

Probability for transition of business cycle and pricing of options with correlated credit risk

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Abstract

In this paper we propose the transition probability of business cycle for the pricing of options with credit risk. In order to describe business cycles of markets, the regime switching model is considered. We provide the probability density functions of the occupation time of the high volatility regime via Laplace transforms. Using these functions we derive the analytic valuation formulae for options with correlated credit risk and business cycle. We also illustrate the important properties of options with numerical graphs.

Keywords: Business cycle, Option pricing, Credit risk, Occupation time.

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1. Introduction

In this paper we study the business cycle model for valuing options with credit risk. It is assumed that the financial event occurs at some time in the market. This should lead to the transition of volatilities of both the underlying stock and the option issuer's asset. The financial events are often modeled by the regime switching model to capture the changes of the market environment by the unanticipated events (see, e.g., Hamilton [8], Bollen [2], Buffington and Elliott [4], Boyle and Draviam [3], Zhang et al. [15], Zhu et al. [16], Elliott et al. [7]). Based on this approach, we model the business cycle by a continuous-time two-state regime switching.

The traditional option pricing based on Black-Scholes model [1] has been used the assumption that options have no default risk. However, there exists the default risk of the option writers in the over-the-counter (OTC) markets. OTC markets have grown rapidly in size in recent years. That is, in the OTC markets, the counterparty default risk is very important and should be considered for pricing of options.

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Johnson and Stulz [10] proposed the valuation of options with credit risk, which is called *Vulnerable option*. In their model, the options depend on the liabilities of the option issuers. If the default of the counterparty occurs at the maturity, the option holder takes all assets of the counterparty. Their model also considers the correlation between the option issuer's asset and the underlying asset. Klein [11] developed the result of Johnson and Stulz [10] by allowing for the proportional recovery of nominal claims in default. Klein and Inglis [12] dealt with options with credit risk employing the stochastic interest model of Vasicek [14]. Hui et al. [9] extended a vulnerable option valuation model that incorporates a stochastic default barrier which reflects the expected leverage level of the option issuer. Chang and Hung [5] provided analytic formulae to evaluate vulnerable American options under the assumptions of Klein's model. In the recent study, Shiu et al [13] proposed a closed-form approximation for valuing European basket warrants with credit risk. However, none of the studies consider options with credit risk under the varying market environment.

The rest of the paper is organized as follows. Section 2 presents the business cycle modeling by using regime switching. In particular, we provide the probability density function of the occupation time of high volatility in a given time period. Section 3 gives the formulae for the arbitrage-free price of options with credit risk as integral under our model. Finally, we provide the numerical examples with various graphs to show the properties of option prices in section 4.

2. The model

We assume that a given filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ satisfies the usual conditions, where Q presents a risk neutral measure[‡] and the filtration $\{\mathcal{F}_t\}$ is generated by Brownian motions and two independent Poisson point processes. Based on the settings of Klein [11], we model the correlated evolutions of the option issuer's asset value process V_t and the underlying stock process S_t as the following:

$$(2.1) \quad dS_t = rS_t dt + \sigma_1(t)S_t dW_t^1,$$

$$(2.2) \quad dV_t = rV_t dt + \sigma_2(t)V_t dW_t^2,$$

where r is a riskless interest rate, $\sigma_i(t)$, $(i = 1, 2)$ are the time-varying volatilities of each process and W_t^i , $(i = 1, 2)$ are standard Brownian motions under a risk neutral measure Q with correlation ρ . Here, we model the business cycle by the volatilities with two regimes.

We refer to two regimes as the high volatility and the low volatility. The high volatility region presents the economic contraction period when the market is stressed by some financial event. On the other hands, the low volatility region presents the economic expansion period, where the market has the stable economic environment. For modeling these, we assume that $\sigma_1(t)$ and $\sigma_2(t)$ are governed by two independent Poisson point processes \mathcal{P}_1 and \mathcal{P}_0 with a two state continuous-time Markov chain.

Let \mathcal{P}_1 and \mathcal{P}_0 be two independent Poisson point processes with intensity λ_1 and λ_0 , respectively. If we are in the high regime, issuer's asset's volatility is $\sigma_1 + \delta_1$. We observe the high volatility Poisson point processes \mathcal{P}_1 . If we get a signal from this high volatility point processes, issuer's asset's volatility is changed from $\sigma_1 + \delta_1$ to σ_1 . If we are in the low regime, issuer's asset's volatility is σ_1 . We observe the low volatility Poisson point processes \mathcal{P}_0 . If we get a signal from this low volatility point processes, issuer's asset's volatility is changed from σ_1 to $\sigma_1 + \delta_1$. Surely, the volatility $\sigma_2(t)$ of underlying stock is

[‡]Elliott et al. [6] show the existence of an equivalent martingale measure in the regime switching model. So, we can get the risk-neutral valuation under our model.

affected by the same Poisson point processes as well. By \mathcal{P}_0 and \mathcal{P}_1 , the volatility $\sigma_2(t)$ moves between σ_2 and $\sigma_2 + \delta_2$.

Let U_t be the occupation time in the high regime from 0 to option's maturity T . Then it is defined by

$$(2.3) \quad U_t := \int_0^t \varepsilon(s) ds$$

$$\text{with } \varepsilon(t) = \begin{cases} 0 & \text{Economic expansion regime } (\sigma_i(t) = \sigma_i), \\ 1 & \text{Economic contraction regime } (\sigma_i(t) = \sigma_i + \delta_i), \end{cases} \quad i = 1, 2,$$

where $\varepsilon(t)$ is the random variable with two regimes 0 (= *Low volatility*) and 1 (= *High volatility*). The following Proposition gives the probability density function of U_t conditioned on $\varepsilon(0)$.

2.1. Proposition. *For a given time T , the probability density functions of U_t conditioned on $\varepsilon(0)$ are given by*

$$(2.4) \quad \begin{aligned} P(U_t = u | \varepsilon(0) = 1) &:= f_1(u; T) = e^{-\lambda_1 T} \delta_0(T - u) + \lambda_1 e^{-(\lambda_1 - \lambda_0)u - \lambda_0 T} \\ &\times [{}_0F_1(2; \lambda_0 \lambda_1 u(T - u)) \lambda_0 u + {}_0F_1(1; \lambda_0 \lambda_1 u(T - u))], \quad 0 < u < T \end{aligned}$$

$$(2.5) \quad \begin{aligned} P(U_t = u | \varepsilon(0) = 0) &:= f_0(u; T) = e^{-\lambda_0 T} \delta_0(u) + \lambda_0 e^{-(\lambda_1 - \lambda_0)u - \lambda_0 T} \\ &\times [{}_0F_1(1; \lambda_0 \lambda_1 u(T - u)) + {}_0F_1(1; \lambda_0 \lambda_1 u(T - u)) \lambda_1 - \lambda_1], \quad 0 < u < T \end{aligned}$$

where ${}_0F_1(a; z)$ is the generalized hypergeometric function defined by

$${}_0F_1(a; z) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!},$$

with the rising factorial $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1)$. And

$$\delta_x(y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

Proof. Let $f_j(u; T)$ be the probability density function of U_t over $[0, T]$. Then, by the Laplace transform,

$$(2.6) \quad m_j(r; T) := \mathbb{E}[e^{-rU_T} | \varepsilon(0) = j] = \mathcal{L}_r(f_j(\cdot; T)).$$

We also consider the two cases $\tau_j > T$ and $\tau_j < T$, where τ_j is the random time of the leaving state j satisfying $P(\tau_j > t) = e^{-\lambda_j t}$ for each state $j \in \{0, 1\}$. We then have

$$(2.7) \quad m_1(r; T) = e^{-rT} e^{-\lambda_1 T} + \int_0^T e^{-\lambda_1 u} \lambda_1 m_0(r; T - u) e^{-ru} du,$$

$$(2.8) \quad m_0(r; T) = e^{-\lambda_0 T} + \int_0^T e^{-\lambda_0 u} \lambda_0 m_1(r; T - u) du.$$

Taking the Laplace transform of the above equations gives

$$(2.9) \quad \widehat{m}_j(r; s) := \mathcal{L}_s(m_j(r; \cdot)) = \mathcal{L}_s[\mathcal{L}_r(f_j(\cdot; T))(r; \cdot)].$$

Then we have

$$(2.10) \quad \widehat{m}_1(r; s) = \frac{s + \lambda_0 + \lambda_1}{rs + r\lambda_0 + s^2 + s\lambda_1 + s\lambda_0},$$

$$(2.11) \quad \widehat{m}_0(r; s) = \frac{r + s + \lambda_0 + \lambda_1}{rs + r\lambda_0 + s^2 + s\lambda_1 + s\lambda_0}.$$

The equation (2.10) is equal to

$$\begin{aligned}
& \int_0^\infty e^{-rx} \frac{s + \lambda_0 + \lambda_1}{s + \lambda_0} e^{-\frac{s(s+\lambda_0+\lambda_1)}{s+\lambda_0}x} dx \\
&= \int_0^\infty e^{-rx} e^{-(s+\lambda_1)x} \left(1 + \frac{\lambda_1}{s + \lambda_0}\right) \sum_{n=0}^\infty \left(\frac{\lambda_0 \lambda_1 x}{s + \lambda_0}\right)^n \frac{1}{n!} dx \\
&= \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} e^{-\lambda_1 x} \\
&\quad \times \left(\delta_x(y) + \sum_{n=1}^\infty \frac{(x\lambda_0\lambda_1)^n e^{-\lambda_0(y-x)} (y-x)^{n-1}}{n!(n-1)!} 1_{\{x < y\}} \right) dy dx \\
&\quad + \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} e^{-\lambda_1 x} \sum_{n=0}^\infty \frac{\lambda_1 (x\lambda_0\lambda_1)^n e^{-\lambda_0(y-x)} (y-x)^n}{n!n!} 1_{\{x < y\}} dy dx \\
&= \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} \left(e^{-\lambda_1 x} \delta_x(y) + \lambda_1 e^{-(\lambda_1 - \lambda_0)x - \lambda_0 y} \right. \\
&\quad \left. \times [{}_0F_1(2; \lambda_0 \lambda_1 x(y-x)) \lambda_0 x + {}_0F_1(1; \lambda_0 \lambda_1 x(y-x))] 1_{\{x < y\}} \right) dy dx.
\end{aligned}$$

Substituting (u, T) for (x, y) yields the equation (2.4). Similarly, from the equation (2.11), we have

$$\begin{aligned}
& \int_0^\infty e^{-rx} \left(\frac{\delta_0(x)}{s + \lambda_0} + \frac{\lambda_0(s + \lambda_0 + \lambda_1)}{(s + \lambda_0)^2} e^{-\frac{s(s+\lambda_0+\lambda_1)}{s+\lambda_0}x} \right) dx \\
&= \int_0^\infty e^{-rx} \delta_0(x) \int_0^\infty e^{-sy} e^{-\lambda_0 y} dy dx \\
&\quad + \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} e^{-\lambda_1 x} \sum_{n=0}^\infty \frac{\lambda_0 (x\lambda_0\lambda_1)^n e^{-\lambda_0(y-x)} (y-x)^n}{n!n!} 1_{\{x < y\}} dy dx \\
&\quad + \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} e^{-\lambda_1 x} \\
&\quad \times \left(\sum_{n=0}^\infty \frac{\lambda_0 \lambda_1 (x\lambda_0\lambda_1)^n e^{-\lambda_0(y-x)} (y-x)^n}{n!n!} - \lambda_0 \lambda_1 e^{-\lambda_0(y-x)} \right) 1_{\{x < y\}} dy dx \\
&= \int_0^\infty e^{-rx} \int_0^\infty e^{-sy} \left(e^{-\lambda_0 y} \delta_0(x) + \lambda_0 e^{-(\lambda_1 - \lambda_0)x - \lambda_0 y} [{}_0F_1(1; \lambda_0 \lambda_1 x(y-x)) \right. \\
&\quad \left. + {}_0F_1(1; \lambda_0 \lambda_1 x(y-x)) \lambda_1 - \lambda_1] 1_{\{x < y\}} \right) dy dx.
\end{aligned}$$

In a same way, substituting (u, T) for (x, y) in above equation completes the proof. \square

For given $U_t = u$ we also can obtain the following solutions of equation (2.1) and equation (2.2), respectively,

$$(2.12) \quad S_t = S_0 e^{(rt - \frac{1}{2}\eta_1(u,t) + \int_0^t \sigma_1(s) dW_s^1)}, V_t = V_0 e^{(rt - \frac{1}{2}\eta_2(u,t) + \int_0^t \sigma_2(s) dW_s^2)},$$

where $\eta_i(u, t) = \sigma_i^2 t + (2\sigma_i \delta_i + \delta_i^2)u$, $i = 1, 2$.

In order to handle the above processes, we need to verify the properties of

$$J_1(t) := \int_0^t \sigma_1(s) dW_s^1, J_2(t) := \int_0^t \sigma_2(s) dW_s^2.$$

If U_i is known, we can find the properties of $J_i(t)$, ($i = 1, 2$). The results are presented by the following lemmas.

2.2. Lemma. *Conditioned on $U_t = u \leq t$, $J_i(t)$ has the normal distribution with mean 0 and variance $\eta_i(u, t)$, for each $i \in \{1, 2\}$.*

Proof. Let us consider the decomposition of $J_1(t)$ as

$$J_1(t) = \delta_1 \int_0^t \varepsilon(s) dW_s^1 + \sigma_1 W_t^1 := \delta_1 X_1(t) + \sigma_1 W_t^1.$$

For some k , $k\varepsilon(t)$ is a bounded simple function. So, the Novikov condition of $\mathbf{E}[e^{\frac{1}{2} \int_0^t (k\varepsilon(s))^2 ds}]$ is satisfied and $e^{\int_0^t k\varepsilon(s) dW_s^1 - \frac{u^2}{2} \int_0^t \varepsilon(s)^2 ds}$ is a martingale for given $U_t = u$. Therefore,

$$\begin{aligned} \mathbf{E}[e^{\int_0^t k\varepsilon(s) dW_s^1 - \frac{k^2}{2} \int_0^t \varepsilon(s)^2 ds} | U_t = u] &= \mathbf{E}[e^{\int_0^t k\varepsilon(s) dW_s^1 - \frac{k^2}{2} u} | U_t = u] \\ &= \mathbf{E}[e^{kX_1(t) - \frac{k^2}{2} u} | U_t = u] = 1. \end{aligned}$$

For given $U_t = u$, since $\mathbf{E}[e^{kX_1(t)}] = e^{\frac{k^2}{2} u}$, $X_1(t)$ has the normal distribution with mean 0 and variance u . We also can calculate the covariance of $X_1(t)$ and W_t^1 as following:

$$\begin{aligned} &\mathbf{E}[X_1(t)W_t^1 | U_t = u] \\ &= \mathbf{E} \left[\lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{(k-1)t/n}^{kt/n} 1_{\{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}\}} dW_s^1 W_t^1 | U_t = u \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\left(\sum_{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}} W_{\frac{kt}{n}}^1 - W_{\frac{(k-1)t}{n}}^1 \right) W_t^1 | U_t = u \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\left(\sum_{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}} W_{\frac{kt}{n}}^1 W_t^1 \right) \right. \\ &\quad \left. - \left(\sum_{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}} W_{\frac{(k-1)t}{n}}^1 W_t^1 \right) | U_t = u \right] \\ &= \lim_{n \rightarrow \infty} \mathbf{E} \left[\sum_{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}} \frac{t}{n} | U_t = u \right] \\ &= \mathbf{E} \left[\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{t}{n} 1_{\{\varepsilon(s)=1, \frac{(k-1)t}{n} \leq s \leq \frac{kt}{n}\}} | U_t = u \right] = u. \end{aligned}$$

Hence, for given $U_t = u$, $J_1(t)$ has the normal distribution with mean 0 and variance $\sigma_1^2 t + (2\sigma_1 \delta_1 + \delta_1^2)u$. In a same way, $J_2(t)$ has the the normal distribution with mean 0 and variance $\sigma_2^2 t + (2\sigma_2 \delta_2 + \delta_2^2)u$ as well. \square

2.3. Lemma. *Conditioned on $U_t = u \leq t$, the correlation of $J_1(t)$ and $J_2(t)$ is given by*

$$\rho_{12}(u, t) = \frac{[(\sigma_1 \delta_2 + \sigma_2 \delta_1 + \delta_1 \delta_2)u + \sigma_1 \sigma_2 t] \rho}{\sqrt{\eta_1(u, t) \eta_2(u, t)}}.$$

Proof. From the decomposition in Lemma 2.2 and $dW_t^1 dW_t^2 = \rho dt$, the covariance $J_1(t)$ and $J_2(t)$ is given by

$$\begin{aligned} \text{Cov}(J_1(t), J_2(t)) &= \mathbf{E}[(\delta_1 X_1(t) + \sigma_1 W_t^1)(\delta_2 X_2(t) + \sigma_2 W_t^2) | U_t = u] \\ &= \delta_1 \delta_2 \mathbf{E}[X_1(t)X_2(t) | U_t = u] + \sigma_1 \delta_2 \mathbf{E}[W_t^1 X_2(t) | U_t = u] \\ &\quad + \delta_1 \sigma_2 \mathbf{E}[X_1(t)W_t^2 | U_t = u] + \sigma_1 \sigma_2 \mathbf{E}[W_t^1 W_t^2 | U_t = u] \\ &= \delta_1 \delta_2 \rho u + \sigma_1 \delta_2 \rho u + \delta_1 \sigma_2 \rho u + \sigma_1 \sigma_2 \rho t. \end{aligned}$$

Therefore, the correlation of $J_1(t)$ and $J_2(t)$ is obtained by Lemma 2.2. \square

3. Valuation of options with correlated credit risk

In this section we provide the formula of the European call option with credit risk in a business cycle environment. As in Klein [11], we assume that if default or bankruptcy of the option issuer occurs, the option issuer's asset is immediately liquidated and the scrap value at T is $(1 - \alpha)V_T D^{-1}(S_T - K)^+$, where D is a constant value of the option issuer's liabilities and α is a constant showing the ratio of bankruptcy costs of the issuer's asset. We also assume that the option issuer declare default only if $V_T < D$. Then, from the equation (2.12), the discounted expected value of the call option with maturity T is given by

$$(3.1) \quad C(T) = e^{-rT} \mathbf{E}^Q[(S_T - K)^+ 1_{\{V_T \geq D\}} + (1 - \alpha)V_T D^{-1}(S_T - K)^+ 1_{\{V_T < D\}}],$$

where K is the strike price and $0 \leq \alpha \leq 1$. From this equation, we now provide the valuation formula for a option with credit risk and business cycle by applying the Girsanov's theorem repeatedly.

For notational simplicity, we rewrite notations as

$$\eta_1(u) := \eta_1(u, T), \eta_2(u) := \eta_2(u, T), \hat{\rho}(u) := \rho_{12}(u, T), \delta_T := (1 - \alpha)V_T D^{-1}.$$

3.1. Proposition. *Let C_j be the arbitrage free price of a call option with credit risk and initial state j ($j = 0, 1$). Then, the value $C_j(T)$ at time 0 of the option with maturity T is given by*

$$(3.2) \quad C_j(T) = \int_0^T v(u) f_j(u; T) du + \delta_0(j) e^{-\lambda_0 T} v(0) + \delta_1(j) e^{-\lambda_1 T} v(T),$$

where $f_j(u; T)$ ($j = 0, 1$) is defined in Proposition 1. And

$$\begin{aligned} v(u) = & S_0 \Phi_2(a_1(u), a_2(u), \hat{\rho}(u)) - K e^{-rT} \Phi_2(b_1(u), b_2(u), \hat{\rho}(u)) \\ & + S_0 \delta_0 e^{rT + \hat{\rho}(u) \sqrt{\eta_1(u) \eta_2(u)}} \Phi_2(c_1(u), c_2(u), -\hat{\rho}(u)) - K \delta_0 \Phi_2(d_1(u), d_2(u), -\hat{\rho}(u)), \end{aligned}$$

where Φ_2 is the bivariate standard normal cumulative density function and

$$\begin{aligned} a_1(u) &= \frac{\ln(S_0/K) + rT + \frac{1}{2}\eta_1(u)}{\sqrt{\eta_1(u)}}, \\ a_2(u) &= \frac{\ln(V_0/D) + rT - \frac{1}{2}\eta_2(u) + \hat{\rho}(u) \sqrt{\eta_1(u) \eta_2(u)}}{\sqrt{\eta_2(u)}}, \\ b_1(u) &= \frac{\ln(S_0/K) + rT - \frac{1}{2}\eta_1(u)}{\sqrt{\eta_1(u)}}, \\ b_2(u) &= \frac{\ln(V_0/D) + rT - \frac{1}{2}\eta_2(u)}{\sqrt{\eta_2(u)}}, \\ c_1(u) &= \frac{\ln(S_0/K) + rT + \frac{1}{2}\eta_1(u) + \hat{\rho}(u) \sqrt{\eta_1(u) \eta_2(u)}}{\sqrt{\eta_1(u)}}, \\ c_2(u) &= -\frac{\ln(V_0/D) + rT + \frac{1}{2}\eta_2(u) + \hat{\rho}(u) \sqrt{\eta_1(u) \eta_2(u)}}{\sqrt{\eta_2(u)}}, \\ d_1(u) &= \frac{\ln(S_0/K) + rT - \frac{1}{2}\eta_1(u) + \hat{\rho}(u) \sqrt{\eta_1(u) \eta_2(u)}}{\sqrt{\eta_1(u)}}, \\ d_2(u) &= -\frac{\ln(V_0/D) + rT + \frac{1}{2}\eta_2(u)}{\sqrt{\eta_2(u)}}. \end{aligned}$$

Proof. From equation (3.1), the credit-risky call option value $C_j(T)$ at time 0 with maturity T and an initial state j is given by

$$\begin{aligned}
C_j(T) &= e^{-rT} \mathbf{E}^Q[\mathbf{E}^Q[(S_T - K)^+ 1_{\{V_T \geq D\}} + \delta_T (S_T - K)^+ 1_{\{V_T < D\}} | U_t = u]] \\
&= e^{-rT} \int_0^T \mathbf{E}^Q[(S_T - K)^+ (1_{\{V_T \geq D\}} + \delta_T 1_{\{V_T < D\}}) | U_t = u] f_j(u; T) du \\
&\quad + \delta_0(j) e^{-(r+\lambda_0)T} \mathbf{E}^Q[(S_T - K)^+ (1_{\{V_T \geq D\}} + \delta_T 1_{\{V_T < D\}}) | U_t = 0] \\
(3.3) \quad &\quad + \delta_1(j) e^{-(r+\lambda_1)T} \mathbf{E}^Q[(S_T - K)^+ (1_{\{V_T \geq D\}} + \delta_T 1_{\{V_T < D\}}) | U_t = T].
\end{aligned}$$

Let us consider the first term of the equation (3.3). For a fixed u , the conditional expectation in the integral is divided into four terms as

$$\begin{aligned}
&e^{-rT} \mathbf{E}^Q[(S_T - K)^+ (1_{\{V_T \geq D\}} + \delta_T 1_{\{V_T < D\}}) | U_t = u] \\
&= e^{-rT} \mathbf{E}^Q[S_T 1_{\{S_T > K, V_T \geq D\}} | U_T = u] - e^{-rT} \mathbf{E}^Q[K 1_{\{S_T > K, V_T \geq D\}} | U_T = u] \\
&\quad + e^{-rT} \mathbf{E}^Q[S_T \delta_T 1_{\{S_T > K\}} 1_{\{V_T < D\}} | U_T = u] - e^{-rT} \mathbf{E}^Q[K \delta_T 1_{\{S_T > K, V_T < D\}} | U_T = u] \\
&:= I_1 - I_2 + I_3 - I_4.
\end{aligned}$$

Under the measure Q , the first term I_1 can be expressed as

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_0 e^{-\frac{1}{2} \eta_1(u) + \sqrt{\eta_1(u)} z_1} 1_{\{S_T > K\}} 1_{\{V_T \geq D\}} \\
(3.4) \quad &\quad \times \frac{1}{2\pi \sqrt{1 - \hat{\rho}(u)}} e^{-\frac{1}{2(1 - \hat{\rho}(u))} (z_1^2 - 2\hat{\rho}(u) z_1 z_2 + z_2^2)} dz_1 dz_2,
\end{aligned}$$

where $z_1 = J_T^1 / \sqrt{\eta_1(u)}$ and $z_2 = J_T^2 / \sqrt{\eta_2(u)}$ are the standard normal variables with correlation $\hat{\rho}(u)$. Then, by the change of variables with $\tilde{z}_1 = z_1 - \sqrt{\eta_1(u)}$, $\tilde{z}_2 = z_2 - \hat{\rho}(u) \sqrt{\eta_1(u)}$, we have

$$\begin{aligned}
(3.5) \quad I_1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_0 1_{\{S_T > K\}} 1_{\{V_T \geq D\}} \\
&\quad \times \frac{1}{2\pi \sqrt{1 - \hat{\rho}(u)}} e^{-\frac{1}{2(1 - \hat{\rho}(u))} (\tilde{z}_1^2 - 2\hat{\rho}(u) \tilde{z}_1 \tilde{z}_2 + \tilde{z}_2^2)} d\tilde{z}_1 d\tilde{z}_2.
\end{aligned}$$

Let \tilde{Q} be the new equivalent probability measure defined by

$$(3.6) \quad \frac{d\tilde{Q}}{dQ} = \exp\left(\int_0^T \theta(s) dW_s - \frac{1}{2} \int_0^T |\theta(s)|^2 ds\right),$$

where W is vector in R^2 and $\theta(s) = (\sigma_1(s), \hat{\rho}(u) \sigma_1(s))$. Then, by Girsanov's theorem,

$$\begin{pmatrix} d\tilde{W}_t^1 \\ d\tilde{W}_t^2 \end{pmatrix} = \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} - \theta(t) dt$$

is a R^2 -valued standard Brownian motion under the equivalent measure \tilde{Q} .

We consider the equation (3.5) under the measure \tilde{Q} . Then, by applying Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
I_1 &= \mathbf{E}^{\tilde{Q}}[S_0 1_{\{S_T > K, V_T > D\}} | U_T = u] \\
&= S_0 \tilde{P} \left(S_0 e^{(rT - \frac{1}{2}\eta_1(u) + \int_0^T \sigma_1(s) dW_s^1)} > K, V_0 e^{(rT - \frac{1}{2}\eta_2(u) + \int_0^T \sigma_2(s) dW_s^2)} > D \right) \\
&= S_0 \tilde{P} \left(\tilde{J}_T^1 > - \left(\ln \frac{S_0}{K} + rT + \frac{1}{2}\eta_1(u) \right), \right. \\
&\quad \left. \tilde{J}_T^2 > - \left(\ln \frac{V_0}{D} + rT - \frac{1}{2}\eta_2(u) + \hat{\rho}(u) \sqrt{\eta_1(u)\eta_2(u)} \right) \right) \\
(3.7) &= S_0 \Phi_2(a_1(u), a_2(u), \hat{\rho}(u)).
\end{aligned}$$

where $\tilde{J}_T^1 = \int_0^T \sigma_1(s) d\tilde{W}_s^1$ and $\tilde{J}_T^2 = \int_0^T \sigma_2(s) d\tilde{W}_s^2$.

In a similar way, without the change of measure, I_2 can be found.

For the evaluation I_3 , we change the variables as $\tilde{z}_1 = z_1 - \sqrt{\eta_1(u)} - \hat{\rho}(u)\sqrt{\eta_2(u)}$, $\tilde{z}_2 = z_2 - \sqrt{\eta_2(u)} - \hat{\rho}(u)\sqrt{\eta_1(u)}$. And, define the equivalent measure by $\frac{d\tilde{Q}}{dQ} = \exp\left(\int_0^T \theta(s) dW_s - \frac{1}{2} \int_0^T |\theta(s)|^2 ds\right)$, where $\theta(s) = (\sigma_1(s) + \hat{\rho}(u)\sigma_2(s), \sigma_2(s) + \hat{\rho}(u)\sigma_1(s))$. Then, by Girsanov's theorem, we have

$$\begin{aligned}
I_3 &= \mathbf{E}^{\tilde{Q}}[e^{rT} S_0 \delta_0 e^{\hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)}} 1_{\{S_T > K, V_T < D\}} | U_T = u] \\
&= e^{rT} S_0 \delta_0 e^{\hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)}} \\
&\quad \times \tilde{P} \left(\tilde{J}_T^1 > - \left(\ln \frac{S_0}{K} + rT + \frac{1}{2}\eta_1(u) + \hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)} \right), \right. \\
&\quad \left. \tilde{J}_T^2 > \left(\ln \frac{V_0}{D} + rT + \frac{1}{2}\eta_2(u) + \hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)} \right) \right) \\
(3.8) &= S_0 \delta_0 e^{rT + \hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)}} \Phi_2(c_1(u), c_2(u), -\hat{\rho}(u)).
\end{aligned}$$

Again from the Radon-Nikodym derivative (3.6) that allows the change of probability measure, we change the measure with $\theta(s) = (\hat{\rho}(u)\sigma_2(s), \sigma_2(s))^T$. Then, under an equivalent measure \tilde{Q} , I_4 is evaluated as

$$\begin{aligned}
I_4 &= K \delta_0 \mathbf{E}^{\tilde{Q}}[1_{\{S_T > K, V_T < D\}} | U_T = u] \\
(3.9) &= K \delta_0 \Phi_2(d_1(u), d_2(u), -\hat{\rho}(u)).
\end{aligned}$$

Also one can obtain the second term and the third term of the equation (3.3) from above results. This completes the proof. \square

In a similar way, the following Proposition provides the price of the put option with credit risk.

3.2. Proposition. *Let P_j be the arbitrage free price of a put option with credit risk and initial state j ($j = 0, 1$). Then, the value $P_j(T)$ at time 0 of the option with maturity T is given by*

$$(3.10) \quad P_j(T) = \int_0^T v(u) f_j(u; T) du + \delta_0(j) e^{-\lambda_0 T} v(0) + \delta_1(j) e^{-\lambda_1 T} v(T),$$

where

$$\begin{aligned}
v(u) &= -S_0 \Phi_2(-a_1(u), a_2(u), -\hat{\rho}(u)) + K e^{-rT} \Phi_2(-b_1(u), b_2(u), -\hat{\rho}(u)) \\
&\quad - S_0 \delta_0 e^{rT + \hat{\rho}(u)\sqrt{\eta_1(u)\eta_2(u)}} \Phi_2(-c_1(u), c_2(u), \hat{\rho}(u)) + K \delta_0 \Phi_2(-d_1(u), d_2(u), \hat{\rho}(u)).
\end{aligned}$$

Here, all parameters are given in Proposition 3.1.

4. Numerical example

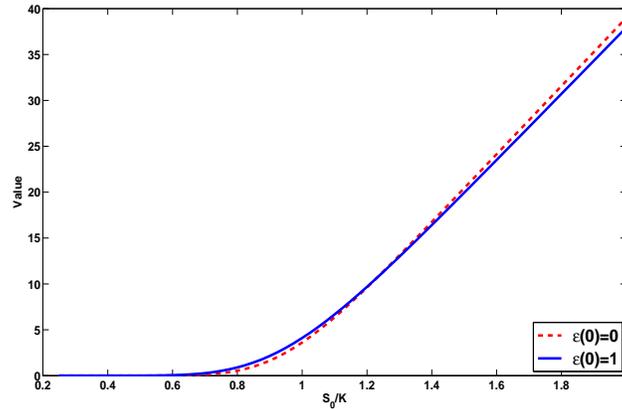


Figure 1. Vulnerable call value for different moneyness (S_0/K) and $\varepsilon(0)$

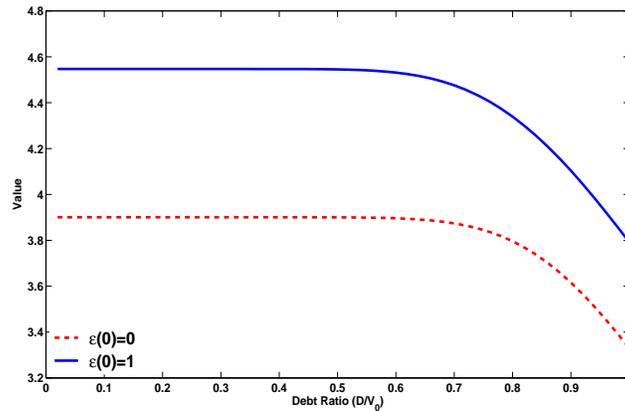


Figure 2. Vulnerable call value for different debt ratio (D/V_0) and $\varepsilon(0)$

In the previous section, we provide the option formulae represented as an integral form under our model. In order to calculate these option formulae, we employ the Gauss-Legendre quadrature as a numerical approximation method. Based on the values reported by Boyle and Draviam [3] and Klein and Inglis [12], we use the following parameters unless stated otherwise: $S_0 = K = 40$, $V_0 = 100$, $D = 90$, $r = 0.05$, $T = 1$, $\alpha = 0.25$, $\rho = 0$, $\sigma_1 = \sigma_2 = 0.15$, $\delta_1 = \delta_2 = 0.1$, $\lambda_0 = \lambda_1 = 1$ and $\varepsilon(0) = 0$.

Fig. 1 illustrates how the prices of a vulnerable call option for two initial states change with the moneyness (S_0/K). We can observe that the option with $\varepsilon(0) = 0$ procedure has higher prices than the option with $\varepsilon(0) = 1$ in the high moneyness region as expected.

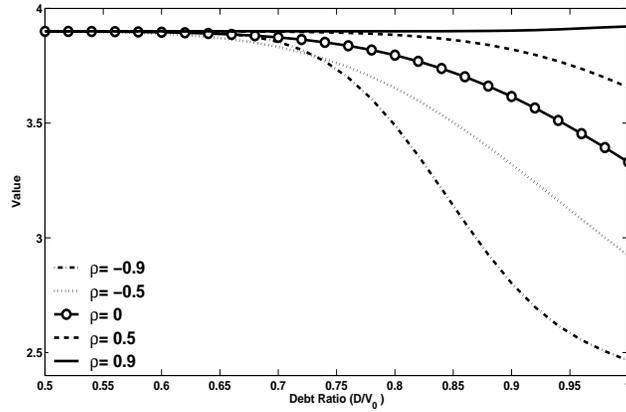


Figure 3. Vulnerable call value for different debt ratio (D/V_0) and ρ

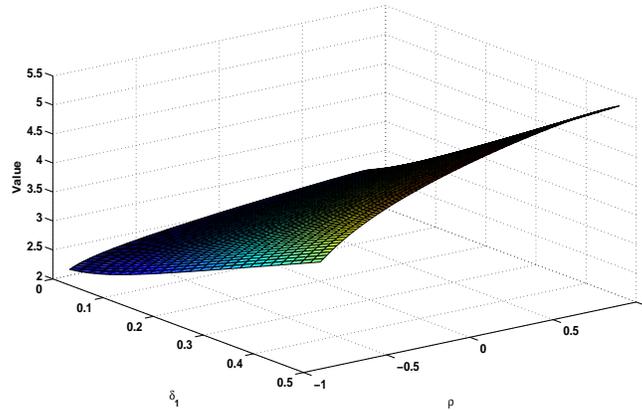


Figure 4. Vulnerable call value for different δ_1 and ρ

Fig. 2 and Fig. 3 illustrate how the prices change when the debt ratio (D/V_0) vary. Fig. 2 shows decreasing trends of prices for different initial states. Here, the option with $\varepsilon(0) = 0$ has always lower prices than the option with $\varepsilon(0) = 1$. We also can see that the negative correlation ρ between underlying asset and firm value processes leads to lower option prices in Fig. 3.

Fig. 4 and Fig. 5 illustrate the sensitivities of the options with respect to the shock sizes δ_i , ($i = 1, 2$) of the volatilities and the correlation. Both Fig. 4 and Fig. 5 show increasing trends of the option prices with respect to the correlation ρ . In Fig. 4, the shock size δ_1 of the underlying asset also leads to an increasing trend. In contrast, an decreasing trend of the option values with respect to the shock size δ_2 of the firm value process is found in Fig. 5. In addition, for a negative ρ , we can see a sharp decreasing of the option values.

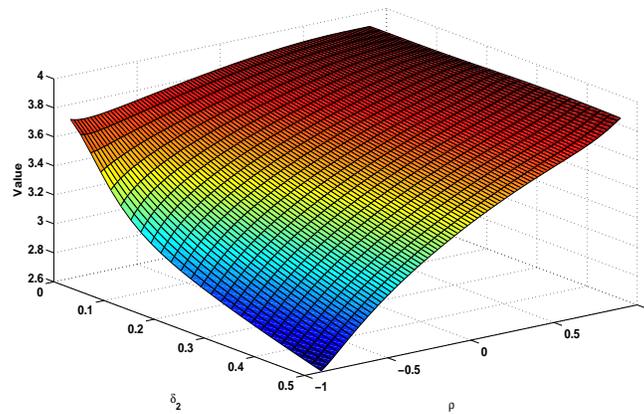


Figure 5. Vulnerable call value for different δ_2 and ρ

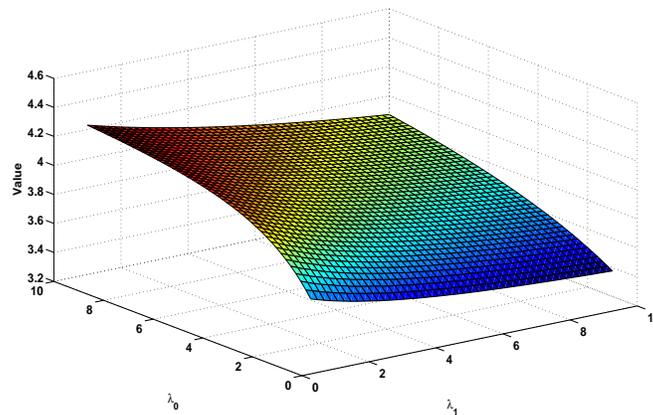


Figure 6. Vulnerable call value for different λ_1 and λ_2

Finally, Fig. 6 illustrates how the option values have the contrary trends with respect to intensities. Consequently, these results show the changes of the option values when the intensities vary by business cycle.

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