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The gamma half-Cauchy distribution: properties and applications

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Abstract

A new distribution, namely, the *Gamma-Half-Cauchy* distribution is proposed. Various properties of the *Gamma-Half-Cauchy* distribution are studied in detail such as limiting behavior, moments, mean deviations and Shannon entropy. The model parameters are estimated by the method of maximum likelihood and the observed information matrix is obtained. Two data sets are used to illustrate the applications of *Gamma-Half-Cauchy* distribution.

Keywords: Folded Cauchy distribution, half-Cauchy distribution, gamma distribution, Shannon entropy.

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1. Introduction

Half-Cauchy distribution is the folded standard Cauchy distribution around the origin so that positive values are observed. Modeling with half (or folded) distributions has been proposed and five folded distributions have been reported so far in literature, namely, the students' t, normal, normal-slash, logistic and Cauchy. These folded distributions have been used in Bayesian paradigm when a proper prior is necessary. Although some applications of the half Cauchy distribution exist in the literature, but the fact that the finite moments of order greater than or equal to one do not exist, the central limit theorem does not hold. This fact reduces the applicability of this distribution in modeling real life scenarios.

A random variable X has the half-Cauchy (HC) distribution with scale parameter $\sigma > 0$, if its cumulative distribution function (cdf) is given by

$$F(x) = \frac{2}{\pi}\tan^{-1}(x/\sigma), \quad x > 0. \tag{1.1}$$

The probability density function (pdf) corresponding to (1.1) is

$$f(x) = \frac{2}{\pi\,\sigma}\left[1 + (x/\sigma)^2\right]^{-1}. \tag{1.2}$$

Henceforth, we denote by $X \sim$HC(σ), the random variable having the HC density in (1.2) with parameters σ. As a heavy tailed distribution, the HC distribution has been used as an alternative to exponential distribution to model dispersal distances [18] as the former predicts more frequent long distance dispersed events than the later. Paradis *et al.* [16] used the HC distribution to model ringing data on two species of tits (Parus caeruleus and Parus major) in Britain and Ireland.

Few generalizations of the HC distribution exist in the literature, namely, beta-half-Cauchy (BHC) by Cordeiro and Lemonte [9], Kumaraswamy-half-Cauchy (KHC) by Ghosh [11] and Marshall-Olkin half-Cauchy (MOHC) by Jacob and Kayakumar [13]. In this paper, we propose a new generalization of the HC distribution using the technique defined by Alzaatreh *et al.* [7].

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$ for $-\infty \leq a < b \leq \infty$ and let $F(x)$ be the cumulative distribution distribution function (cdf) of a random variable X such that the link function $W(\cdot) : [0,1] \longrightarrow [a, b]$ satisfies the following conditions:

$$\begin{cases} (i) & W(\cdot)\,\text{is differentiable and monotonically non-decreasing, and} \\ (ii) & W(0) \to a\,\text{and}\,W(1) \to b. \end{cases} \tag{1.3}$$

The *T-X family* of distributions defined by Alzaatreh *et al.* [7] as

$$G(x) = \int_a^{W[F(x)]} r(t)\,dt. \tag{1.4}$$

If $T \in (0, \infty)$, X is a continuous random variable and $W[F(x)] = -\log[1 - F(x)]$, then the pdf corresponding to (1.4) is given by

$$g(x) = \frac{f(x)}{1 - F(x)} r\Big(-\log\left[1 - F(x)\right]\Big) = h_f(x) r\Big(H_f(x)\Big), \tag{1.5}$$

where $h_f(x)$ and $H_f(x)$ are, respectively, the hazard and cumulative hazard function corresponding to $f(x)$. For more details about the *T-X family*, one is refer to Alzaatreh *et al.* [3, 6], Alzaatreh and Ghosh [5] and Lee *et al.* [14].

If a random variable T follows the gamma distribution with parameters α and β, $r(t) = \big(\beta^{\alpha}\,\Gamma(\alpha)\big)^{-1} t^{\alpha-1} \mathrm{e}^{-t/\beta}, \quad t \geq 0$. Then from (1.5),the pdf of *Gamma-X* family of distributions is given by

$$(1.6) \qquad g(x) = \frac{1}{\beta^\alpha \, \Gamma(\alpha)} \, f(x) \Big(- \log\big[1 - F(x)\big]\Big)^{\alpha-1} \big[1 - F(x)\big]^{\frac{1}{\beta}-1}.$$

The cdf corresponding to (1.6) is

$$(1.7) \qquad G(x) = \frac{1}{\Gamma(\alpha)} \, \gamma\Big(\alpha, -\beta^{-1} \log\big[1 - F(x)\big]\Big),$$

where $\gamma(\alpha, t) = \int_0^t u^{\alpha-1} \, \mathrm{e}^u \, du$ is the incomplete gamma function. Several properties of gamma-X family have been studied in literature. For more details see Alzaatreh *et al.* [3, **?**, 6, 4, 8].

The paper is unfolded as follows. In Section 2, we define a new generalization of the HC distribution, namely, *Gamma-half-Cauchy* (GHC) distribution. In Section 3, some properties of the GHC are investigated. The density of the order statistics is obtained in Section 4. In Section 5, the model parameters are estimated by the method of maximum likelihood and the observed information matrix is determined. In Section 6, we explore the usefulness of the proposed distribution by means of two real data sets. Finally, Section 7 offers some concluding remarks.

2. The gamma-half Cauchy (GHC) distribution

From (1.1), (1.2), (1.6) and (1.7), it follows that the pdf and cdf of the GHC are given by

$$\begin{aligned} g(x) &= \frac{2}{\pi \, \sigma \, \Gamma(\alpha) \, \beta^\alpha} \, \big[1 + (x/\sigma)^2\big]^{-1} \, \Big(- \log\big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]\Big)^{\alpha-1} \\ (2.1) \qquad & \quad \times \big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]^{\frac{1}{\beta}-1} \end{aligned}$$

and

$$(2.2) \qquad G(x) = \frac{1}{\Gamma(\alpha)} \, \gamma\Big(\alpha, -\beta^{-1} \, \log\big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]\Big),$$

respectively. Henceforth, a random variable having pdf in (2.2) is denoted by $X \sim \mathrm{GHC}(\alpha, \beta, \sigma)$.

Special cases of GHC distribution:

(i) If $\alpha = \beta = 1$ in (2.2), the GHC distribution reduces to the HC distribution with parameter σ.

(ii) If $\alpha = 1$ in (2.2), the GHC distribution reduces to the exponentiated HC distribution with parameters β and σ.

(iii) If $\alpha = n + 1$ and $\beta = 1$ in (2.2), the density of GHC reduces to the density of the nth upper record of the HC distribution.

Note that the special case in (ii) does not exist in the literature and it is considered another generalization of the HC distribution.

The survival function (sf), $S(x)$, hazard rate function (hrf), $h(x)$, and cumulative hazard rate function (chrf), $H(x)$, of X are, respectively, given by

$$S(x) = 1 - \frac{1}{\Gamma(\alpha)} \, \gamma\Big(\alpha, -\beta^{-1} \, \log\big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]\Big),$$

$$h(x) = \frac{2\Big(- \log\big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]\Big)^{\alpha-1} \big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]^{\frac{1}{\beta}-1}}{\pi \, \sigma \, \beta^\alpha \, (1 + (x/\sigma)^2) \, \Big\{\Gamma(\alpha) - \gamma\Big(\alpha, -\beta^{-1} \, \log\big[1 - 2\,\pi^{-1} \, \tan^{-1}(x/\sigma)\big]\Big)\Big\}}$$

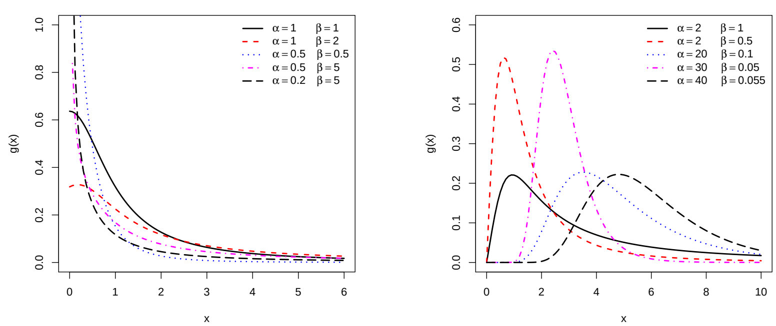
and

$$H(x) = -\ \log\left[1 - \frac{1}{\Gamma(\alpha)}\,\gamma\Big(\alpha, -\beta^{-1}\ \log\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\right]\Big)\right].$$

2.1. Asymptotic behavior of the pdf. The limit of the pdf of X as $x \to \infty$ is 0. Further, the limits of the pdf of X as $x \to 0^+$ are given by

$$\lim_{x\to 0^+} g(x) = \begin{cases} \infty, & \text{if } \alpha < 1 \\ \frac{2}{\pi\,\sigma\,\beta}, & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha > 1. \end{cases}$$

In Figures 1 and 2, various graphs of the density when $\sigma = 1$ and for different values of α and β are displayed. Figure 1 indicates that the GHC distribution is well-suited for right-skewed data. For fixed $\alpha \leq 1$, the density is always reversed-J shaped. For fixed $\alpha > 1$, the peakedness increases as β decreases. Also, Figure 2 shows that the hazard function of the GHC distribution has DFR (decreasing failure) or UBT (upside down bathtub) properties.



(a) (b)

Figure 1. Plots of the GHC densities for various values of α and β.

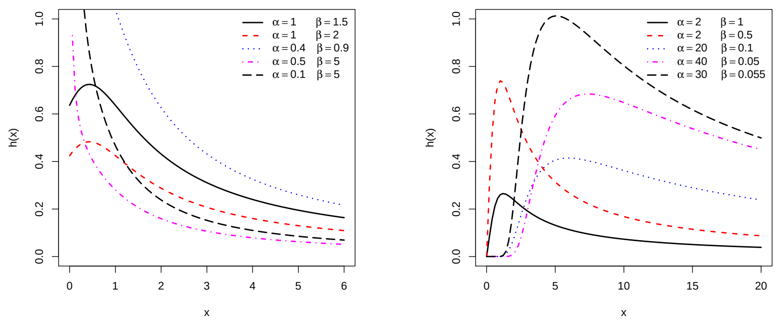
3. Properties of the GHC distribution

In this section, we provide some properties of the GHC distribution. Some proofs are omitted in case of trivial results.

The following Lemma gives the relation between GHC and gamma distributions.

3.1. Transformation.

3.1. Lemma. *If a random variable Y follows the gamma distribution with parameters α and β, then* $X = \sigma \cot\left(\frac{\pi}{2}\,\mathrm{e}^{-Y}\right) \sim \mathrm{GHC}(\alpha, \beta, \sigma)$.



(a) (b)

Figure 2. Plots of GHC hazard rates for various values of α and β.

3.2. Mode.

3.2. Lemma. *The mode of GHC distribution is the solution of* $k(x) = 0$*, where*

$$k(x) = -\frac{x}{\sigma} + \pi^{-1}\Big[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\Big]^{-1} \times \left\{\frac{\alpha - 1}{\log\left[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\right]^{-1}} - \frac{1}{\beta} + 1\right\}. \tag{3.1}$$

Proof. Setting $g'(x)$ is equivalent to,

$$g'(x) = \frac{4\left[1 + (x/\sigma)\right]^{-2}}{\pi\sigma^2\Gamma(\alpha)\beta^\alpha}\left(-\log\left[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\right]\right)^{\alpha-1} \times\Big[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\Big]^{\frac{1}{\beta}-1} \times k(x), \tag{3.2}$$

where

$$k(x) = -(x/\sigma) + \pi^{-1}\left[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\right]^{-1} \times\left\{\frac{\alpha - 1}{\log\left[1 - 2\pi^{-1} \tan^{-1}(x/\sigma)\right]^{-1}} - \frac{1}{\beta} + 1\right\}. \tag{3.3}$$

Hence the critical values of $g(x)$ is the solution of $k(x) = 0$. □

Note that equation implies the following; when $\alpha = \beta = 1$, the mode of GHC is at $x = 0$ which is the mode of HC distribution. When $\alpha < 1$, implies that $x < 0$ and as $x \to 0^+$, $k(x) \to \infty$. Also, when $\alpha = 1$, $x = 0$ is a modal point and as $x \to 0^+$, $k(x) \to \frac{2}{\pi\,\sigma\,\beta}$. Hence, when $\alpha \le 1$, GHC has a unique mode at $x = 0$.

3.3. Quantile function. The following Lemma gives the quantile function for the GHC distribution.

3.3. Lemma. *The mode of GHC distribution is given by*

$$Q(\lambda) = \sigma \cot\left(0.5\,\pi\,\mathrm{e}^{-\beta\,\gamma^{-1}\left(\alpha,\,\lambda\,\Gamma(\alpha)\right)}\right). \tag{3.4}$$

Proof. Follows by inverting equation 2.1. □

3.4. Shannon entropy.

3.4. Theorem. *The Shannon entropy for the GHC distribution is given by*

$$\eta_X = 3\log(0.5\,\pi) + \alpha(\beta-1) + \log\big(\beta\,\Gamma(\alpha)\big) + (1-\alpha)\,\psi(\alpha) - 2\sum_{k=1}^{\infty} w_k\,(1+2k\beta)^{-\alpha}, \tag{3.5}$$

where $\psi(\cdot)$ *is the digamma function and* $w_k = \frac{(-1)^k\,(\pi)^{2k}\,B_{2k}}{2k\,(2k)!}$.

Proof. Based on Alzaatreh *et al.* [8], the Shannon entropy for the gamma-X family is given by

$$\eta_X = -\mathbb{E}\left\{\log f\left(F^{-1}\left(1-\mathrm{e}^{-T}\right)\right)\right\} + \alpha(1-\beta) + \log\big(\beta\,\Gamma(\alpha)\big) + (1-\alpha)\,\psi(\alpha), \tag{3.6}$$

where $T \sim Gamma(\alpha,\beta)$.

We first need to find $-\mathbb{E}\left\{\log f\left(F^{-1}\left(1-\mathrm{e}^{-T}\right)\right)\right\}$, where $f(x)$ and $F(x)$ are the pdf and cdf of HC distribution. It follows immediately that $\log f\left(F^{-1}\left(1-\mathrm{e}^{-T}\right)\right) = \log(0.5\,\pi) + 2\log\left(\sin(0.5\,\pi\,\mathrm{e}^{-T})\right)$ and hence by using the series expansion for $\log\left(\sin(0.5\,\pi\,\mathrm{e}^{-T})\right)$ (see [12]) as

$$\log\left(\sin(0.5\,\pi\,\mathrm{e}^{-T})\right) = \log(0.5\,\pi) - T + \sum_{k=1}^{\infty} \underbrace{\frac{(-1)^k\,(\pi)^{2k}\,B_{2k}}{2k\,(2k)!}}_{w_k}\,\mathrm{e}^{-2kT}, \tag{3.7}$$

where B_{2k} is the Bernoulli number.

Therefore,

$$-\mathbb{E}\left\{\log f\left(F^{-1}\left(1-\mathrm{e}^{-T}\right)\right)\right\} = 3\,\log(0.5\,\pi) - 2\mathbb{E}(T) + 2\sum_{k=1}^{\infty} w_k\,\mathbb{E}\big(\mathrm{e}^{-2kT}\big). \tag{3.8}$$

The results in (3.5) followed by substituting (3.8) in (3.6) and noting that $\mathbb{E}(T) = \alpha\beta$ and $\mathbb{E}\big(\mathrm{e}^{-2kT}\big) = (1+2k\beta)^{-\alpha}$. □

3.5. Moments. By using the Lemma 3.1, the rth moments of GHC distribution can be written as

$$\mathbb{E}(X^r) = \frac{\sigma^r}{\beta^\alpha\,\Gamma(\alpha)} \int_0^\infty \Big(\cot(0.5\,\pi\,\mathrm{e}^{-u})\Big)^r\,u^{\alpha-1}\,\mathrm{e}^{-u/\beta}\,du. \tag{3.9}$$

A series expansion for $\cot(0.5\,\pi\,\mathrm{e}^{-u})$ can be obtained from [12] as follows

$$\cot(0.5\,\pi\,\mathrm{e}^{-u}) = \sum_{k=0}^{\infty} v_k\,\mathrm{e}^{-(2k-1)u}, \tag{3.10}$$

where $v_k = \frac{2\,(-1)^k\,(\pi)^{2k-1}\,B_{2k}}{(2k)!}$.

Hence,

$$\Big(\cot(0.5\,\pi\,\mathrm{e}^{-u})\Big)^r = \sum_{k_1,\dots,k_r=0}^{\infty} v_{k_1,\dots,k_r}\,\mathrm{e}^{-(2s_r-1)u},$$

where $v_{k_1,\dots,k^r} = v_{k=1}\,v_{k=2}\,\dots\,v_{k=r}$ and $s_r = k_1 + k_2 + \cdots + k_r$.

Therefore, from (3.9) we get

$$\mathbb{E}(X^r) = \sigma^r \sum_{k_1,\dots,k_r=0}^{\infty} v_{k_1,\dots,k_r}\,\big(2\beta\,s_r - \beta + 1\big)^{-\alpha}. \tag{3.11}$$

3.5. Theorem. *Let* $X \sim GHC(\alpha,\beta,\sigma)$, *then* $\mathbb{E}(X^r)$ *exists iff* $\beta < r^{-1}$.

Proof. The rth moment of GHC can be obtained from

$$\mathbb{E}(X^r) = \int_0^1 x^r\, g(x)\, dx + \int_1^\infty x^r\, g(x)\, dx, \tag{3.12}$$

where $g(x)$ is defined in (2.2).

Without loss of generality assume $\sigma = 1$. From (3.12), the existence of $\mathbb{E}(X^r)$ equivalent to the existence of $\int_1^\infty x^r\, g(x)\, dx$. Now,

$$\int_1^\infty x^r\, g(x)\, dx = \frac{1}{\pi\, \beta^\alpha\, \Gamma(\alpha)}\ \mathbb{I}, \tag{3.13}$$

where

$$\begin{aligned} \mathbb{I} \quad &= \quad \int_1^\infty \frac{x^r}{1+x^2}\, \left\{-\log\left[1-0.5\pi^{-1}\ \tan^{-1}(x)\right]\right\}^{\alpha-1} \\ &\qquad\qquad \times \left[1-0.5\pi^{-1}\ \tan^{-1}(x)\right]^{\frac{1}{\beta}-1}\, dx. \end{aligned} \tag{3.14}$$

Consider the following inequality (Abramowitz and Stegun [1])

$$x < -\log(1-x) < \frac{x}{1-x}, \quad x < 1, x \neq 0. \tag{3.15}$$

Now, for $\alpha \geq 1$, one can use the right hand-side of the inequality in (3.15) to show that

$$I < \int_1^\infty \underbrace{\frac{x^r}{1+x^2}\, \left[0.5\pi^{-1}\ \tan^{-1}(x)\right]^{\alpha-1}\, \left[1-0.5\pi^{-1}\ \tan^{-1}(x)\right]^{\frac{1}{\beta}-\alpha}\, dx}_{\tau_1(x)}\,. \tag{3.16}$$

Let $\tau_2(x) = x^{-\frac{1}{\beta}+\alpha+r-2}$, then $\lim_{x\to\infty} \frac{\tau_1(x)}{\tau_2(x)} = \left(0.5\pi^{-1}\right)^{\frac{1}{\beta}-\alpha}$. Therefore, $\int_1^\infty \tau_1(x)$ exists iff $\int_1^\infty \tau_2(x)$ exists iff $\frac{1}{\beta} > \alpha + r - 1$. Since $\alpha \geq 1$, this implies that $\frac{1}{\beta} > r$. If $\alpha < 1$, the left hand side of the inequality in (3.15) implies that

$$I < \int_1^\infty \frac{x^r}{1+x^2}\, \left[0.5\pi^{-1}\ \tan^{-1}(x)\right]^{\alpha-1}\, \left[1-0.5\pi^{-1}\ \tan^{-1}(x)\right]^{\frac{1}{\beta}-1}\, dx. \tag{3.17}$$

Similarly, one can show the right hand side of the integrand in exists iff $\frac{1}{\beta} > r$. This ends the proof. □

3.6. Mean deviations. The mean deviations about the mean $\left(\delta_1(X) = \mathbb{E}(|X-\mu_1'|)\right)$ and about the median $\left(\delta_2(X) = \mathbb{E}(|X-M|)\right)$ of X can be expressed as

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2m_1(\mu_1') \quad \text{and} \quad \delta_2(X) = 2\mu_1' - 2m_1(M), \tag{3.18}$$

respectively, where $\mu_1' = \mathbb{E}(x)$ can be obtained from (3.11) by setting $r = 1$ and M is the median of the GHC which can be calculated from Lemma 3.3 as

$$M = \sigma \cot\Big(0.5\, \pi\, \mathrm{e}^{-\beta\, \gamma^{-1}\left(\alpha,\, 0.5\, \Gamma(\alpha)\right)}\Big). \tag{3.19}$$

Further, $F(\mu_1')$ can easily be computed from the (2.1) and $m_1(z) = \int_0^z x\, f(x)\, dx$ (the first incomplete moment of X) can be computed from

$$m_1(z) = \int_0^z \cot(0.5\, \pi\, \mathrm{e}^{-u})\, u^{\alpha-1}\, \mathrm{e}^{-u/\beta}\, du. \tag{3.20}$$

The result immediately follows from (3.10) as

$$m_1(z) = \frac{\sigma}{\Gamma(\alpha)} \sum_{k=0}^\infty v_k\, (1+2\beta\, k-\beta)^{-\alpha}\ \gamma\Big(\alpha,\, \frac{z}{\beta}(1+2\beta\, k-\beta)\Big). \tag{3.21}$$

3.7. Mean residual life function. Let X be a random variable with cdf F such that $\mathbb{E}(X) < \infty$. The mean residual life (MRL) function $\boldsymbol{\xi}(x)$ of X is defined by $\boldsymbol{\xi}(x) = \mathbb{E}(X - x|X > x)$. It plays a major rule in many fields such as industrial reliability, life insurance and biomedical science. The following theorem provides an expansion for the MRL for the GHC distribution.

3.6. Theorem. *Let X be a random variable which follows the $GHC(\alpha, \beta, \sigma)$ such that $\beta < 1$, then the MRL function is given by*

$$(3.22) \qquad \boldsymbol{\xi}(x) = \frac{\sigma}{\Gamma(\alpha)\, S(x)} \sum_{k=0}^{\infty} v_k \, \frac{\Gamma(\alpha, (2k + \beta^{-1} - 1)\, x)}{(2\beta\, k - \beta + 1)^{\alpha}} - x$$

where $\Gamma(x, a) = \int_x^{\infty} t^{a-1}\, \mathrm{e}^{-t}\, dt$ is the upper incomplete gamma function and v_k is defined in (3.10) and $S(x)$ is survival function of GHC defined in section 2.

Proof. From Lemma (3.1)

$$\mathbb{E}(X|X > x) = \frac{\sigma}{\beta^{\alpha}\, \Gamma(\alpha) S(x)} \int_y^{\infty} \cot\left(0.5\pi \mathrm{e}^{-y}\right)\, y^{\alpha-1}\, \mathrm{e}^{-y/\beta}\, dy.$$

On using the expansion in (3.10), one can get the result in (3.22). □

3.8. Reliability estimation. The reliability parameter R is defined as $R = P(X > Y)$, where X and Y are independent random variables. Many applications of the reliability parameter have appeared in the literature such as the area of classical stress-strength model and the breakdown of a system having two components. If X and Y are two continuous random variables with cdfs $F_1(x)$ and $F_2(y)$ and their pdfs $f_1(x)$ and $f_2(y)$ respectively. Then, the reliability parameter R can be written as

$$(3.23) \qquad R = P(X > Y) = \int_{-\infty}^{\infty} F_2(x)\, f_1(x)\, dx.$$

3.7. Theorem. *Suppose that X and Y are two independent GHC random variables with parameters α_1, β_1 and α_2, β_2, and fixed scale parameter σ. Then*

$$(3.24) \qquad R = \frac{1}{\Gamma(\alpha_1)\, \Gamma(\alpha_2)} \sum_{k=0}^{\infty} \left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1+k} \frac{(-1)^k\, \Gamma(\alpha_1 + \alpha_2 + k)}{k!\, \Gamma(\alpha_2 + k)}.$$

Proof. On using the following series expansion from [1]

$$(3.25) \qquad \gamma(\alpha, x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{k+\alpha}}{k!\ (k+\alpha)},$$

and then substituting $u = -\log\left[1 - 2\pi^{-1}\, \tan^{-1}(x/\sigma)\right]$, (3.23) reduces to

$$(3.26) \qquad R = \frac{1}{\beta_1^{\alpha_1}\, \Gamma(\alpha_1)\, \Gamma(\alpha_2)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, (\alpha_2 + k)\, \beta_2^{\alpha_2+k}} \int_0^{\infty} u^{\alpha_1+\alpha_2+k-1}\, \mathrm{e}^{-u/\beta_1}\, du.$$

The result in (3.24) follows immediately from (3.26). □

3.9. Mixture representation of GHC density.

3.8. Theorem. *The GHC distribution is the linear combination of infinite exponentiated-HC densities*

$$(3.27) \qquad g(x) = \sum_{k=0}^{\infty} w_{i,j}\, h_{(\alpha+k+i,\, \sigma)}(x),$$

where $h_{(\alpha+k+i,\,\sigma)}(x)$ is the exponentiated-HC density with power parameter $\alpha+k+i$ and

$$w_{i,j}=\sum_{j=0}^{k}\sum_{i=0}^{\infty}\binom{k+1-\alpha}{k}\binom{k}{j}\binom{\frac{1}{\beta}-1}{i}\frac{(-1)^{j+k+i}\ p_{j,k}}{(\alpha-j-1)(\alpha+k+i)\,\Gamma(\alpha)\,\beta^{\alpha}}.$$

Proof. Based on the formula given at
`http:// functions.wolfram.com/ ElementaryFunctions/Log/06/01/04/03/`, we can write

$$\Big(-\log\big[1-2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\big]\Big)^{\alpha-1}=$$

(3.28) $$(\alpha-1)\sum_{k=0}^{\infty}\binom{k+1-\alpha}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}\,p_{j,k}}{(\alpha-j-1)}\Big[\frac{2}{\pi}\ \tan^{-1}(x/\sigma)\Big]^{\alpha+k-1}.$$

Here, the constants $p_{j,k}$ (for $j\ge 0$ and $k\ge 1$) can be determined recursively by

$$p_{j,k}=k^{-1}\sum_{m=1}^{\infty}[k-m(j+1)]\,c_m\ p_{j,k-m},$$

where $p_{j,0}=1$ and $c_k=(-1)^{k+1}\,(k+1)^{-1}$.
Now, using the generalized binomial series expansion

(3.29) $$\Big[1-2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\Big]^{\frac{1}{\beta}-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\frac{1}{\beta}-1}{i}\Big[\frac{2}{\pi}\ \tan^{-1}(x/\sigma)\Big]^{i},$$

where $\binom{a}{i}=a(a-1)\cdots(a-i+1)/i!$.
The result (3.27) follows immediately by substituting (3.28) and (3.29) in (2.2). □

Note that the second summation in $w_{i,j}$ is finite whenever β^{-1} is a natural number.

4. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1,\ldots,X_n$ is a random sample from the GHC distribution. Let $X_{i:n}$ denote the ith order statistic. Then, the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\,f(x)\,F(x)^{i-1}\ \{1-F(x)\}^{n-i}\\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}f(x)\,F(x)^{j+i-1}. \end{aligned}$$

Inserting (2.1) and (2.2) in the last equation and after some algebra, we obtain

$$\begin{aligned} f_{i:n}(x) = \sum_{j=0}^{n-i}\frac{(-1)^{j}\,\Gamma(n+1)\,(i+j)^{-1}}{\Gamma i\ \Gamma(j+1)\Gamma(n-i-j+1)}\Bigg\{&\frac{2}{\pi\,\sigma\,\Gamma(\alpha)\,\beta^{\alpha}}\ \big[1+(x/\sigma)^2\big]^{-1}\\ &\times\Big(-\log\big[1-2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\big]\Big)^{\alpha-1}\Big[1-2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\Big]^{\frac{1}{\beta}-1}\\ &\times\left[\frac{\gamma\Big(\alpha,-\beta^{-1}\log\big[1-2\,\pi^{-1}\ \tan^{-1}(x/\sigma)\big]\Big)}{\Gamma(\alpha)}\right]^{j+i-1}\Bigg\}. \end{aligned}$$

Hence,

(4.1) $$f_{i:n}(x)=\sum_{j=0}^{n-i}\eta_j\ f_{\alpha,\beta,(j+i)}(x),$$

where

$$\eta_j = \frac{(-1)^j\,\Gamma(n+1)}{(i+j)\,\Gamma(i)\,\Gamma(j+1)\,\Gamma(n-i-j+1)}$$

and $f_{\alpha,\beta,(j+i)}(x)$ is the exponentiated-GHC density with parameters $\big(\alpha,\beta,(i+j)\big)$.

Equation (4.1) is the main result of this section. It reveals that the pdf of the GHC order statistics is a linear combination of exponentiated-GHC densities. So, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and several others can be derived from those quantities of the GHC distribution.

5. Estimation and information matrix

In this section, the method of maximum likelihood estimation is used to estimate the GHC distribution parameters. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and regions and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. Let $x_1,\ldots,x_n$ be a sample of size n from the GHC distribution given by (2.2). The log-likelihood function for the vector of parameters $\Theta=(\alpha,\beta,\sigma)^\top$ can be expressed as

$$\begin{aligned}\ell &= n\,\log\left[\frac{2}{\pi\,\sigma\,\Gamma(\alpha)\,\beta^{\alpha}}\right]-\sum_{i=1}^{n}\log\left[1+(x_i/\sigma)^2\right]\\ &\quad+(\alpha-1)\sum_{i=1}^{n}\log\Big(-\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]\Big)\\ &\quad+\Big(\tfrac{1}{\beta}-1\Big)\sum_{i=1}^{n}\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]\end{aligned}$$

The components of the score vector $J(\Theta)$ are given by

$$\begin{aligned}J_\alpha &= -n\,\psi(\alpha)-n\,\log\beta+\sum_{i=1}^{n}\log\Big(-\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]\Big),\\ J_\beta &= -n\,\alpha\,\beta^{-1}-\beta^{-2}\sum_{i=1}^{n}\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right],\\ J_\sigma &= -n\,\sigma^{-1}+2\sigma^{-3}\sum_{i=1}^{n}x_i^2\,\left[1+(x_i/\sigma)^2\right]^{-1}\\ &\quad-2(\alpha-1)\,\pi^{-1}\,\sigma^{-2}\sum_{i=1}^{n}\left\{\frac{x_i\,\tan^{-1\prime}(x_i/\sigma)\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]^{-1}}{-\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]}\right\}\\ &\quad+2\left(\tfrac{1}{\beta}-1\right)\,\pi^{-1}\,\sigma^{-2}\sum_{i=1}^{n}\left\{\frac{x_i\,\tan^{-1\prime}(x_i/\sigma)}{\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]}\right\}.\end{aligned}$$

Setting these equations to zero and solving them simultaneously yield the maximum likelihood estimates (MLEs) of the model parameters. Numerical methods can be used to obtain the MLE $\widehat{\Theta}$. For example, the Newton-Raphson iterative technique could be applied to solve the likelihood equations and obtain $\widehat{\Theta}$ numerically. For interval estimation of the parameters, we require the 3×3 observed information matrix $J(\Theta)=\{-J_{rs}\}$ (for $r,s=\alpha,\beta,\sigma$) given in Appendix A. The observed information matrix can be determined numerically from standard maximization routines, which provide the observed information matrix as part of their output; e.g., one can use the R functions `optim` or `nlm`, the Ox function `MaxBFGS`, the SAS procedure `NLMixed`, among others, to compute $J(\Theta)$ numerically.

Under standard regularity conditions, the multivariate normal $N_3(0, J(\widehat{\Theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\gamma)\%$ confidence intervals for α, β and σ are given by $\hat{\alpha} \pm z_{\gamma^*/2} \times \sqrt{var(\hat{\alpha})}$, $\hat{\beta} \pm z_{\gamma^*/2} \times \sqrt{var(\hat{\beta})}$ and $\hat{\sigma} \pm z_{\gamma^*/2} \times \sqrt{var(\hat{\sigma})}$, respectively, where the $var(\cdot)$'s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma^*/2}$ is the quantile $(1-\gamma^*/2)$ of the standard normal distribution.

The likelihood ratio (LR) statistic can be used to check if the GHC distribution is strictly "superior" to the HC distribution for a given data set. The test of $H_0: \alpha = \beta = 1$ versus $H_1: H_0$ *is not true* is equivalent to compare the GHC and HC distributions and the statistic $w = -2\log\lambda = 2\{\ell(\widehat{\alpha}, \widehat{\beta}, \widehat{\sigma}) - \ell(1, 1, \tilde{\sigma})\}$, where $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\sigma}$ are the MLEs under H_1 and $\tilde{\sigma}$ is the MLE under H_0, is asymptotically follows chi-square distribution with 2 degrees of freedom. Similarly, the test of $H_0: \alpha = 1$ versus $H_1: \alpha \neq 1$ is equivalent to compare the GHC and exponentiated HC distributions with the statistic $w = 2\{\ell(\widehat{\alpha}, \widehat{\beta}, \widehat{\sigma}) - \ell(1, \tilde{\beta}, \tilde{\sigma})\}$, where $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\sigma}$ are the MLEs under H_1 and $\tilde{\beta}$ and $\tilde{\sigma}$ are the MLEs under H_0. In this case w is asymptotically follows chi-square distribution with 1 degrees of freedom.

5.1. Simulation study. We evaluate the performance of the maximum likelihood method for estimating the GHC parameters using Monte Carlo simulation for a total of twenty four parameter combinations and the process is repeated 200 times. Two different sample sizes $n = 100$ and 300 are considered. The MLEs and the standard deviations of the parameter estimates are listed in Table 1. The MLEs of α, β and σ are determined by solving the nonlinear equations $U(\Theta) = \mathbf{0}$. From Table 1, we note that the ML method performs well for estimating the model parameters. Also, as the sample size increases, the biases and the standard deviations of the MLEs decrease as expected.

Table 1: MLEs and standard deviations for various parameter values.

Sample size	Actual values			Estimated values			Standard deviations		
n	α	β	σ	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\sigma}$	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\sigma}$
100	0.5	0.5	1	0.5267	0.4094	3.6791	0.0060	0.0272	0.6534
	0.5	1.0	2	0.5212	0.9324	2.9044	0.0080	0.0308	0.4838
	0.5	1.5	1	0.5315	1.4004	1.1285	0.0085	0.0329	0.0556
	0.5	2.0	2	0.5168	1.9218	2.36179	0.0100	0.0426	0.1342
	1.0	0.5	1	1.0416	0.4409	2.2191	0.0164	0.0176	0.4728
	1.0	1	2	1.0741	0.9578	2.1989	0.0605	0.0186	0.0939
	1.0	1.5	1	1.3303	1.4166	1.0224	0.1236	0.0274	0.0513
	1.0	2.0	2	1.4304	1.8939	1.9073	0.1399	0.0424	0.1084
	1.5	0.5	1	1.7037	0.4683	1.3396	0.0992	0.0111	0.3024
	1.5	1.0	2	2.2656	0.9118	2.0189	0.2288	0.0194	0.1082
	1.5	1.5	1	2.1739	1.3711	0.9861	0.1726	0.0315	0.0570
	1.5	2.0	2	2.1626	1.8253	2.1688	0.1758	0.0455	0.2187
300	0.5	0.5	1	0.5070	0.4529	1.7515	0.0020	0.0100	0.1402
	0.5	1	2	0.5040	0.9787	2.1165	0.0022	0.0095	0.0311
	0.5	1.5	1	0.5075	1.4764	1.0328	0.0027	0.0124	0.0153
	0.5	2.0	2	0.5014	1.9610	2.1106	0.0026	0.0138	0.0291
	1.0	0.5	1	1.0140	0.5001	1.0231	0.0052	0.0047	0.0159
	1.0	1.0	2	1.0120	0.9854	2.0763	0.0069	0.0061	0.0299
	1.0	1.5	1	1.0196	1.4891	1.0263	0.0077	0.0075	0.0148
	1.0	2.0	2	1.0281	1.9801	2.0308	0.0084	0.0107	0.0314
	1.5	0.5	1	1.5326	0.4970	1.0183	0.0104	0.0036	0.0157
	1.5	1.0	2	1.6108	0.9887	1.9955	0.035	0.0059	0.0381
	1.5	1.5	1	1.7063	1.4497	0.9605	0.0479	0.0109	0.0193
	1.5	2.0	2	1.6754	1.9499	1.9397	0.0335	0.0160	0.0475

6. Applications

In this section, we provide two applications to real data to illustrate the importance of the GHC distribution. The model parameters are estimated by the method of maximum likelihood and three well-recognized goodness-of-fit statistics are calculated to compare the GHC distribution with other competing models.

The first data set represents the annual food discharge rates for the 39 years (1935-1973) at Floyd River located in James, Iowa, USA. The Floyd River data were reported by Mudholkar and Hutson [15] and Akinsete *et al.* [2]. The second data set consists of the waiting times between 65 consecutive eruptions of the Kiama Blowhole (da Silva *et al.* [10]; Pinho *et al.* [17]). The Kiama Blowhole is a tourist attraction located nearly 120km to the south of Sydney. The swelling of the ocean pushes the water through a hole bellow a cliff. The water then erupts through an exit usually drenching whoever is nearby. The times between eruptions of a 1340 hours period starting from July 12th of 1998 were recorded using a digital watch. Both data sets are reported in Appendix B.

We fitted the GHC model to the three data sets and compared it with other models: the BHC, KHC, EHC and HC. The measures of goodness-of-fit statistics including the log-likelihood function evaluated at the MLEs $(-\log \hat{\ell})$, Akaike information criterion (AIC) and Kolmogrov-Smirnov (K-S) are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out using the R-software.

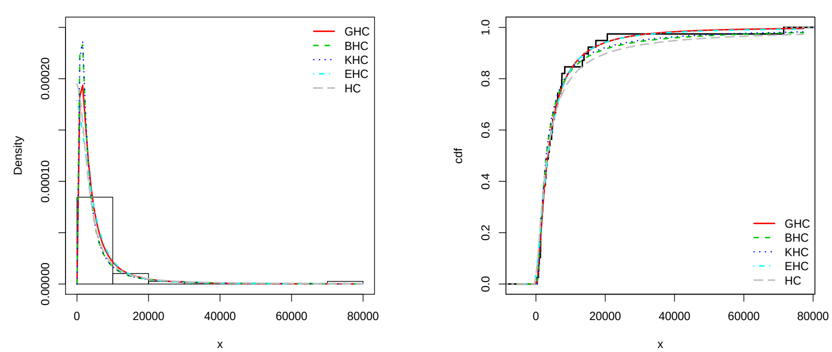
Table 2: The statistics $-\log \hat{\ell}$, AIC and K-S for the data sets 1, 2 and 3.

Distribution	α	β	σ	$-\log \hat{\ell}$	AIC	K-S	K-S p-value
			Data set 1				
GHC	45.9778	0.1554	4.4487	**376.3683**	**7758.7366**	**0.0648**	**0.9932**
	(101.8881)	(0.1758)	(34.8257)				
BHC	61.8037	1.1002	59.2418	377.9875	761.9750	0.0856	0.9141
	(158.0451)	(0.2233)	(146.9170)				
KHC	73.2921	1.1512	50.0133	377.8883	761.7766	0.0833	0.9287
	(322.4964)	(0.2615)	(215.1563)				
EHC	-	0.5947	5992.2102	378.5194	761.0387	0.1154	0.6351
	-	(0.2178)	(2563.2997)				
HC	-	-	3262.2630	379.6545	761.3090	0.1388	0.4029
	-	-	(661.1149)				
			Data set 2				
GHC	26.0412	0.1670	0.5759	**293.8255**	**593.6509**	**0.0962**	**0.5938**
	(40.6471)	(0.1346)	(1.9159)				
BHC	41.9366	1.6173	1.1937	294.9065	595.8130	0.1020	0.5189
	(91.5773)	(0.2709)	(2.4699)				
KHC	38.3343	1.7084	1.1251	294.8059	595.6118	0.1030	0.5059
	(87.5243)	(0.3251)	(2.4761)				
EHC	-	0.1283	211.1940	299.3473	602.6947	0.1576	0.0833
	-	(0.1585)	(251.4769)				
HC	-	-	28.3486	306.4299	614.8597	0.1595	0.0771
	-	-	(4.5011)				

Table 2 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters for data sets 1 and 2. The numerical values of the model selection statistics $-\log \hat{\ell}$, AIC and K-S, and p-values are listed in Table 2. In general, the results from Table 2 indicate that the GHC distribution provides the best fit among the BHC, KHC, EHC and HC models. The histogram of the data sets 1 and 2, and the estimated pdfs and cdfs of the GHC distribution and its competitive models are displayed in Figures 3 and 4. These Figures support the results in Table 2. To compare the GHC distribution with its sub-models, EHC and HC distributions, the LR test is used for both data sets 1 and 2. When comparing the fits between GHC and EHC (HC) for data 1, $w = 4.2902$ ($w = 6.5608$) with p-value=0.0383 (p-value=0.0376). For data 2, $w = 11.0436$ ($w = 12.6044$) with p-value=0.0009 (p-value=0.0018). These values suggest that GHC performs significantly better for both data sets when comparing it with the sub-models EHC and HC distributions.

7. Concluding remarks

In this paper, we propose a generalization of half-Cauchy distribution called the *gamma-half-Cauchy* distribution. We study some properties of gamma-half Cauchy distribution including quantile function, moments, mean deviations and Shannon entropy. The maximum likelihood method is used for estimating the model parameters and the observed information matrix is analytically derived. We fit the gamma-half-Cauchy to two real data sets to demonstrate its usefulness. The new model provides consistently better fit than other competing models.



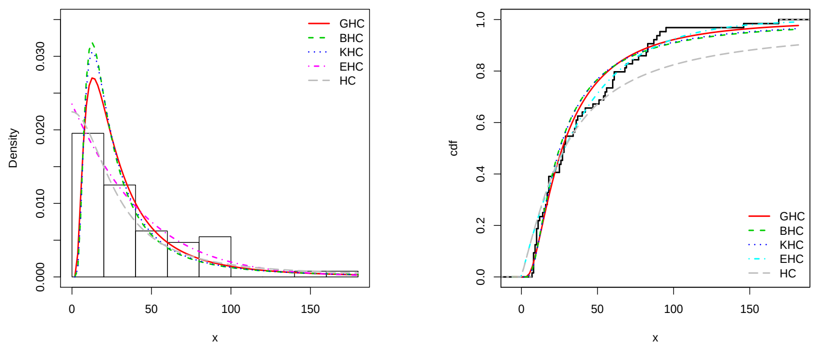
(a) Estimated pdfs (b) Estimated cdfs

Figure 3. Plots of the estimated pdfs and cdfs of the GHC, BHC, KHC and EHC models for data set 1.

Appendix A

The observed information matrix for the parameter vector $\Theta = (\alpha, \beta, \sigma)^\top$ is given by

$$J(\Theta) = -\frac{\partial^2 \ell(\Theta)}{\partial\,\Theta\,\partial\,\Theta^\top} = -\begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\sigma} \\ . & J_{\beta\beta} & J_{\beta\sigma} \\ . & . & J_{\sigma\sigma} \end{pmatrix},$$



(a) Estimated pdfs (b) Estimated cdfs

Figure 4. Plots of the estimated pdfs and cdfs of the GHC, BHC, KHC and EHC models for data set 2.

whose elements are

$$
\begin{aligned}
J_{\alpha\alpha} &= -n\,\psi'(\alpha)\ ,\\
J_{\alpha\beta} &= -\frac{n}{\beta}\ ,\\
J_{\alpha\sigma} &= \frac{2}{\pi\,\sigma^2}\sum_{i=1}^{n}\left\{\frac{x_i\,\tan^{-1\prime}(x_i/\sigma)\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]^{-1}}{-\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]}\right\}\ ,\\
J_{\beta\beta} &= \frac{n\alpha}{\beta}+\frac{2}{\beta^3}\sum_{i=1}^{n}\log\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]\ ,\\
J_{\beta\sigma} &= -\frac{2}{\pi\,\sigma^2\beta^2}\ \sum_{i=1}^{n}\left\{\frac{x_i\,\tan^{-1\prime}(x_i/\sigma)}{\left[1-2\,\pi^{-1}\,\tan^{-1}(x_i/\sigma)\right]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
J_{\sigma\sigma} \;=\; & \frac{2}{\sigma^2} + \sum_{i=1}^{n} \left\{ \frac{4\,x_i^4}{\sigma^6\,[1+(x_i/\sigma)^2]^2} - \frac{6\,x_i^2}{\sigma^4\,[1+(x_i/\sigma)^2]} \right\} \\
& - \left(\tfrac{1}{\beta} - 1\right) \sum_{i=1}^{n} \left\{ \frac{4\,x_i\ \tan^{-1\prime}(x_i/\sigma)}{\pi\,\sigma^3 \left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]} \right. \\
& \left. + \frac{4\,x_i^2\ \tan^{-1\prime}(x_i/\sigma)^2}{\pi^2\,\sigma^4 \left\{\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]\right\}^2} + \frac{2\,x_i^2\ \tan^{-1\prime\prime}(x_i/\sigma)}{\pi\,\sigma^4 \left\{\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]\right\}} \right\} \\
& - (\alpha - 1) \\
& \sum_{i=1}^{n} \left\{ \frac{4\,x_i\ \tan^{-1\prime}(x_i/\sigma)}{\pi\,\sigma^3\ \log\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]\left(1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right)} \right. \\
& + \frac{4\,x_i^2\ \tan^{-1\prime}(x_i/\sigma)^2}{\pi^2\,\sigma^4\ \log\left\{\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]\right\}^2 \left\{\left(1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right)\right\}^2} \\
& + \frac{4\,x_i^2\ \tan^{-1\prime}(x_i/\sigma)^2}{\pi^2\,\sigma^4\ \log\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right] \left\{\left(1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right)\right\}^2} \\
& \left. + \frac{2\,x_i^2\ \tan^{-1\prime\prime}(x_i/\sigma)}{\pi\,\sigma^4\ \log\left[1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right]\left(1 - 2\,\pi^{-1}\ \tan^{-1}(x_i/\sigma)\right)} \right\},
\end{aligned}
$$

where $\psi(\alpha) = \frac{\partial \log\,\Gamma(\alpha)}{\partial \alpha} = \frac{\Gamma(\alpha)'}{\Gamma(\alpha)}$ is the polygamma function and $\psi'(\alpha) = \frac{\partial^2\ \log\,\Gamma(\alpha)}{(\partial \alpha)^2} = \frac{\partial\,\psi(\alpha)}{\partial\,\alpha}$ is the trigamma function.

Appendix B

The first data set are: 1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440, 5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250, 2260, 318, 1330, 970, 1920, 15100, 2870, 20600, 3810, 726, 7500, 7170, 2000, 829, 17300, 4740, 13400, 2940, 5660.

The second data set were reported by professor Jim Irish and can be obtained at http://www.statsci.org/data/oz/kiama.html. The data are: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35,47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

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