

Reliability analysis under constant-stress partially accelerated life tests using hybrid censored data from Weibull distribution

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Abstract

This article discusses the estimation of Weibull distribution parameters based on hybrid censored data under constant-stress partially accelerated test model. Two estimation methods; maximum likelihood (ML) and percentile bootstrap (PB) are used to make statistical inference on the Weibull distribution parameters and the acceleration factor. The mean square errors of the estimators are calculated to evaluate their performances through a Monte Carlo simulation study. Moreover, the confidence intervals lengths (CILs) and their associated coverage probabilities (CPs) are obtained. Finally, to demonstrate the proposed methodology, an arithmetic example is given.

Keywords: Statistics; reliability; percentile bootstrap; confidence interval; coverage rate; hybrid censoring; mean square error.

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1. Introduction

The ordinary life testing methods of high reliability products usually need a long period to gain sufficient failure data required to do inferences. So, to perform reliability analysis, accelerated life tests (ALTs) are the most common ways to measure such products' life. Under such test settings, products are tested at higher-than-usual levels of stress to induce failures rapidly and economically. Applying ALTs depends on a life-stress relationship. The parameters of life can be estimated via this relationship by using the failure data obtained under accelerated conditions. However, in some cases such a relationship can't be known or assumed. Thus, ALTs can't be applied and the partially accelerated life tests (PALTs) come to be a good appliance to implement the needed life tests.

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The stress can be used in different techniques, frequently applied techniques are constant-stress and step-stress. Under step-stress PALTs, a test unit is first run at use condition and, if it does not fail for a definite time, then it is run at accelerated condition until failure happens or the observation is stopped. But the constant-stress PALTs run each unit at either use condition or accelerated condition only, i.e. each unit is run at a constant-stress level until the test is finished. Accelerated stresses include higher than normal temperature, power, pressure, load, etc., for more details see Nelson [28].

In this article, we deal with hybrid censored constant-stress PALTs when the lifetime of test unit follows Weibull distribution. PALTs have been considered under Type-I and Type-II censoring schemes by numerous authors, for example, see Goel [18], DeGroot and Goel [12], Bhattacharyya and Soejoeti [9], Bai and Chung [7], Bai et al. [8], Abdel-Ghaly et al. [1], Abdel-Ghaly et al. [2], Abdel-Ghaly et al. [3], Abdel-Ghani [4], Abdel-Ghani [5], Ismail [21], Aly and Ismail [6], Ismail and Sarhan [25], Ismail et al. [20] and Ismail [23].

In general, accelerated tests are frequently ended before all items fail. The estimates from the censored data are less precise than those from complete data. However, this is more than offset by the reduced test time and cost. The most used censoring schemes are Type-I and Type-II censoring. Consider n units placed on life test. In traditional Type-I censoring, the experiment lasts up to a pre-specified time C_1 . Any failures that happen after that time are not witnessed. The end point C_1 of the experiment is supposed to be s-independent of the failure times. But in traditional Type-II censoring, the experimenter finishes the experiment after a pre-identified number of units $R \leq n$ fail. In this situation, only the lowest lifetimes are noticed. In Type-I censoring, the number of failures witnessed is random and the endpoint of the experiment is fixed. But in failure-censoring R is fixed and the termination time is random. Several previous works have considered the reliability analysis using the traditional time- and failure- censoring schemes under different life distributions, for more details one can see Cohen [11].

Concerning hybrid censoring scheme it can be applied as follows. Consider a life testing experiment in which n units are placed on test concurrently. Failure times are noticed and the test is finished either at a pre-specified time C_1 or based on a pre-determined number R of failures acquired by a time; say C_2 whichever is happened first. Such a combination of Type-I and Type-II censoring schemes is identified as hybrid censoring scheme. So, sampling according to the hybrid censoring scheme is finished at $\min(C_1, C_2)$. It is noted that the traditional time- and failure- censoring schemes can be found as special cases of hybrid censoring scheme by taking $R = n$ and $C_1 = \infty$, respectively. The most important benefit of applying hybrid censoring scheme is that it preserves the probable experiment time and cost. Several authors have discussed the statistical inference problem about the parameters for sampling schemes Type-I and Type-II censoring. In this work the estimation of parameters is studied under constant-stress partially accelerated life tests (CSPALTs) with hybrid censored data supposing Weibull distribution. It is also supposed that the failed items are not exchanged.

Although the hybrid censoring scheme is applicable, most of preceding works under PALTs were studied using the usual time- and failure- censoring schemes and no consideration has been provided in examining hybrid censored data. All papers prepared under hybrid censoring were correlated with ordinary or fully accelerated tests, see, for example, Fairbanks et al. [17], Draper and Guttman [14], Chen and Bhattacharyya [10], Ebrahimi [15], Gupta and Kundu [19], Kundu [26], Xie [31], Park and Balakrishnan [29] and Zhang et al. [32]. Recently, only two papers made by Ismail [22] and Ismail [24] have considered the hybrid censoring scheme under step-stress PALTs.

The rest of the paper is structured as follows. In Section 2 the model and the hybrid censored data are designated. The maximum likelihood (ML) and percentile bootstrap

(PB) estimations of the CSPALTs model parameters are considered in Section 3. Section 4 covers the simulation results. Section 5 presents an illustrative example. Conclusion is yielded in Section 6.

2. Model description

In this study, it is assumed that the lifetime of a test unit say X under normal condition has Weibull distribution with probability density function (PDF) given by

$$(2.1) \quad f(x; \beta, \eta) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^\beta}, \quad x > 0, \beta > 0, \eta > 0,$$

In fact, Weibull distribution has high flexibility compared to other distributions. Its failure rate function can be increasing, decreasing and constant according to the value of the shape parameter. For more information, see Dimitri [13].

The survival function of this distribution is given by

$$(2.2) \quad R(x) = e^{-(x/\eta)^\beta},$$

The matching failure rate function is

$$(2.3) \quad h(x) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1}.$$

Constant-stress PALTs can be processed according to the following steps and assumptions.

- (1) n_1 units randomly selected among n test units sampled are assigned to run under normal stress and $n_2 (= n - n_1)$ items are allotted to run under severe stress.
- (2) Each item is tested until the censoring times C_1 or C_2 is realized whichever is smaller or the item fails.
- (3) The lifetimes $X_i, i = 1, \dots, n_1$ of units consigned to normal (use) stress, are i.i.d. r.v.'s.
- (4) The lifetimes $Y_j, j = 1, \dots, n_2$ of units assigned to severe stress, are i.i.d r.v.'s.

Now, for a unit subjected to accelerated condition, the PDF of its lifetime say Y is provided by

$$(2.4) \quad f(y; \lambda, \beta, \eta) = \frac{\lambda\beta}{\eta} \left(\frac{\lambda y}{\eta}\right)^{\beta-1} e^{-(\lambda y/\eta)^\beta}, \quad y > 0, \lambda > 1, \beta > 0, \eta > 0,$$

where $Y = \lambda^{-1}X$ and λ is the acceleration factor.

Because the test in Type-I censoring is finished when a pre-specified time C_1 is attained and in failure-censoring the test is ended based on a pre-defined number R of failures gained by a time C_2 ; say. Accordingly, the failure times $x_{(??)} \leq \dots \leq x_{(n_u)} \leq C_1$ (or C_2) and $y_{(??)} \leq \dots \leq y_{(n_a)} \leq C_1$ (or C_2) are ordered failure times at use and accelerated conditions respectively, where $n_u (< n_1)$ and $n_a (< n_2)$ are the numbers of items failed at use and accelerated conditions, respectively.

Under hybrid censoring scheme, supposing that R and C_1 are predetermined, we can observe the following data.

If $C_1 < C_2$, the sample is $x_{(??)} \leq \dots \leq x_{(n_u)} \leq C_1$ and $y_{(??)} \leq \dots \leq y_{(n_a)} \leq C_1$.

If not, the sample is $x_{(??)} \leq \dots \leq x_{(n_u)} \leq C_2$ and $y_{(??)} \leq \dots \leq y_{(n_a)} \leq C_2$.

3. Estimation process

Here, the maximum likelihood estimates (MLEs) of the CSPALTs model parameters under hybrid censoring as well as their confidence limits are considered.

3.1. ML point estimation. Now, let us define the indicator functions: $\delta_{ui} \equiv I(X_i \leq C_1 \text{ (or } C_2))$ and $\delta_{aj} \equiv I(Y_j \leq C_1 \text{ (or } C_2))$. Then the total likelihood function for $(x_1; \delta_{u1}, \dots, x_{n_1}; \delta_{un1}, y_1; \delta_{a1}, \dots, y_{n_2}; \delta_{an2})$ under CSPALTs is given by

$$\begin{aligned} L(\underline{x}, \underline{y} | \lambda, \beta, \eta) &= \prod_{i=1}^{n_1} L_{ui}(x_i, \delta_{ui} | \beta, \eta) \cdot \prod_{j=1}^{n_2} L_{aj}(y_j, \delta_{aj} | \lambda, \beta, \eta) \\ &= \prod_{i=1}^{n_1} \left[\frac{\beta}{\eta} \left(\frac{x_i}{\eta} \right)^{\beta-1} \exp\{-(x_i/\eta)^\beta\} \right]^{\delta_{ui}} \left[\exp\{-(C_1^{\delta_{1R}} C_2^{\delta_{2R}}/\eta)^\beta\} \right]^{\bar{\delta}_{ui}} \\ &\quad \times \prod_{j=1}^{n_2} \left[\frac{\lambda\beta}{\eta} \left(\frac{\lambda y_j}{\eta} \right)^{\beta-1} \exp\{-(\lambda y_j/\eta)^\beta\} \right]^{\delta_{aj}} \left[\exp\{-(\lambda C_1^{\delta_{1R}} C_2^{\delta_{2R}}/\eta)^\beta\} \right]^{\bar{\delta}_{aj}}, \end{aligned}$$

(3.1)

where,

L_{ui} and L_{aj} denote the contributions of the items i , $i = 1, \dots, n_1$ and j , $j = 1, \dots, n_2$ to the total likelihood function under use and accelerated conditions, respectively; and $\bar{\delta}_{ui} = 1 - \delta_{ui}$, $\bar{\delta}_{aj} = 1 - \delta_{aj}$, $\delta_{1R} = 1$ if $C_2 > C_1$ and 0 otherwise, and $\delta_{2R} = 1$ if $C_2 < C_1$ and 0 otherwise.

The value of $\hat{\eta}$ can be found by

$$(3.2) \quad \hat{\eta} = \left\{ \frac{\psi}{n_u + n_a} \right\}^{\frac{1}{\hat{\beta}}},$$

where

$$\psi = \sum_{i=1}^{n_1} \delta_{ui} x_i^\beta + \lambda^\beta + \sum_{j=1}^{n_2} \delta_{aj} y_j^\beta + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^\beta (n_1 - n_u) + (\lambda C_1^{\delta_{1R}} C_2^{\delta_{2R}})^\beta (n_2 - n_a).$$

Now, we have two ML non-linear equations which can be extracted as follows.

$$(3.3) \quad \frac{n_a \hat{\beta}}{\hat{\lambda}} - \left[\frac{(n_u + n_a) \hat{\beta} \hat{\lambda}^{\hat{\beta}-1}}{\psi} \right] \left[\sum_{j=1}^{n_2} \delta_{aj} y_j^{\hat{\beta}} + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_2 - n_a) \right] = 0,$$

$$(3.4) \quad \frac{n_u + n_a}{\hat{\beta}} + \sum_{i=1}^{n_1} \delta_{ui} \ln x_i + \sum_{j=1}^{n_2} \delta_{aj} \ln y_j - (n_u + n_a) \ln \left(\frac{\psi}{n_u + n_a} \right)^{1/\hat{\beta}} + n_a \ln \hat{\lambda} + \hat{\beta} \left(\frac{n_u + n_a}{\psi} \right)^{1/\hat{\beta}} = 0.$$

From equation (3.3), the value of $\hat{\lambda}$ is easily determined by the following formula.

$$(3.5) \quad \hat{\lambda} = \left\{ \frac{n_a [\sum_{i=1}^{n_1} \delta_{ui} x_i^{\hat{\beta}} + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_1 - n_u)]}{n_u [\sum_{j=1}^{n_2} \delta_{aj} y_j^{\hat{\beta}} + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_2 - n_a)]} \right\}^{\frac{1}{\hat{\beta}}}.$$

After substituting for $\hat{\lambda}$, the equation (3.4) can be expressed by

$$(3.6) \quad \begin{aligned} & \frac{n_u + n_a}{\hat{\beta}} + \sum_{i=1}^{n_1} \delta_{ui} \ln x_i + \sum_{j=1}^{n_2} \delta_{aj} \ln y_j \\ & - n_u \frac{\sum_{i=1}^{n_1} \delta_{ui} x_i^{\hat{\beta}} \ln x_i + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_1 - n_u) \ln (C_1^{\delta_{1R}} C_2^{\delta_{2R}})}{\sum_{i=1}^{n_1} \delta_{ui} x_i^{\hat{\beta}} + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_1 - n_u)} \\ & - n_a \frac{\sum_{j=1}^{n_2} \delta_{aj} y_j^{\hat{\beta}} \ln y_j + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_2 - n_a) \ln (C_1^{\delta_{1R}} C_2^{\delta_{2R}})}{\sum_{j=1}^{n_2} \delta_{aj} y_j^{\hat{\beta}} + (C_1^{\delta_{1R}} C_2^{\delta_{2R}})^{\hat{\beta}} (n_2 - n_a)} = 0. \end{aligned}$$

To get the value of $\hat{\beta}$, the Newton-Raphson method is utilized to solve the non-linear equation (3.6), numerically. Consequently, based on the value of $\hat{\beta}$, the values of $\hat{\eta}$ and $\hat{\lambda}$ can be simply determined from (3.2) and (3.5) respectively.

3.2. ML interval estimation. In this subsection, the approximate confidence bounds of the parameters are obtained based on the asymptotic distribution of the MLEs of the elements of the vector of unknown parameters $\Omega = (\beta, \eta, \lambda)$. It is known that the asymptotic distribution of the MLEs of Ω is given by; see Miller [27],

$$((\hat{\beta} - \beta), (\hat{\eta} - \eta), (\hat{\lambda} - \lambda)) \rightarrow N(0, \mathbf{I}^{-1}(\beta, \eta, \lambda))$$

where $\mathbf{I}^{-1}(\beta, \eta, \lambda)$ is the variance-covariance matrix of the unknown parameters $\Omega = (\beta, \eta, \lambda)$. The elements of the 3×3 matrix \mathbf{I}^{-1} , $I_{ij}(\beta, \eta, \lambda)$, $i, j = 1, 2, 3$; can be approximated by $I_{ij}(\hat{\beta}, \hat{\eta}, \hat{\lambda})$, where

$$I_{ij}(\hat{\Omega}) = -\frac{\partial^2 \ln L(\Omega)}{\partial \Omega_i \partial \Omega_j} \Big|_{\Omega = \hat{\Omega}}$$

Thus, the approximate $100(1 - \gamma)\%$ two sided confidence intervals of β , η and λ are, respectively, yielded by

$$\pm Z_{\gamma/2} \sqrt{I_{11}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})}, \quad \hat{\eta} \pm Z_{\gamma/2} \sqrt{I_{22}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})} \quad \text{and} \quad \hat{\lambda} \pm Z_{\gamma/2} \sqrt{I_{33}^{-1}(\hat{\beta}, \hat{\eta}, \hat{\lambda})}.$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$ th percentile of a standard normal distribution.

3.3. Percentile bootstrap estimation. In this section, we use a parametric bootstrap method to construct CIs for the unknown parameters β , η and λ . The bootstrap is a re-sampling technique for statistical inference. It is frequently used to estimate CIs. Also, it can be used to estimate bias and variance of an estimator. It has the advantage of computational ease especially for large sample sizes. We present the percentile bootstrap CIs (PBCIs) proposed by Efron [16]. The following steps can be proceeded to obtain bootstrap samples for the proposed method.

- (1) Using the original hybrid censored sample, $x_{(??)} \leq \dots \leq x_{(n_u)} \leq C_1$ and $y_{(??)} \leq \dots \leq y_{(n_a)} \leq C_1$ if $C_1 < C_2$ or $x_{(??)} \leq \dots \leq x_{(n_u)} \leq C_2$ and $y_{(??)} \leq \dots \leq y_{(n_a)} \leq C_2$ if $C_2 < C_1$, obtain $\hat{\beta}, \hat{\eta}$ and $\hat{\lambda}$.
- (2) Using the values of n_1 and n_2 , generate two independent samples of sizes n_1 and n_2 from Weibull distribution, $\underline{x}^* = (x_1^* < x_2^* < \dots < x_{n_1}^*)$ and $\underline{y}^* = (y_1^* < y_2^* < \dots < y_{n_2}^*)$.
- (3) As in step 1 based on \underline{x}^* and \underline{y}^* compute the bootstrap sample estimates of $\hat{\beta}, \hat{\eta}$ and $\hat{\lambda}$ say, $\hat{\beta}^*, \hat{\eta}^*$ and $\hat{\lambda}^*$.
- (4) Repeat the above steps 2 and 3 N ($=10,000$) times representing N different bootstrap samples.
- (5) Arrange all $\hat{\beta}^*, \hat{\eta}^*$ and $\hat{\lambda}^*$ in an ascending order to obtain the bootstrap sample $\hat{\varphi}_\ell^{*[1]}, \hat{\varphi}_\ell^{*[2]}, \dots, \hat{\varphi}_\ell^{*[N]}$, $\ell = 1, 2, 3$, where $\varphi_1^* = \beta^*, \varphi_2^* = \eta^*$ and $\varphi_3^* = \lambda^*$.

To obtain PBCIs, let $G(z) = P(\hat{\varphi}_\ell^* \leq z)$ be CDF of $\hat{\varphi}_\ell^*$. Define $\hat{\varphi}_{\ell boot}^* = G^{-1}(z)$ for given z . The approximate bootstrap $100(1 - \gamma)\%$ CI of $\hat{\varphi}_\ell^*$ is given by $(\hat{\varphi}_{\ell boot}^*(\frac{\gamma}{2}), \hat{\varphi}_{\ell boot}^*(1 - \frac{\gamma}{2}))$.

4. Simulation studies

In this section simulation studies are made to evaluate the performances of the MLEs in terms of their mean square errors (MSEs) for various choices of n , R and C_1 values. Also, the 95 % asymptotic confidence bounds based on the asymptotic distribution of the MLEs are constructed and their lengths are computed and presented with the associated coverage probabilities (CPs). For different hybrid censored data sets, the average values of the MSEs, confidence interval lengths (CILs) and CPs are calculated using 10,000 replications and the results are given in Tables 1-6. In each Table, the odd rows represent the results of the ML estimation for β , η and λ , respectively, while the even ones denote the results of the percentile bootstrap estimation (between brackets) for the three parameters respectively.

From Tables 1-6 some notes can be discovered concerning the two approaches as follows.

- (1) For fixed n and R , the MSEs decrease as C_1 increases.
- (2) For fixed n and C_1 , the MSEs decrease as R increases.
- (3) For fixed R and C_1 , the MSEs decrease as n increases.
- (4) For fixed R and C_1 , the CILs decrease as n increases.
- (5) For fixed n and C_1 , the CILs decrease as R increases.
- (6) For fixed n and R , the CILs decrease as C_1 increases.

Also, we observed that the computed CPs of the confidence bounds for each parameter are very close to the nominal level as n increases. The same pattern is noticed as R or C_1 increases. That is, the procedure is successfully working.

Now, when we compare between the two methods of estimation, it is observed that for relatively small and moderate sample sizes, percentile bootstrap method works better than the ML method. It provides smaller MSEs, narrower CILs with closest CPs to the nominal level. The method of bootstrap is recommended to use even for large samples for computational ease and high precision.

Moreover, point and 95 % confidence interval estimations for the survival function at mission times 3, 5, 7 and 10 are obtained using the two methods of estimation. The

estimations of the true survival are calculated via the following expressions:

$$(x) = \exp\{- (x/\hat{\eta})^{\hat{\beta}}\}, \text{ for items run under use condition,}$$

or

$$(y) = \exp\{- (\hat{\lambda}y/\hat{\eta})^{\hat{\beta}}\}, \text{ for items run under accelerated condition.}$$

As Soliman [30] shows, "estimation of the reliability function of some equipment is one of the main problems of reliability theory. In most practical applications and life-test experiments, the distributions with positive domain, e.g., Weibull, Burr-XII, Pareto, Beta, and Rayleigh, are quite appropriate models".

The estimation results of the true survival function are introduced in Tables 7 and 8. It can be observed that the percentile bootstrap method gives reliability estimations better than the ML method.

Table 1: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 1.5$, $\eta = 2$ and $\lambda = 2.5$ when $C_1 = 20$ and $n = 25$ ($n_1=12$, $n_2=13$).

R = 10	R = 15	R = 20
0.038, 1.235, 0.947 (0.026), (1.127), (0.948)	0.017, 0.992, 0.948 (0.014), (0.842), (0.949)	0.007, 0.851, 0.952 (0.004), (0.762), (0.951)
0.047, 1.786, 0.945 (0.031), (1.514), (0.946)	0.023, 1.415, 0.946 (0.018), (1.217), (0.947)	0.012, 1.194, 0.948 (0.009), (0.988), (0.949)
0.054, 1.911, 0.943 (0.041), (1.817), (0.944)	0.037, 1.549, 0.944 (0.031), (1.311), (0.946)	0.029, 1.352, 0.946 (0.018), (1.132), (0.948)

Table 2: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 1.5$, $\eta = 2$ and $\lambda = 2.5$ when $C_1 = 20$ and $n = 35$ ($n_1=17$, $n_2=18$).

R = 15	R = 20	R = 25
0.011, 0.715, 0.953 (0.008), (0.689), (0.951)	0.006, 0.661, 0.950 (0.004), (0.541), (0.950)	0.002, 0.526, 0.950 (0.001), (0.418), (0.950)
0.019, 1.218, 0.947 (0.015), (1.115), (0.948)	0.011, 0.907, 0.949 (0.007), (0.833), (0.950)	0.007, 0.789, 0.951 (0.003), (0.640), (0.950)
0.033, 1.355, 0.945 (0.026), (1.172), (0.947)	0.021, 1.141, 0.948 (0.013), (0.917), (0.949)	0.013, 0.911, 0.951 (0.009), (0.763), (0.950)

Table 3: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 0.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 30$ and $n = 25$ ($n_1=12$, $n_2=13$).

R = 10	R = 15	R = 20
0.021, 1.015, 0.948 (0.016), (0.985), (0.949)	0.015, 0.910, 0.949 (0.011), (0.852), (0.950)	0.005, 0.754, 0.951 (0.003), (0.669), (0.950)
0.032, 1.487, 0.946 (0.024), (1.311), (0.948)	0.019, 1.321, 0.947 (0.0014), (1.103), (0.949)	0.011, 0.988, 0.952 (0.005), (0.901), (0.951)
0.041, 1.802, 0.945 (0.036), (1.587), (0.947)	0.027, 1.463, 0.945 (0.021), (1.298), (0.947)	0.018, 1.076, 0.954 (0.0012), (0.992), (0.952)

Table 4: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 0.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 30$ and $n = 35$ ($n_1=17$, $n_2=18$).

R = 15	R = 20	R = 25
0.006, 0.411, 0.950 (0.004), (0.392), (0.950)	0.004, 0.286, 0.950 (0.002), (0.203), (0.950)	0.001, 0.197, 0.950 (0.002), (0.163), (0.950)
0.010, 0.762, 0.949 (0.007), (0.611), (0.949)	0.007, 0.531, 0.951 (0.004), (0.498), (0.950)	0.003, 0.312, 0.951 (0.002), (0.277), (0.950)
0.024, 1.117, 0.948 (0.019), (0.996), (0.949)	0.012, 0.820, 0.952 (0.008), (0.758), (0.951)	0.008, 0.524, 0.951 (0.005), (0.469), (0.950)

Table 5: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 1.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 35$ and $n = 50$ ($n_1=20$, $n_2=30$).

R = 15	R = 20	R = 25
0.003, 0.397, 0.950 (0.002), (0.364), (0.950)	0.002, 0.254, 0.950 (0.001), (0.226), (0.950)	0.001, 0.182, 0.950 (0.001), (0.165), (0.950)
0.007, 0.748, 0.949 (0.006), (0.711), (0.950)	0.005, 0.503, 0.950 (0.003), (0.489), (0.950)	0.002, 0.292, 0.950 (0.001), (0.203), (0.950)
0.019, 1.001, 0.949 (0.014), (0.964), (0.950)	0.007, 0.611, 0.950 (0.004), (0.522), (0.950)	0.003, 0.479, 0.950 (0.002), (0.445), (0.950)

Table 6: The results of MSEs, CILs and CPs using the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 0.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 35$ and $n = 50$ ($n_1=20$, $n_2=30$).

R = 15	R = 20	R = 25
0.004, 0.402, 0.950 (0.003), (0.387), (0.950)	0.003, 0.261, 0.950 (0.002), (0.207), (0.950)	0.001, 0.182, 0.950 (0.001), (0.147), (0.950)
0.008, 0.751, 0.949 (0.006), (0.620), (0.950)	0.006, 0.508, 0.950 (0.004), (0.433), (0.950)	0.003, 0.304, 0.950 (0.002), (0.287), (0.950)
0.020, 1.004, 0.949 (0.016), (0.981), (0.950)	0.009, 0.627, 0.951 (0.004), (0.489), (0.950)	0.004, 0.481, 0.950 (0.003), (0.423), (0.950)

Table 7: Average values of point and interval estimations for the survival function at different mission times 3, 5, 7 and 10 according to the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 1.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 35$ and $n = 50$ ($n_1=20$, $n_2=30$).

R = 15	R = 20	R = 25
0.746, 0.688, 0.879 (0.797), (0.708, 0.893)	0.767, 0.722, 0.896 (0.812), (0.742, 0.898)	0.782, 0.744, 0.920 (0.831), (0.758, 0.932)
0.731, 0.678, 0.864 (0.782), (0.711, 0.883)	0.752, 0.694, 0.887 (0.801), (0.734, 0.883)	0.771, 0.721, 0.895 (0.822), (0.744, 0.912)
0.694, 0.647, 0.822 (0.741), (0.683, 0.852)	0.746, 0.688, 0.873 (0.789), (0.723, 0.876)	0.766, 0.706, 0.889 (0.804), (0.723, 0.896)
0.658, 0.625, 0.794 (0.719), (0.667, 0.814)	0.728, 0.671, 0.868 (0.763), (0.692, 0.871)	0.748, 0.697, 0.874 (0.781), (0.711, 0.883)

Table 8: Average values of point and interval estimations for the survival function at different mission times 3, 5, 7 and 10 according to the methods of ML and percentile bootstrap (between brackets), respectively, with true parameter values set at $\beta = 0.5$, $\eta = 0.7$ and $\lambda = 3$ when $C_1 = 35$ and $n = 50$ ($n_1=20$, $n_2=30$).

R = 15	R = 20	R = 25
0.741, 0.682, 0.873 (0.789), (0.702, 0.882)	0.760, 0.704, 0.882 (0.802), (0.728, 0.891)	0.773, 0.728, 0.887 (0.825), (0.746, 0.904)
0.728, 0.673, 0.859 (0.776), (0.694, 0.871)	0.747, 0.694, 0.870 (0.788), (0.709, 0.879)	0.762, 0.694, 0.877 (0.815), (0.738, 0.894)
0.685, 0.638, 0.804 (0.733), (0.675, 0.838)	0.738, 0.676, 0.856 (0.778), (0.687, 0.864)	0.743, 0.682, 0.863 (0.791), (0.692, 0.871)
0.643, 0.611, 0.781 (0.692), (0.642, 0.798)	0.713, 0.664, 0.843 (0.748), (0.663, 0.832)	0.728, 0.672, 0.841 (0.767), (0.687, 0.849)

5. A demonstrative example

To demonstrate the proposed methodology, a demonstrative example via hybrid censored data set from Weibull distribution is considered. We use $n = 75$ ($n_1=25$, $n_2=50$), $\beta = 2$, $\eta = 2.5$ and $\lambda = 3$ when $C_1 = 40$ and $R = 20$. The number of failures observed at use and accelerated conditions are $n_u=11$ and $n_a=39$, respectively, with censored items $n_c=25$. The MSEs associated with the MLEs of the parameters β , η and λ are 0.002, 0.003 and 0.005, respectively, while those associated with the percentile bootstrap estimation are respectively 0.001, 0.002 and 0.004. In addition, a 95% CILs of the model parameters β , η and λ using the two approaches ML and PB are 0.241, 0.462, 0.581 and 0.212, 0.409, 0.523, respectively. Moreover, the CPs associated with ML and PB are respectively 0.948, 0.947, 0.949 and 0.950, 0.951, 0.950. Finally, the point and interval estimations for the survival function at a mission time 6 according to the methods of ML and PB (between brackets) are, respectively, 0.749, 0.691, 0.883 and (0.795), (0.734, 0.887).

6. Conclusion

In this article, the likelihood and percentile bootstrap estimation methods has been applied to the CSPALTS model parameters assuming Weibull distribution under hybrid censoring. The performance of the estimators has been examined in terms of their MSEs via simulation studies for the two methods of estimation. Also, the CILs of the model parameters have been obtained as well as their CPs. It is observed that for small and moderate sample sizes, percentile bootstrap method works better than the approximate

method. It provides smaller MSEs, narrower CILs with closest CPs to the nominal level. The method of bootstrap is recommended to use even for large samples for computational ease and high precision. Finally, an illustrative example has been given.

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