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Transmuted Dagum distribution: A more flexible and broad shaped hazard function model

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Abstract

In this article, we introduce an extended Dagum distribution, named as transmuted Dagum distribution which can be used for income distribution, actuarial, survival and reliability analysis. The main motivation for generalizing the standard distribution is to provide more flexible distribution to model a variety of data. The extended distribution has been expressed using quadratic rank transmutation map and its tractable properties like moments, moment generating, quantile, reliability and hazard functions are derived. The transmuted Dagum model provides the broader range of hazard behavior than the Dagum model. The densities of its order statistics, generalized TL-moments with its special cases are also studied. The parameters of the new model are estimated by maximum likelihood using Newton-Raphson approach and the information matrix and confidence intervals are also obtained. To illustrate utility and potentiality of the proposed model, it has been applied to rainfall data for the city of Islamabad, Pakistan.

Keywords: Transmuted Dagum distribution, Moments, Order statistics, TL-moments, Newton Raphson, Parameter estimation .

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1. Introduction

Dagum distribution is widely used for modeling a wide range of data in several fields. It is very worthwhile for analyzing income distribution, actuarial, metrological data and equally preferable for survival analysis. Moreover, it is considered to be the most suitable choice as compared to other three parameter distributions in several cases. It belongs to the generalized Beta distribution and is generated from generalized Beta-II by considering a shape parameter one and referred as inverse Burr distribution. Dagum [5] and Fattorini and Lemmi [13] derived the Dagum distribution independently. Dagum [7] studied the income and income related data by Dagum distributions. Dagum [6] also fitted this distribution on 1978 family incomes data for the United States and proved that its performance is the best among all the models. Bordley, McDonald, and Mantrala [4] also studied the United States family income data by probability distributions along with the Dagum distribution. Bandourian, McDonald, and Turley [2] revealed after the study of 23 countries' income data, the Dagum distribution is the best among the two and three parameter distributions. Quintano and Dagostino [21] studied single-person income distribution of European countries data and found that the Dagum distribution performs better to model the each country data separately. Perez and Alaiz [20] analyzed the personal income data of Spain by Dagum distribution. Alwan, Baharum, and Hassan [1] tried more than fifty distributions to model the reliability of the electrical distribution system and finally the Dagum distribution was considered as the best choice. We have cited very few studies but various other related studies also confirm the better performance of the Dagum distribution. Recently Dagum distribution is found to be quite useful and popular in modeling the skewed data.

Domma, Giordano and Zenga [11] and Domma [8] estimated the parameters of Dagum distribution with censored samples and the right-truncated Dagum distribution by maximum likelihood estimation. McGarvey, et al [17] studied the estimation and skewness test for the Dagum distribution. Shahzad and Asghar [22] estimated the parameter of this distribution by TL-moments. Oluyede and Rajasooriya [18] introduced the Mc-Dagum distribution. Oluyede and Ye [19] presented the class of weighted Dagum and related distributions and Domma and Condino [9] proposed the five parameter beta-Dagum distribution. In this study, we present the transmuted Dagum distribution that is the extension of the Dagum distribution.

Rest of the paper is organized as follows. Section 2 is about the quadratic rank transmutation map, mathematical derivation of the probability density function (pdf) and probability distribution function (cdf) of transmuted Dagum distribution with their graphical presentation. In section 3, rth moment and moment generating function are derived and the expression for the coefficient of variation, skewness and kurtosis are also reported. Section 4 is about the quantile function, median and random number generating process for transmuted Dagum distribution. Reliability function, hazard function and their mathematical and graphical presentation are given in Section 5. Section 6 is related to order statistics: the lowest, highest and joint order densities of transmuted Dagum distribution are specified. Section 7 contains the generalized TL-moments and its special cases, such as L-moments, TL-moments, LL-moments, and LH-moments. Methodology for parameter estimation, Newton-Raphson algorithm for maximum likelihood is discussed in Section 8. To compare the suitability of transmuted Dagum distribution with its related distributions, rainfall data is selected and its goodness of fit on empirical data is tested by using likelihood function, AIC, AICC, BIC, KS test, LR test and PP-plots in section 9.

2. Transmuted Dagum Distribution

A random variable follows the transmuted distribution, if it satisfies the following relationship that is proposed by Shaw et al. [24] named as quadratic rank transmutation map

$$F(y) = G(y)\big[(1+\lambda) - \lambda G(y)\big] \tag{2.1}$$

Where $G(y)$ is the cdf of the parent distribution and λ is the additional parameter that is called transmuted parameter. Due to the transmuted parameter the distribution becomes more flexible distribution to model even the complex data sets.
The pdf of the Dagum (parent) distribution is as

$$g(y;\alpha,\beta,\rho) = \frac{\alpha\rho y^{\alpha\rho-1}}{\beta^{\alpha\rho}\big(1+(y/\beta)^{\alpha}\big)^{\rho+1}}, \quad 0 \le x \le \infty;\ \alpha,\beta,\rho > 0 \tag{2.2}$$

and its cdf is as

$$G(y;\alpha,\beta,\rho) = \big(1+(y/\beta)^{-\alpha}\big)^{-\rho}. \tag{2.3}$$

Where α and ρ are the shape parameters, β is the scale parameter and all the three parameters are positive. Now substituting the (2.3) in (2.1), we obtained the cdf of the transmuted Dagum distribution in the following form

$$F(y;\alpha,\beta,\rho,\lambda) = \big(1+(y/\beta)^{-\alpha}\big)^{-\rho}[1+\lambda-\lambda\big(1+(y/\beta)^{-\alpha}\big)^{-\rho}], \tag{2.4}$$

and its respective pdf of transmuted Dagum distribution is given by

$$f(y;\alpha,\beta,\rho,\lambda) = \frac{\alpha\rho y^{2\alpha\rho-1}\big[(1+\lambda)\big(1+(y/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]}{\beta^{2\alpha\rho}\big(1+(y/\beta)^{\alpha}\big)^{2\rho+1}}. \tag{2.5}$$

The parameter λ has the support $-1 \le y \le 1$ and simply taking $\lambda = 0$ in above pdf and cdf, transmuted distribution reduces to the parent distribution. Dagum distribution due to quadratic rank transmutation map becomes more flexible. The shapes of this density and distribution function assuming various combinations of parameters are illustrated in the Figure 1 and Figure 2, respectively.

3. Moments and moments ratio

In this section, main statistical properties such as rth moments, mean, variance, and moment generating function for transmuted Dagum distribution are derived and discussed.

3.1. Theorem. *Let the random variable Y follows the transmuted Dagum distribution, then its rth moment has the following form*

$$E(Y^r) = \beta^r\Gamma\Big(1-\frac{r}{\alpha}\Big)\Big[\frac{(1+\lambda)\Gamma(\rho+\frac{r}{\alpha})}{\Gamma(\rho)} - \frac{\lambda\Gamma(2\rho+\frac{r}{\alpha})}{\Gamma(2\rho)}\Big]. \tag{3.1}$$

Proof. Let the rth moments is given by

$$m_r^{'} = E(Y^r) = \int_0^{\infty} \frac{\alpha\rho y^{2\alpha\rho+r-1}\big[(1+\lambda)\big(1+(y/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]}{\beta^{2\alpha\rho}\big(1+(y/\beta)^{\alpha}\big)^{2\rho+1}}\,\mathrm{d}y$$

$$= \int_0^{\infty} \frac{\alpha\rho y^{2\alpha\rho+r-1}(1+\lambda)\big(1+(y/\beta)^{-\alpha}\big)^{\rho}}{\beta^{2\alpha\rho}\big(1+(y/\beta)^{\alpha}\big)^{2\rho+1}}\,\mathrm{d}y - \int_0^{\infty} \frac{2\lambda\alpha\rho y^{2\alpha\rho+r-1}}{\beta^{2\alpha\rho}\big(1+(y/\beta)^{\alpha}\big)^{2\rho+1}}\,\mathrm{d}y$$

For convenience substitute $x = (y/\beta)^{\alpha}$, hence

$$m_r^{'} = \beta^r(1+\lambda)\int_0^{\infty} x^{\rho+r/\alpha-1}(1+x)^{-(\rho+1)}\mathrm{d}x - 2\alpha\rho\beta^r\int_0^{\infty} x^{\rho+r/\alpha-1}(1+x)^{-(\rho+1)}\mathrm{d}x$$

$$= \beta^r\Big[(1+\lambda)B\Big(\rho+\frac{r}{\alpha}, 1-\frac{r}{\alpha}\Big) - \lambda B\Big(2\rho+\frac{r}{\alpha}, 1-\frac{r}{\alpha}\Big)\Big],$$

where $B(.,.)$ is the beta type-II function, defined by

$$B(\theta_1,\theta_2) = \int_0^{\infty} z_1^{\theta}(1+z)^{-(\theta_1+\theta_2)}\mathrm{d}z; \quad \theta_1,\theta_2 > 0$$

after one step simplification, we obtain the result given in (3.1).

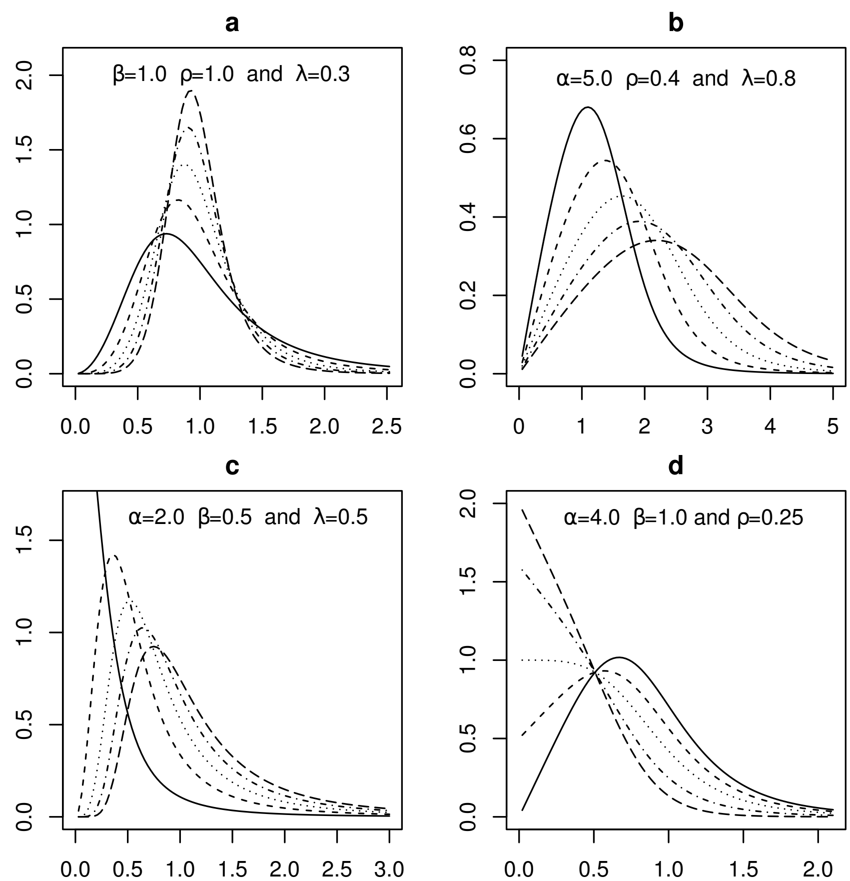

Figure 1. The pdf's of various transmuted Dagum distributions for values of parameters: *a)* α = 3.0[0.5]5.5; *b)* β = 2.0[0.25]3.25; *c)* ρ = 0.5, 0.75, 1.0[1.0]4.0; *d)* λ = -1.0[0.4]1.0 with solid, dashed, dotted, dotdash and longdash lines, respectively.

In particular, by setting $r = 1$ and $r = 2$ in (3.1), we obtain mean and varianc (σ^2) by taking usual steps

$$E(Y) = \beta\Gamma\Big(1-\frac{1}{\alpha}\Big)\Big[\frac{(1+\lambda)\Gamma(\rho+\frac{1}{\alpha})}{\Gamma(\rho)} - \frac{\lambda\Gamma(2\rho+\frac{1}{\alpha})}{\Gamma(2\rho)}\Big] \tag{3.2}$$

and

$$\sigma^2 = \beta^2\Gamma\Big(1-\frac{2}{\alpha}\Big)\big[(1+\lambda)P_{12}-\lambda P_{22}\big]-\beta^2\Big[\Gamma\Big(1-\frac{1}{\alpha}\Big)\Big]^2\big[(1+\lambda)P_{11}-\lambda P_{21}\big], \tag{3.3}$$

where $P_{ir}=\Gamma(i\rho+\frac{r}{\alpha})/\Gamma(i\rho)$.
The following expression can be used to obtain the moment ratios for transmuted Dagum distribution such as Cofficient of vartiation (CV), Skewness (Sk) and Kurtosis (Kr) using $m_r^{'}(r=1,2,3,4)$.

$$CV=\frac{\sigma}{m_1'},$$

$$Sk=\frac{m_3'-2m_2'm_1'+2\left(m_1'\right)^3}{\sigma^3},$$

$$Kr=\frac{m_4'-4m_3'm_1'+6m_2'\left(m_1'\right)^2-3\left(m_1'\right)^4}{\sigma^4}.$$

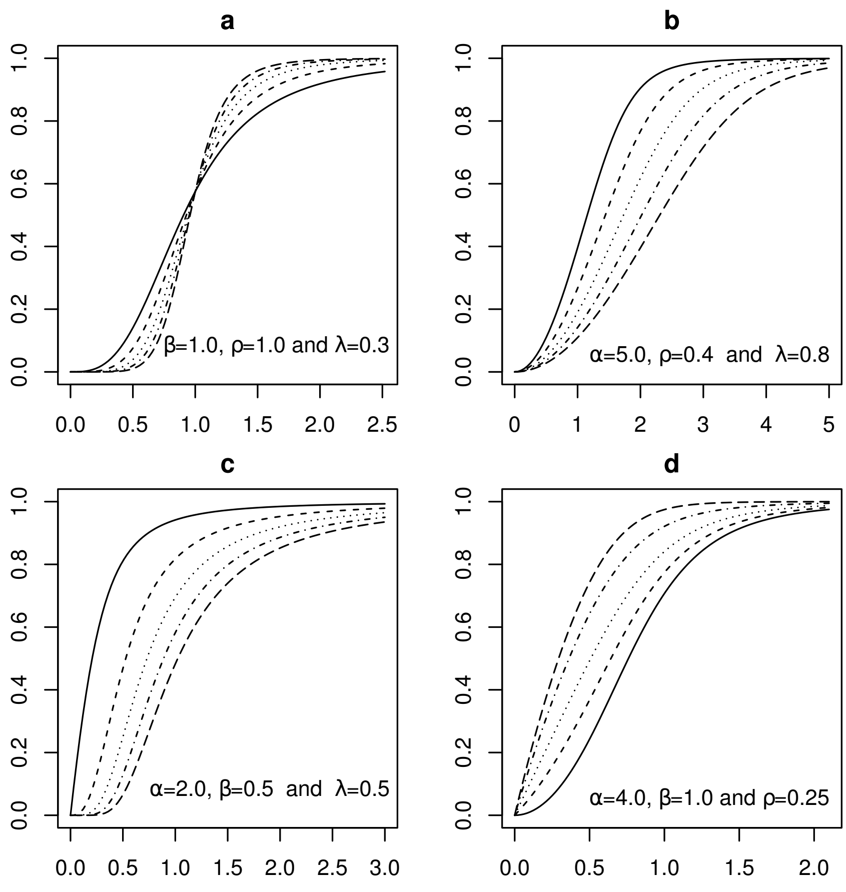

Figure 2. The cdf's of various transmuted Dagum distributions for values of parameters: *a*) α = 3.0[0.5]5.5; *b*) β = 2.0[0.25]3.25; *c*) ρ = 0.5, 0.75, 1.0[1.0]4.0; *d*) λ = -1.0[0.4]1.0 with solid, dashed, dotted, dotdash and longdash lines, respectively.

3.2. Theorem. *The moment generating function of Y , $M_y(t)$ when random variable follows the transmuted Dagum distribution is*

$$M_y(t) = \sum_{r=0}^{\infty} \frac{t^r \beta\Gamma(1-1/\alpha)}{r!}\big[(1+\lambda)P_{11} - \lambda P_{21}\big] \tag{3.4}$$

Proof. Let the moment generating function for Y is given by

$$\begin{aligned} M_Y(t) = E(e^{ty}) &= \int_0^{\infty} e^{ty} f(y)\mathrm{d}y \\ &= \int_0^{\infty}\left(1 + ty + \frac{t^2y^2}{2!} + ... + \frac{t^ny^n}{n!} + ...\right) f(y)\mathrm{d}y \\ &= \sum_{r=0}^{\infty} \frac{t^r E\left(Y^r\right)}{r!} \\ &= \sum_{r=0}^{\infty} \frac{t^r \beta\Gamma(1-1/\alpha)}{r!}\big[(1+\lambda)P_{11} - \lambda P_{21}\big]. \end{aligned}$$

4. Quantile function and random number generation

Hyndman and Fan [16] defined the quantile function for any distribution is in the form

$$Q(q) = F^{-1}(q) = \inf\{y : F(y) \geqslant q\} \quad 0 < q < 1, \tag{4.1}$$

where $F(y)$ is the distribution function. Quantile function divides the ordered data into q equal sized portions. The smallest and largest value of the ordered data corresponds to probability 0 and 1, respectively. The qth quantile of transmuted Dagum distribution is obtained using (2.4) and (4.1) is given as

$$Q(q) = \beta\Big[\Big(\frac{1+\lambda+\sqrt{(1+\lambda)^2 - 4\lambda q}}{2q}\Big)^{1/\rho} - 1\Big]^{1/\alpha}. \tag{4.2}$$

Median is the 50th percentile, hence median of transmuted Dagum distribution is obtained from (4.2) as below

$$\text{Median} = \beta\Big[\Big(1+\lambda+\sqrt{1+\lambda^2}\Big)^{1/\rho} - 1\Big]^{1/\alpha}.$$

The expression (4.2) can also be used to find the tertiles, quartiles, quintiles, sextiles, deciles, percentiles and permilles. To generate the random numbers for the transmuted Dagum distribution, let suppose that the U is the standard uniform variate in (4.2) rather than q. Then the random variable

$$y = \beta\Big[\Big(\frac{1+\lambda+\sqrt{(1+\lambda)^2 - 4\lambda u}}{2u}\Big)^{1/\rho} - 1\Big]^{1/\alpha} \tag{4.3}$$

follows the transmuted Dagum distribution. Now (4.3) is ready to generate the random number for the distribution, taking α, β, ρ and λ known.

5. Reliability analysis

The reliability function $R(t)$ gives the probability of surviving of an item at least reach the age of t time. The cdf $F(t)$ and reliability function are reverse of each other as $R(t) + F(t) = 1$. The reliability function for transmuted Dugam distribution is given by

$$\begin{aligned} R(t) = P(T > t) &= \int_t^{\infty} f(t)\mathrm{d}t = 1 - F(t) \\ &= 1 + \big(1 + (t/\beta)^{-\alpha}\big)^{-2\rho}\Big[\lambda - (1+\lambda)\big(1 + (t/\beta)^{-\alpha}\big)^{\rho}\Big]. \end{aligned}$$

With various choices of parametric values the Figure 3 illustrates the reliability function pattern of transmuted Dagum distribution.

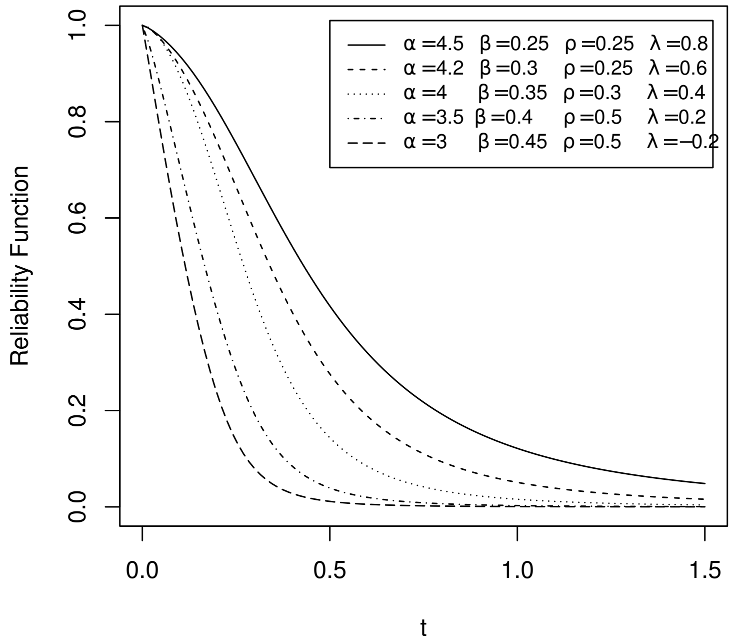

Figure 3. The various shapes of reliability function of transmuted Dagum distribution.

An important property of a random variable is the hazard function, it measure the inclination towards failure rate. The probability approaches to failure increases as the value of the hazard function increase. Mathematically, the hazard function and the hazard function of transmuted Dagum distribution is defined as

$$\begin{aligned} h(t) &= \frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)} \\ &= \frac{\alpha\rho t^{2\alpha\rho-1}\left[(1+\lambda)\left(1+(t/\beta)^{-\alpha}\right)^{\rho}-2\lambda\right]}{\beta^{2\alpha\rho}\left(1+(t/\beta)^{\alpha}\right)^{2\rho+1}\left[1+\left(1+(t/\beta)^{-\alpha}\right)^{-2\rho}\left[\lambda-(\lambda+1)\left(1+(t/\beta)^{-\alpha}\right)^{\rho}\right]\right]}. \end{aligned}$$

The hazard function of the transmuted Dagum distribution is attractively flexible. Therefore, it is useful and suitable for the real life situations. As in the case of transmuted Dagum distribution when $\lambda = 0$ is the Dagum distribution. Domma [10] and Domma, Giordano and Zenga [11] using Glaser's theorem [14] proved the proposition of the hazard function of the Dagum distribution. So taking these propositions and Glaser's theorem [14], we concentrate on the additional parameter λ and find out the following four behaviour of the hazard function on the combinations of parameters.

(1) The hazard function of transmuted Dagum distribution is decreasing if
   (a) $\rho = 2/\alpha - 1$, $\alpha < 2$, $\beta > 0$ and -1$\leq \lambda \leq$1.
   (b) $\alpha\rho = 1$, $\rho < 2/\alpha - 1$, $\alpha < 1$, $\beta > 0$ and -1$\leq \lambda \leq$1
   (c) $\alpha < 1$, $\rho(\alpha^{-1}, 2/\alpha - 1)$, $\beta > 0$ and -1$\leq \lambda \leq$1
(2) It is upside down bathtub (increasing-decreasing) if

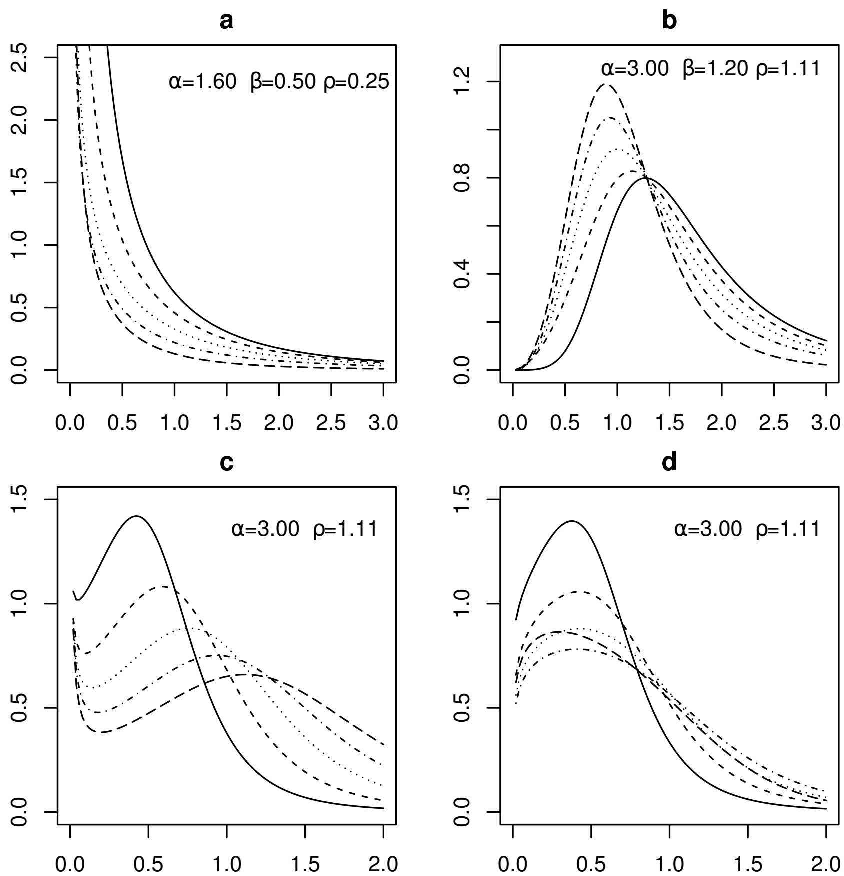

Figure 4. The behaviour of the hazard rate function of the transmuted Dagum distributions for various parameters values such as : *a*) λ = -1.0[0.5]1.0; *b*) λ = -1.0[0.5]1.0; *c*) β = 0.75[0.25]1.75, λ = -0.8[0.1]-0.4; *d*) β = 0.75[0.25]1.75, λ = -0.2[0.2]0.8 with solid, dashed, dotted, dotdash and longdash lines, respectively.

   (a) $\alpha\rho > 1$, $\rho \neq 2/\alpha - 1$, $\beta > 0$ and -1$\leq \lambda \leq$1
   (b) $\alpha\rho = 1$, $\rho > 2/\alpha - 1$, $\alpha > 1$, $\beta > 0$ and -1$\leq \lambda \leq$1

(3) It is bathtub and upside down bathtub if
   (a) $\alpha \in (1, 3)$, $\rho \in \left(\frac{3-\alpha}{\alpha+1}, \frac{2}{\alpha} - 1\right)$, $\beta > 0$ and -1$\leq \lambda <$ -0.4
   (b) $\alpha \geq 3$, $\rho \in \left(\frac{2}{\alpha} - 1\right)$, $\beta > 0$ and -1$\leq \lambda <$ -0.4

(4) It is upside down bathtub if
   (a) $\alpha \in (1, 3)$, $\rho \in \left(\frac{3-\alpha}{\alpha+1}, \frac{2}{\alpha} - 1\right)$, $\beta > 0$ and -0.4$\leq \lambda \leq$1
   (b) $\alpha \geq 3$, $\rho \in \left(\frac{2}{\alpha} - 1\right)$, $\beta > 0$ and -0.4$\leq \lambda \leq$1

The graphical pesentation of the behaviour of the hazard rate function for transmuted Dagum distribution is sketched in Figure 4 for various choices of parametric values.

6. Order statistics of transmuted Dagum distribution

In probability statistics the distribution of extremes (smallest and/or largest), median and joint order statistics are the most important functions of a random variable. This is only the order statistics that help us to study the peaks of the data to understand the pattern of the extremes. Mathematically the order statistics is defined as, let $Y_1, Y_2, ..., Y_n$ be any real valued random variables and its ordered values denoted as $Y_{(1)} \leq Y_{(2)} \leq ... \leq Y_{(n)}$ then the values $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$ are the order statistics of random variable .
The density of the nth ordered statistics, that follows the transmuted Dagum distribution is derived in the following form

$$\begin{aligned} f_{(n)}(y_{(n)}) &= n[F(y_{(n)})]^{n-1} f_{(}y_{(n)}) \\ &= \frac{n\alpha\rho\beta^{\alpha}}{y_{(n)}^{\alpha+1}} \Big[(1+\lambda)\Big(1+\Big(\frac{y_{(n)}}{\beta}\Big)^{-\alpha}\Big) - 2\lambda\Big] \sum_{j=0}^{n-1} \left(\begin{array}{c} n-1 \\ j \end{array} \right) \\ &\times (-\lambda)^j (1+\lambda)^{n-j-1} \Big(1+\Big(\frac{y_{(n)}}{\beta}\Big)^{-\alpha}\Big)^{-\rho(n+j+1)-1}. \end{aligned}$$

Let suppose that the smallest values also follows the transmuted Dagum distribution, then the denity of the smallest order statistic, is obtained as

$$\begin{aligned} f_{(1)}(y_{(1)}) &= n[1 - F(y_{(1)})]^{n-1} f_{(}y_{(1)}) \\ &= \frac{n\alpha\rho\beta^{\alpha}}{y_{(1)}^{\alpha+1}} \Big[(1+\lambda)\Big(1+\Big(\frac{y_{(1)}}{\beta}\Big)^{-\alpha}\Big) - 2\lambda\Big] \sum_{i=0}^{n-1} \sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) \left(\begin{array}{c} n-1 \\ i \end{array} \right) \\ &\times (-1)^{i+j} (-\lambda)^j (1+\lambda)^{i-j} \Big(1+\Big(\frac{y_{(1)}}{\beta}\Big)^{-\alpha}\Big)^{-\rho(i+j+1)-1}. \end{aligned}$$

Generally the pdf of the rth order statistics is given by

$$\begin{aligned} f_{(r)}(y_{(r)}) &= \frac{n!}{(r-1)!(n-r)!} [F(y_{(r)})]^{r-1} [1 - F(y_{(r)})]^{n-r} f_{(}y_{(r)}) \\ &= \frac{n!\alpha\rho\beta^{\alpha} y_{(r)}^{-(\alpha+1)}}{(r-1)!(n-r)!} \Big[(1+\lambda)\Big(1+\Big(\frac{y_{(r)}}{\beta}\Big)^{-\alpha}\Big) - 2\lambda\Big] \sum_{i=0}^{n-r} \sum_{j=0}^{r+i-1} \left(\begin{array}{c} r+i-1 \\ j \end{array} \right) \\ &\times \left(\begin{array}{c} n-r \\ i \end{array} \right) (-1)^{i+j} (\lambda)^j (1+\lambda)^{r+i-j-1} \Big(1+\Big(\frac{y_{(r)}}{\beta}\Big)^{-\alpha}\Big)^{-\rho(r+i+j+1)-1}. \end{aligned}$$

Sometimes interest is in the joint pdf such as to find the joint breaking strength of certain equipment, for the transmuted Dagum distribution the pdf of $Y_{(r)}$ and $Y_{(s)}$, when $1 \leq r < s \leq n$ is obtained as

$$\begin{aligned} f_{(r,s)}(u,v) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F(u)]^{r-1} [F(v) - F(u)]^{s-r-1} \\ &\times [1 - F(v)]^{n-s} f(u) f(v) \\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \Big(\frac{\alpha\beta\rho}{uv}\Big)^2 \Big[(1+\lambda)\Big(1+\Big(\frac{u}{\beta}\Big)^{-\alpha}\Big) - 2\lambda\Big] \\ &\times \sum_{i=0}^{S} \sum_{j=0}^{s-r-1} \sum_{k=0}^{n-s} \sum_{l=0}^{r+j-1} \left(\begin{array}{c} r+j-1 \\ l \end{array} \right) \left(\begin{array}{c} n-s \\ k \end{array} \right) \left(\begin{array}{c} s-r-1 \\ j \end{array} \right) \left(\begin{array}{c} S \\ j \end{array} \right) \\ &\times (-1)^{i+j+k+l} (\lambda)^{i+l} (1+\lambda)^{r+i+j-1} \Big(1+\Big(\frac{u}{\beta}\Big)^{-\alpha}\Big)^{-\rho(r+i+j+1)-1} \\ &\times \Big(1+\Big(\frac{v}{\beta}\Big)^{-\alpha}\Big)^{-\rho(S+i+2)-1}, \end{aligned}$$

where $S = s + k - r - j - 1$.

7. Generalized TL-moments of transmuted Dagum distribution

Hosking [15] introduced the L-moments and now these moments are frequently used for extreme value analysis. Elamir and Seheult [12] extended these moments and presented the TL-moments. These moments based on the order statistics used to describe the shape of the probability distribution by evaluating all descriptive statistics including parameter estimation and hypothesis testing. The rth generalized TL-moments with s smallest and t largest trimming is defined as follows

$$T_r^{(s,t)} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\left(Y_{r+s-k:r+s+t}\right), \tag{7.1}$$

where $T_r^{(s,t)}$ is a linear function of the expectations of the order statistics and $r = 1, 2, ...; t, s = 0, 1, 2....$
The expression for the expected value of the $(r + s - k)$th order statistics of the random sample of size $(r + s + t)$ is as

$$E\left(Y_{r+s-k:r+s+t}\right) = C \int_0^\infty [F(y)]^{r+s-k-1} [1 - F(y)]^{t+k} \, dF(y). \tag{7.2}$$

where $C = \frac{(r+s+t)!}{(r+s-k-1)!(t+k)!}$ and F is the cdf of the transmuted Dagum distribution, and by substitute expression (7.1) into expression (7.2), we obtain $T_r^{(s,t)}$ as

$$T_r^{(s,t)} = \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{C}{r} (-1)^k \int_0^\infty [F(y)]^{r+s-k-1} [1 - F(y)]^{t+k} \, dF(y)$$

Having the cdf and pdf of transmuted Dagum distribution the generalized TL-moments is given by

$$\begin{aligned} T_r^{(s,t)} = {} & \frac{\beta\rho}{r} \sum_{k=0}^{r-1} \sum_{i=0}^{t+k} \sum_{j=0}^{s+r-k+i-1} \binom{s+r-k+i-1}{j} \binom{t+k}{i} \\ & \times \binom{r-1}{k} C(-1)^{i+j+k} \lambda^j (1+\lambda)^{I+s-1} \Gamma(1 - 1/\alpha) \\ & \times \left[(1+\lambda) \frac{\Gamma[\rho(I+s) + 1/\alpha]}{\Gamma[\rho(I+s)+1]} - 2\lambda \frac{\Gamma[\rho(I+s+1) + 1/\alpha]}{\Gamma[\rho(I+s+1)+1]} \right], \end{aligned} \tag{7.3}$$

where $I = r - k + i + j$.
This expression of the generalized TL-moments used to obtain its special cases such as L-moments, TL-moments, LH-moments and LL-moments. First two TL-moments $T_1^{(s,t)}$ and $T_2^{(s,t)}$ are used to calculate the location and dispersion of the data, respectively. The ratio of TL-moments $T_{CV}^{(s,t)} = T_2^{(s,t)} \big/ T_1^{(s,t)}$, $T_{Sk}^{(s,t)} = T_3^{(s,t)} \big/ T_2^{(s,t)}$ and $T_{Kr}^{(s,t)} = T_4^{(s,t)} \big/ \lambda_2^{(s,t)}$ are the coefficient of variation, skewness and kurtosis characteristic of the probability distribution, respectively.

7.1. The TL-moments $(s = t = 1)$**.** Generally it is possible to trim any number of smallest and largest values from the ordered observation. As a special case, if only one extreme value from both sides $(s = t = 1)$ are trimmed then expression (7.3) becomes the rth TL-moments and we get

$$T_r^{(1)} = \sum_{k=0}^{r-1}\sum_{i=0}^{k+1}\sum_{j=0}^{r-k+i} \begin{pmatrix} r-k+i \\ j \end{pmatrix} \begin{pmatrix} k+1 \\ i \end{pmatrix} \begin{pmatrix} r-1 \\ k \end{pmatrix} \frac{(r+2)!\Gamma(1-1/\alpha)\beta\rho}{r(r-k)!(k+1)!} \times (-1)^{i+j+k}\lambda^j(1+\lambda)^I\Big[(1+\lambda)\frac{\Gamma[\rho(I+1)+1/\alpha]}{\Gamma[\rho(I+1)+1]} - 2\lambda\frac{\Gamma[\rho(I+2)+1/\alpha]}{\Gamma[\rho(I+2)+1]}\Big].$$

7.2. The L-moments $(s = t = 0)$**.** When none of the observation is trimmed from the ordered sample, the TL-moments reduced to L-moments and basically L-moments and related moments are due the Hosking [15] methodology. The rth L-moments of transmuted Dagum distribution is as

$$T_r^{(0)} = \sum_{k=0}^{r-1}\sum_{i=0}^{k}\sum_{j=0}^{r-k+i-1} \begin{pmatrix} r-k+i-1 \\ j \end{pmatrix} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} r-1 \\ k \end{pmatrix} \frac{(r)!\Gamma(1-1/\alpha)\beta\rho}{r(r-k-1)!(k)!} \times (-1)^{i+j+k}\lambda^j(1+\lambda)^{I-1}\Big[(1+\lambda)\frac{\Gamma[\rho(I)+1/\alpha]}{\Gamma[\rho(I)+1]} - 2\lambda\frac{\Gamma[\rho(I+1)+1/\alpha]}{\Gamma[\rho(I+1)+1]}\Big].$$

7.3. The LL-moments $(s = 0, t = t)$**.** LL-moments progressively reflect the characteristics of the lower part of distribution. Bayazit and Onoz [3] introduced these moments and later it became the special case of the TL-moments, when $s = 0$ and $t = t$. Following is the LL-moments

$$T_r^{(0,t)} = \sum_{k=0}^{r-1}\sum_{i=0}^{t+k}\sum_{j=0}^{r-k+i-1} \begin{pmatrix} r-k+i-1 \\ j \end{pmatrix} \begin{pmatrix} t+k \\ i \end{pmatrix} \begin{pmatrix} r-1 \\ k \end{pmatrix} \times \frac{(r+t)!\beta\rho}{r(r-k-1)!(t+k)!}(-1)^{i+j+k}\lambda^j(1+\lambda)^{I-1}\Gamma(1-1/\alpha) \times \Big[(1+\lambda)\frac{\Gamma[\rho(I)+1/\alpha]}{\Gamma[\rho(I)+1]} - 2\lambda\frac{\Gamma[\rho(I+1)+1/\alpha]}{\Gamma[\rho(I+1)+1]}\Big].$$

7.4. The LH-moments $(s = s, t = 0)$**.** LH moments proposed by Wang [26], these moments describe the upper part of the data more precisely. These moments give more weight to the larger values and the theoretical LH-moments for the transmuted Dagum distribution are defined as

$$T_r^{(s,0)} = \sum_{k=0}^{r-1}\sum_{i=0}^{k}\sum_{j=0}^{r+s-k+i-1} \begin{pmatrix} r+s-k+i-1 \\ j \end{pmatrix} \begin{pmatrix} k \\ i \end{pmatrix} \begin{pmatrix} r-1 \\ k \end{pmatrix} \times \frac{(r+t)!\beta\rho}{r(r-k-1)!(t+k)!}(-1)^{i+j+k}\lambda^j(1+\lambda)^{I-1}\Gamma(1-1/\alpha) \times \Big[(1+\lambda)\frac{\Gamma[\rho(I+s)+1/\alpha]}{\Gamma[\rho(I+s)+1]} - 2\lambda\frac{\Gamma[\rho(I+s+1)+1/\alpha]}{\Gamma[\rho(I+s+1)+1]}\Big].$$

8. Parameter estimation

In this section, interest is to estimate the parameters of transmuted Dagum distribution by maximum likelihood estimation. Let $Y_1, Y_2, ..., Y_n$ be i.i.d random variables of transmuted Dagum distribution of size n. Then the sample likelihood function and log-likelihood function for this distribution are obtained as follows

$$L(x;.) = \frac{\alpha\rho}{\beta^{2\alpha\rho}} \prod_{i=1}^{n} y_i^{2\alpha\rho-1} \big(1 + (y_i/\beta)^{\alpha}\big)^{2\rho+1} \big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big] \tag{8.1}$$

and

$$\begin{aligned} \ell(x;.) = {} & n \ln \alpha + n \ln \rho - 2n\alpha\rho \ln \beta - (2\alpha\rho + 1) \sum_{i=1}^{n} \ln \big(1 + (y_i/\beta)^{\alpha}\big) \\ & + (2\alpha\rho + 1) \sum_{i=1}^{n} \ln y_i + \sum_{i=1}^{n} \ln \big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big], \end{aligned} \tag{8.2}$$

respectively.
To find the parameter estimates, now we take the first order derivatives of (8.2) with respect to parameter (α, β, ρ and λ) and equating them equal to zero, respectively,

$$\begin{aligned} & \frac{n}{\alpha} - 2n\rho \ln \beta + 2\rho \sum_{i=1}^{n} \ln y_i - (2\rho + 1) \sum_{i=1}^{n} \frac{(y_i/\beta)^{\alpha} \ln(y_i/\beta)}{\big(1 + (y_i/\beta)^{\alpha}\big)} \\ & \qquad - \rho(1+\lambda) \sum_{i=1}^{n} \frac{(y_i/\beta)^{-\alpha}\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho-1} \ln(y_i/\beta)}{\big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]} = 0, \end{aligned}$$

$$\begin{aligned} & -\frac{2n\alpha\rho}{\beta} + \alpha(2\rho + 1) \sum_{i=1}^{n} \frac{y_i (y_i/\beta)^{\alpha-1}}{\beta^2 \big(1 + (y_i/\beta)^{\alpha}\big)} \\ & \qquad - \alpha\lambda(1+\lambda) \sum_{i=1}^{n} \frac{(y_i/\beta)^{-\alpha}\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho-1} \ln(y_i/\beta)}{\big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]} = 0, \end{aligned}$$

$$\begin{aligned} & \frac{n}{\rho} - 2n\alpha \ln \beta - 2 \sum_{i=1}^{n} \ln \big(1 + (y_i/\beta)^{\alpha}\big) \\ & \qquad + (1+\lambda) \sum_{i=1}^{n} \frac{\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} \ln \big(1 + (y_i/\beta)^{-\alpha}\big)}{\big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]} = 0, \end{aligned}$$

$$\sum_{i=1}^{n} \frac{\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2}{\big[(1+\lambda)\big(1 + (y_i/\beta)^{-\alpha}\big)^{\rho} - 2\lambda\big]} = 0.$$

The exact solution to derive the estimator for unknown parameters is not possible, so the estimates $(\hat{\alpha}, \hat{\beta}, \hat{\rho}, \hat{\lambda})'$ are obtained by solving the above four nonlinear equations simultaneously. This solution of nonlinear system is easier by Newton-Raphson approach. The Newton-Raphson approach used the jth element of the gradient and the (j,k)th elements of the Hessian matrix and these elements are $g_j = \partial\ell(\theta)/\partial\theta_j$ and $H_{jk} = \partial^2\ell(\theta)/\partial\theta_j\partial\theta_k$, respectively, whereas j,k =1,2,3,4, due to the four parameters of transmuted Dagum distribution. The information matrix, $I(\theta) = I_{jk}(\theta) = -E(H_{jk})$ and then its inverse of matrix $I(\theta)^{-1}$ provides the variances and covariances, diagonal and off diagonal entries, respectively. Asymptotically these estimates of parameters approaches to normality and the z-score are approximately standard normal, which can be used to find the $100(1-r)\%$ two sided confidence interval for α, β, ρ and λ.

9. Application

In this section, the performance of the transmuted Dagum distribution is compared with Dagum distribution and some other related distributions. Monthly maximum precipitation data of Islamabad city is considered for the comparison. Islamabad is the capital city of the Pakistan. The geographical location of this city has Latitude 33.71 and Longitude 73.07 with humid subtropical climate and has five seasons. This area receives heavy rainfall during monsoon season. The data of monthly precipitation retrieved from the Regional Meteorological Center (RMC) Lahore and from Pakistan Metrological Department (PMD) Islamabad. The length of data is 640 recorded from January 1954 to December 2013 excluding some unobserved or unreported months and the summary statistics are given in Table 1 and Table 3.

Table 1. Summary Statistics for monthly maximum precipitation data of the Islamabad, Pakistan.

Length	Average	Minimum	Maximum	Q_1	Median	Q_3	S.D
640	86.25	0.10	641.00	20.35	49.90	101.90	94.98

In order to compare the transmuted Dagum and its related distribution, we consider criteria like log-likelihood (ℓ), Akaike information criterion (AIC), Akaike information corrected criterion (AICC), Bayesian information criteria (BIC) and Kolmogoro-Smirnov (KS) goodness of fit test for the data sets. The better distribution have corresponds to smaller ℓ, AIC, AICC , BIC and KS values. Where

$$\text{AIC}=2\text{k}-2\ell,$$
$$\text{AICC}=\text{AIC}+2k(k+1)/(n-k-1),$$
$$\text{BIC}=2\ell+\text{k}\log(n)$$

and

$$\text{KS}=\max_{i\leq i\leq n}[F(Y_i)-(i-1)/n, i/n-F(Y_i)].$$

Here k is the number of parameters in each distribution, and n is the sample size.
It is better to test the superiority of the transmuted Dagum distribution over the Dagum distribution before analyzing the data. We employed the likelihood ratio (LR) statistic for this purpose. To perform this test the maximized restricted and unrestricted log-likelihoods can be computed under the null and alternative hypothesis
$H_0 : \lambda = 0$ (restricted, Dagum model is true for the data set)
versus
$H_1 : \lambda \neq 0$ (unrestricted, transmuted Dagum model is true for the data set).
The LR statistic for testing the hypothesis is computed by $\omega = 2(\ell(\hat{\theta}_0) - \ell(\hat{\theta}_1))$, where $\hat{\theta}_0$ and $\hat{\theta}_1$ are the maximum likelihood estimates under H_0 and H_1, respectively. The LR statistic is asymptotically distributed as chi-square ($\chi^2_{v,r}$). The computed value of LR statistic under the hypothesis is ω =22.74. We may observe that the $\omega > \chi^2_{1,0.05}(3.84)$, so we reject the null hypothesis and found that the transmuted Dagum model is best for the data set.
Variance covariance matrix of the MLEs under the transmuted Dagum distribution is obtained as

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 0.0407 & 2.7555 & -0.0097 & 0.0094 \\ 2.7555 & 1360.2 & -0.5489 & 13.361 \\ -0.0097 & -0.5489 & 0.0028 & 0.0004 \\ 0.0094 & 13.361 & 0.0004 & 0.4924 \end{pmatrix}.$$

Table 2. Estimated parameters of the transmuted Dagum and related distributions.

Model	Estimates	$\ell(., y)$	AIC	AICC	BIC	KS
Transmuted Dagum	$\hat{\alpha} = 2.2198$ $\hat{\beta} = 132.94$ $\hat{\rho} = 0.3981$ $\hat{\lambda} = 0.3565$	3452.71	6913.42	6913.48	6931.27	0.0280
Dagum	$\hat{\alpha} = 2.1302$ $\hat{\beta} = 105.49$ $\hat{\rho} = 0.5827$	3464.08	6934.16	6934.20	6947.544	0.1646
Transmuted Pareto	$\hat{a} = 0.1000$ $\hat{b} = 0.2374$ $\hat{\lambda} = -0.962$	4002.16	8010.32	8010.36	8023.70	0.3332
Pareto	$\hat{a} = 0.1000$ $\hat{b} = 0.1657$	4177.59	8361.18	8361.19	8368.10	0.4245
Fisk	$\hat{a} = 1.3577$ $\hat{b} = 46.452$	3476.16	6958.32	6956.32	6965.24	0.9718
Inverse Lomax	$\hat{a} = 31.223$ $\hat{b} = 1.3335$	3508.12	7020.24	7020.25	7029.16	0.1209

Thus, the variances of the ML estimates are, $var(\hat{\alpha}) = 0.2019$, $var(\hat{\beta}) = 36.8808$, $var(\rho) = 0.0527$ and $var(\hat{\lambda}) = 0.3863$. Therefore, confidence interval for α, β, ρ and λ are [1.8240, 2.6156], [60.657, 205.23], [0.2946, 0.5015] and [-0.4007, 1.1136], respectively.

The results in Table 2 indicates that the proposed transmuted Dagum distribution fits well as it has the smallest $\ell(., y)$, AIC, AICC and BIC as compared to the Dagum distribution and the others considered distributions. The KS goodness fit test is employed to evaluate the best fitted model for the precipitation data. The calculated value of this test is 0.0280, whereas the tabled critical two-tailed values at 0.05 and 0.01 significance levels are 0.0538 and 0.0644, respectively. According to Sheskin [25], if the value of KS statistic is greater or equal to the critical value then the null hypothesis should be rejected. Thus the null hypothesis cannot be rejected for the transmuted Dagum distribution as the value of the KS-test is not greater or equal to the critical values.

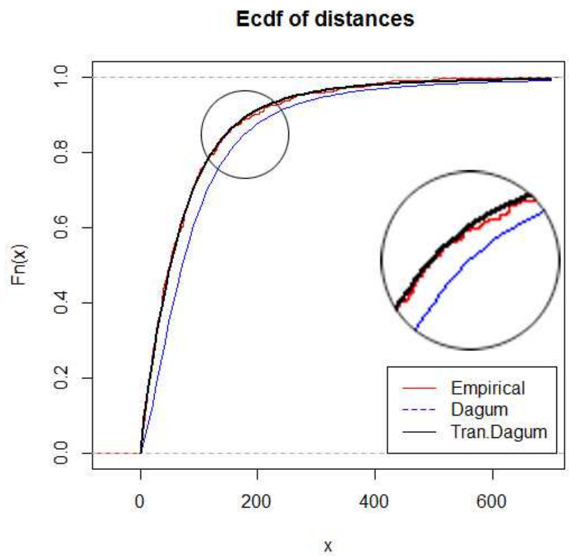

Figure 5. Empirical, fitted transmuted Dagum and Dagum cdf of the data set with maximum distance highlight.

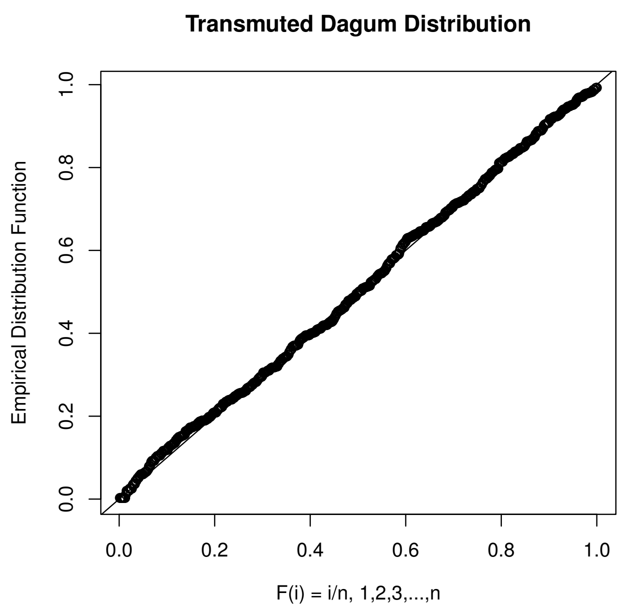

Figure 6. PP plots for fitted transmuted Dagum distribution.

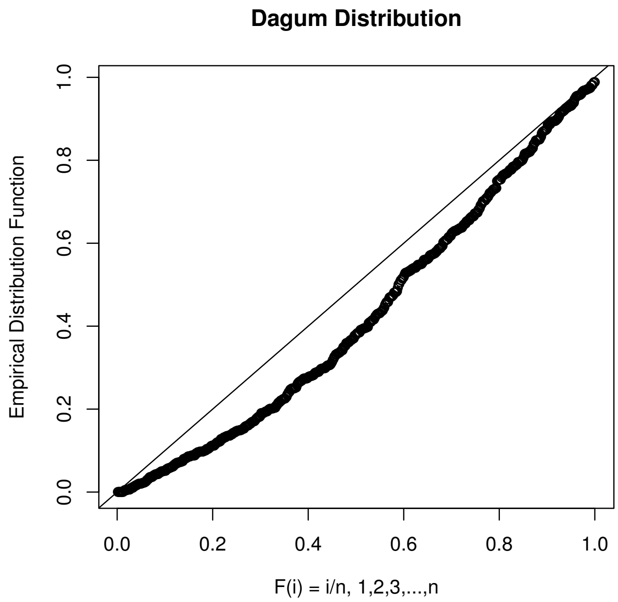

Figure 7. PP plots for fitted Dagum distribution

Both empirical cdf and PP-plots also indicate that the transmuted Dagum distribution is better than its competitor Dagum distribution to model the rainfall data. As transmuted Dagum distribution exactly follow the empirical pattern of the data and more closest view showed in circle in Figure 5 and similarly in PP-plot the transmuted Dagum distribution lies almost perfectly on the 45^o line. So we conclude that the transmuted Dagum distribution fulfills all the goodness of fit criteria for the data set.

TL-moments evaluate the basic characteristics of data in a better way and show the true picture of the data. First and second moments show the average value and variation in data, respectively. Consistency, symmetry and peakness evaluated by the coefficient of variation CV, Sk and Kr using the 2nd, 3rd and 4th moments. These moments and coefficients are calculated and reported in the Table 3 using Islamabad precipitation data set.

Table 3. Moments and moment ratios for monthly maximum precipitation data of the Islamabad, Pakistan

Model	Moments	L-moments	TL-moments	LL-moments	LH-moments
1st	86.2486	86.2486	114.8730	68.2673	104.2290
2nd	45730.1	17.9811	24.3523	34.9539	7.9823
3rd	-2.15×10^6	-28.6242	-15.3700	-4.0121	-34.1535
4th	-4.24×10^8	-22.6061	-16.8617	-16.4118	-11.8458
CV	2.4794	0.2085	0.2120	0.5120	0.0765
Sk	-1.1691	-1.5919	-0.6311	-0.4115	-4.2786
Kr	1.4962	-1.2572	-0.6924	-0.4695	-1.4840

10. Conclusion

The transmuted Dagum distribution proposed in this study, is the generalization of the Dagum distribution. This distribution is quite flexible and its application diversities increased due to the fourth transmuted parameter as compared to the standard Dagum distribution. To show the flexibility of new density the plots of the pdf, cdf, reliability function and hazard functions are presented. We derived moments and other basic properties of the proposed distribution. The densities of the lowest, highest, rth order statistics, the joint density of the two order statistics and TL-moments are also studied. The parameter estimation is obtained by the maximum likelihood estimation via Newton-Raphson approach. To evaluate its worth five goodness of fit criterion are considered for the selection of most appropriate model among transmuted Dagum, Dagum, transmuted Pareto, Pareto, Fisk and inverse Lomax. On all of these criteria, the results of the application show that transmuted Dagum distribution is superior to the Dagum distribution and other related distribution. Finally, we hope that the proposed model will serve better in income distribution, actuarial, meteorological and survival data analysis.

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