.9236	0.9426	0.9541	0.9770
	$\beta^*_{(1/2)}$	0.9730	0.9826	0.9871	0.9898	0.9950
	$\beta^*_{(1)}$	0.8148	0.8788	0.9100	0.9284	0.9646
	$\beta^*_{(3/2)}$	0.6758	0.7768	0.8302	0.8630	0.9304
	$\beta^*_{(2)}$	0.7325	0.8001	0.8412	0.8686	0.9297
	$\beta^*_{(5/2)}$	1.0254	1.0314	1.0250	1.0197	1.0083
2.50	$\beta^*_{(-1)}$	1.2645	1.2227	1.1859	1.1583	1.0892
	$\beta^*_{(-1/2)}$	0.9494	0.9693	0.9781	0.9830	0.9920
	$\beta^*_{(1/2)}$	0.9760	0.9846	0.9887	0.9910	0.9956
	$\beta^*_{(1)}$	1.0123	1.0115	1.0097	1.0082	1.0046
	$\beta^*_{(3/2)}$	1.9130	1.6374	1.4877	1.3946	1.2014
	$\beta^*_{(2)}$	4.4417	3.5723	3.0293	2.6704	1.8810
	$\beta^*_{(5/2)}$	8.4051	7.1672	6.1078	5.3222	3.4012

Similarly, the relative efficiency of the estimator class proposed in equation (3.17) with respect to estimator given in equation (3.3) is calculated for different values of n, p and λ with the help of equation (3.24). These calculated values are given in Table 2.

Table 2 shows that $\beta^*_{(-1/2)}$ and $\beta^*_{(1/2)}$ estimators are better than the unbiased estimators for all λ values. Furthermore, when $0.50 \leq \lambda \leq 1.50$, the all proposed biased estimators better than the unbiased estimators. But the efficiency of the proposed biased estimator class with respect to the unbiased estimator decrease as λ values differ

Table 3. The relative biases of equation (3.4) and equation (3.17) estimators for different n and p values

p	**Sample Size n**				
	10	**15**	**20**	**25**	**50**
-1	0.6111	0.5714	0.5526	0.5417	0.5204
-1/2	0.2185	0.2120	0.2089	0.2070	0.2035
1/2	4.5769	4.7175	4.7880	4.8303	4.9151
1	1.6364	1.7500	1.8095	1.8462	1.9216
3/2	1.1841	1.2952	1.3569	1.3961	1.4799
2	0.9985	1.1009	1.1623	1.2030	1.2940
5/2	0.9108	0.9958	1.0530	1.0931	1.1882

from 1. In addition increased p values cause a decreased efficiency of the proposed biased estimator class with respect to the unbiased estimator. Moreover, when the estimators given in Table 1 are compared to the estimators given in Table 2, it is observed that the efficiency range of the estimator class introduced in equation (3.17) with respect to the estimator given in equation (3.3) is larger than that of the estimator class introduced in equation (3.4) with respect to estimator given in equation (3.3).

In addition to the MSE criteria, bias has an important role in comparison of estimators. A relative bias can be calculated by dividing equation (3.7) to equation (3.19). The relative bias is given in equation (4.1).

$$\frac{Bias(\beta^{\star}_{(p)})}{Bias(\beta^{*}_{(p)})} = \frac{1-k_{(p)}}{1-w_{(p)}} \ . \tag{4.1}$$

The relative bias values are calculated by means of equation (4.1) for different n and p values and given in Table 3.

In Table 3, it is seen that the biases of $\beta^{\star}_{(p)}$ estimators are smaller than those of $\beta^{*}_{(p)}$ estimators when p has a negative value. Furthermore, it can be mentioned that $Bias(\beta^{\star}_{(p)})/Bias(\beta^{*}_{(p)})$ values decrease when there is an increase on positive values of p. However, it can be noted that$\beta^{*}_{(p)}$ estimators have smaller bias than $\beta^{\star}_{(p)}$estimators if p is near 1.

5. Simulation Study

In this section, we generated a data set for the Pareto distribution with the shape parameter $\beta = 5$ and the scale parameter $\alpha = 1$. The scale parameter α was taken as 1 because the same results were obtained from experiments for $\alpha = 1,\ 1.5,\ 2, \ldots$. The shape parameter should be greater than 2 so that the variance of a data set from the Pareto distribution could be calculated. Also Thompson [1,2] used the proportion 1/5 between two descriptive parameters of the normal distribution in his study. The relative efficiency of the obtained estimator classes with respect to the unbiased estimator for the shape parameter of the Pareto distribution is calculated using different values of n, p and λ. First we calculated the MSE values to obtain the relative efficiency. These MSE values were calculated by the means of Monte Carlo Simulation study where the number of iterations was 25000. We obtained relative efficiencies similar to that of previous section. The simulation study results which are given in Table 4 support to the theoretical results.

Table 4. The relative efficiency of the estimator class proposed equation(3.4) and equation(3.17) with respect to estimator given by equation (3.3)

		Sample Size n		
λ	**Estimator**	**5**	**15**	**50**
0.50	$\beta^{\star}_{(-1)}$	0.7797	0.9339	0.9808
	$\beta^{*}_{(-1)}$	0.7124	0.8857	0.9633
	$\beta^{\star}_{(-1/2)}$	0.9715	0.9916	0.9976
	$\beta^{*}_{(-1/2)}$	0.8841	0.9609	0.9882
	$\beta^{\star}_{(1)}$	0.7123	0.8857	0.9633
	$\beta^{*}_{(1)}$	0.7796	0.9339	0.9808
1.00	$\beta^{\star}_{(-1)}$	0.5625	0.7901	0.9596
	$\beta^{*}_{(-1)}$	0.4444	0.6694	0.9231
	$\beta^{\star}_{(-1/2)}$	0.9396	0.9726	0.9949
	$\beta^{*}_{(-1/2)}$	0.7610	0.8778	0.9751
	$\beta^{\star}_{(1)}$	0.4444	0.6694	0.9231
	$\beta^{*}_{(1)}$	0.5625	0.7901	0.9596
2.50	$\beta^{\star}_{(-1)}$	0.1242	0.6646	0.8973
	$\beta^{*}_{(-1)}$	0.0195	0.4575	0.8075
	$\beta^{\star}_{(-1/2)}$	0.8471	0.9545	0.9869
	$\beta^{*}_{(-1/2)}$	0.4475	0.7948	0.9363
	$\beta^{\star}_{(1)}$	0.0195	0.4575	0.8075
	$\beta^{*}_{(1)}$	0.1242	0.6646	0.8973

6. Conclusion and Suggestions

When the biased estimators give smaller MSE than unbiased estimators, the biased estimators can be preferred to the unbiased estimators. In this study, considering this case, two different biased estimator classes are proposed. These estimators are generated by minimizing MSE.

In section 4, the cases in which the biased estimators have smaller MSE than the unbiased estimator are assessed. When $0.50 \leq \lambda \leq 1.50$, the biased estimator class which is given in equation(3.4) is better than the unbiased estimator. However the efficiency of the proposed biased estimator class with respect to the unbiased estimator decreases as the λ values differ from 1. Increased p values cause a decrease in efficiency of the proposed biased estimator class with respect to the unbiased estimator. Similarly, when $0.50 \leq \lambda \leq 1.50$, the biased estimator class given in equation (3.17) is better than the unbiased estimators. However the efficiency of the proposed biased estimator class with respect to the unbiased estimator decreases as λ values differ from 1. Further increased p values cause decrease in efficiency of the proposed biased estimator class with respect to the unbiased estimator. When the relative efficiency values given in Table 1 and Table 2 are considered, it is shown that the both biased estimators classes have almost the same efficiency range. Besides, if both biased estimator classes are considered as an efficient range, it is observed that the efficiencies of biased estimators with respect to biased estimators decrease when n increases. In addition to the relative efficiency values in both tables, the efficiency range of the estimator class introduced in equation (3.17) is greater than that of the estimator class given in equation (3.4) as shown in case study 1 and 3. It is observed that the estimator class given in equation (3.4) is more efficient than the others when p is a negative real number, while the estimator class given equation (3.17) is more efficient than others when p is a positive real number.

In conclusion, it is possible to obtain estimators that give a smaller MSE than the unbiased estimator for the shape parameter of Pareto distribution using the estimator class given in equation (3.4) if p is a negative number near zero, while it is reasonable to use the estimator class given in equation (3.17) if p is a positive number near zero, when λ values are near 1.

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