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Inference on $Pr(X > Y)$ Based on Record Values from the Burr Type X Distribution

Bahman Tarvirdizade *† and Hossein Kazemzadeh Gharehchobogh*

Abstract

Our interest is in estimating the stress-strength reliability $Pr(X > Y)$ based on lower record values when X and Y are two independent but not identically distributed Burr type X random variables. The maximum likelihood estimator, Bayes and empirical Bayes estimators using Lindleys approximations, are obtained and their properties are studied. The exact confidence interval, as well as the Bayesian credible sets are obtained. Two examples are presented in order to illustrate the inferences discussed in the previous sections. A Monte Carlo simulation study is conducted to investigate and compare the performance of different types of estimators presented in this paper and to compare them with some bootstrap intervals.

Keywords: Likelihood estimation, Bayesian estimation, Burr type X distribution, Record values, Stress-strength reliability, Lindley approximation.

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1. Introduction

The problem of estimating $R = P(X > Y)$ arises in the context of mechanical reliability of a system with strength X and stress Y and R is chosen as a measure of system reliability. The system fails if and only if, at any time the applied stress is grater than its strength. This type of reliability model is known as the stress-strength model. This problem also arises in situations where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other. For example, in biometrical studies, the random variable X may represent the remaining lifetime of a patient treated with a certain drug while Y represent the remaining lifetime when treated by another drug. The estimation of stress-strength reliability is very common in the statistical literature. The reader is referred to Kotz et al. [1] for other applications and motivations for the study of the stress-strength reliability.

*Department of Statistics, Maku Branch, Islamic Azad University, Maku, Iran.
†Corresponding Author. Email: bahtary@gmail.com

Record values arise naturally many real life applications involving data relating to meteorology, hydrology, sports and life-tests. In industry and reliability studies, many products may fail under stress. For example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only values larger (or smaller) than all previous ones are recorded. Data of this type are called record data. Thus, the number of measurements made is considerably smaller than the complete sample size. This measurement saving can be important when the measurements of these experiments are costly if the entire sample was destroyed.

Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called an upper record if its value exceeds all previous observations, i.e. X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower records. Records were first introduced and studied by Chandler [2]. Interested readers may refer to the book by Arnold et al. [3] and the references contained therein.

Burr [4] introduced twelve different forms of cumulative distribution functions for modeling lifetime data or survival data. Among those twelve distribution functions, Burr type X and Burr type XII received the maximum attention. Several aspects of the Burr type X distribution were studied by Sartawi and Abu-Salih [5], Jaheen [6] and Raqab [7]. The cumulative distribution function (cdf) and the probability density function (pdf) of the Burr type X distribution with shape parameter θ, which will be denoted by $Burr(\theta)$, are respectively as follows,

$$F(x) = \left(1 - e^{-x^2}\right)^{\theta}, \quad x > 0, \quad \theta > 0, \tag{1.1}$$

$$f(x) = 2\theta x e^{-x^2}\left(1 - e^{-x^2}\right)^{\theta - 1}, \quad x > 0, \quad \theta > 0. \tag{1.2}$$

The problem of estimating the stress-strength reliability in the Burr type X distribution was considered by Ahmad et al. [8] and Surles and Padgett [9]. Kim and Chung [10] discussed Bayesian estimation of stress-strength reliability from Burr type X model containing spurious observations. We consider the problem of point and interval estimating the stress-strength reliability in the Burr type X distribution based on lower record values. The problem of interval estimating the stress-strength reliability based on record values was considered by Baklizi [11] for the generalized exponential distribution.

The rest of this paper is organized as follows: In Section 2, we discussed likelihood inference for the stress-strength reliability, while in Section 3 we considered Bayesian inference. In Section 4, we presented two numerical examples. A Monte Carlo simulation study is described in Section 5. Finally conclusion of the paper is provided in section 6.

2. Likelihood inference

Let X and Y be independent random variables from the Burr type X distribution with the parameters θ_1 and θ_2 respectively. Let $R = \Pr(X > Y)$ be the stress-strength reliability. then,

$$R = \int_0^{\infty}\int_y^{\infty} 2\theta_1 x e^{-x^2}\left(1 - e^{-x^2}\right)^{\theta_1 - 1} 2\theta_2 y e^{-y^2}\left(1 - e^{-y^2}\right)^{\theta_2 - 1} dx dy = \frac{\theta_1}{\theta_1 + \theta_2}.$$

Our interest is in estimating R based on lower record values on both variables. Let $\underset{\sim}{r} = (r_1, ..., r_n)$ be a set of lower records from $Burr(\theta_1)$ and let $\underset{\sim}{s} = (s_1, ..., s_m)$ be an independent set of lower records from $Burr(\theta_2)$. The likelihood functions are given by (Ahsanullah [12]),

$$L(\theta_1|\underset{\sim}{r}) = f(r_n)\prod_{i=1}^{n-1}\left(\frac{f(r_i)}{F(r_i)}\right), \quad 0 < r_n < ... < r_1 < \infty,$$

$$L(\theta_2|\underset{\sim}{s}) = g(s_m)\prod_{i=1}^{m-1}\left(\frac{g(s_i)}{G(s_i)}\right), \quad 0 < s_m < ... < s_1 < \infty. \tag{2.1}$$

where f and F are the pdf and cdf of $X \sim Burr(\theta_1)$ respectively and g and G are the pdf and cdf of $Y \sim Burr(\theta_2)$ respectively. Substituting f , F , g and G in the likelihood functions and using Equation(2.1), we obtain

$$L(\theta_1|\underset{\sim}{r}) = (2\theta_1)^n\left(1 - e^{-r_n^2}\right)^{\theta_1}\prod_{i=1}^{n}\left(\frac{r_i e^{-r_i^2}}{1 - e^{-r_i^2}}\right),$$

$$L(\theta_2|\underset{\sim}{s}) = (2\theta_2)^m\left(1 - e^{-s_m^2}\right)^{\theta_2}\prod_{i=1}^{m}\left(\frac{s_i e^{-s_i^2}}{1 - e^{-s_i^2}}\right). \tag{2.2}$$

It can be shown that the maximum likelihood estimators (MLE) of θ_1 and θ_2 based on the lower record values are

$$\hat{\theta}_1 = -\frac{n}{\ln(1 - e^{-r_n^2})}, \quad \hat{\theta}_2 = -\frac{m}{\ln(1 - e^{-s_m^2})}. \tag{2.3}$$

Therefore using the invariance properties of the maximum likelihood estimation, the MLE of R is given by

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}.$$

To study the distribution of $\hat{R}$ we need the distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$. Consider first $\hat{\theta}_1 = -\frac{n}{\ln(1-e^{-r_n^2})}$, the pdf of the nth lower record value R_n is given by (Ahsanullah [12]),

$$\begin{aligned} f_{R_n}(r_n) &= \tfrac{1}{(n-1)!}f(r_n)[-\ln F(r_n)]^{n-1} \\ &= \tfrac{2\theta_1^n}{(n-1)!}r_n e^{-r_n^2}\left(1 - e^{-r_n^2}\right)^{\theta_1 - 1}\left(-\ln\left(1 - e^{-r_n^2}\right)\right)^{n-1}, 0 < r_n < \infty. \end{aligned} \tag{2.4}$$

Consequently, the pdf of $Z_1 = \hat{\theta}_1$ is given by,

$$f_{Z_1}(z_1) = \frac{(n\theta_1)^n}{(n-1)!z_1^{n+1}}\exp\left(-\frac{n\theta_1}{z_1}\right), \quad z_1 > 0. \tag{2.5}$$

This is recognized as the inverted gamma distribution, i.e., $Z_1 \sim IGamma(n, n\theta_1)$. Similarly, the pdf of $Z_2 = \hat{\theta}_2$ is given by,

$$f_{Z_2}(z_2) = \frac{(m\theta_2)^m}{(m-1)!z_2^{m+1}}\exp\left(-\frac{m\theta_2}{z_2}\right), \quad z_2 > 0. \tag{2.6}$$

Thus $Z_2 \sim IGamma(m, m\theta_2)$. Therefore we can find the pdf of

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} = \frac{Z_1}{Z_1 + Z_2} = \frac{1}{1 + \frac{Z_2}{Z_1}}.$$

Consider Z_2/Z_1. Note that, by the properties of the inverted gamma distribution and its relation with the gamma distribution we have $(n\theta_1/Z_1) \sim Gamma(n, 1)$ and $(n\theta_2/Z_2) \sim Gamma(m, 1)$. Hence $(2n\theta_1/Z_1) \sim \chi^2_{2n}$ and $(2m\theta_2/Z_2) \sim \chi^2_{2m}$. Note that, by the independence of two random quantities we have

$$\frac{(2n\theta_1/2nZ_1)}{(2m\theta_2/2mZ_2)} = \frac{\theta_1 Z_2}{\theta_2 Z_1} \sim F_{(2n,2m)}.$$

Hence, $(Z_2/Z_1) = (\theta_2/\theta_1)F_{(2n,2m)}$, has a scaled F distribution. It follows that the distribution of $\hat{R}$ is that of $\frac{1}{1+(\theta_2/\theta_1)F_{(2n,2m)}}$ which can be obtained using simple transformation techniques. This fact can be used to construct the following $(1-\alpha)\%$ confidence interval for R,

$$\left(\left(1+\frac{z_2}{z_1 F_{\alpha/2,2n,2m}}\right)^{-1}, \left(1+\frac{z_2}{z_1 F_{1-\alpha/2,2n,2m}}\right)^{-1}\right). \tag{2.7}$$

3. Bayesian inference

Consider the likelihood functions of θ_1 and θ_2 based on the two sets of lower record values from the Burr type X distribution mentioned in Equation (2.2). We have

$$L(\theta_1|\underset{\sim}{r}) \propto \theta_1^n e^{-\theta_1\nu_1(r_n)}, \quad L(\theta_2|\underset{\sim}{s}) \propto \theta_2^m e^{-\theta_2\nu_2(s_m)} \tag{3.1}$$

where $\nu_1(r_n) = -\ln(1-e^{-r_n^2})$ and $\nu_2(s_m) = -\ln(1-e^{-s_m^2})$. These suggest that the conjugate family of prior distributions for θ_1 and θ_2 is the Gamma family of probability distributions,

$$\pi(\theta_1) = \frac{\gamma_1^{\delta_1}\theta_1^{\delta_1-1}e^{-\gamma_1\theta_1}}{\Gamma(\delta_1)}, \theta_1 > 0 \quad \textit{and} \quad \pi(\theta_2) = \frac{\gamma_2^{\delta_2}\theta_2^{\delta_2-1}e^{-\gamma_2\theta_2}}{\Gamma(\delta_2)}, \theta_2 > 0 \tag{3.2}$$

where δ_1, γ_1, δ_2 and γ_2 are the parameters of the prior distributions of θ_1 and θ_2 respectively. It can be shown that $(\theta_1|\underset{\sim}{r}) \sim Gamma\,(n+\delta_1, \gamma_1+\nu_1(r_n))$ and $(\theta_2|\underset{\sim}{s}) \sim Gamma\,(m+\delta_2, \gamma_2+\nu_2(s_m))$. Since the priors θ_1 and θ_2 are independent, then, using standard transformation techniques and after some manipulations, the posterior pdf of R will be

$$f_R(r) = C\frac{r^{n+\delta_1-1}(1-r)^{m+\delta_2-1}}{[r(\gamma_1+\nu_1(r_n))+(1-r)(\gamma_2+\nu_2(s_m))]^{n+m+\delta_1+\delta_2}}, 0<r<1 \tag{3.3}$$

where

$$C = \frac{\Gamma(n+m+\delta_1+\delta_2)}{\Gamma(n+\delta_1)\Gamma(m+\delta_2)}(\gamma_1+\nu_1(r_n))^{n+\delta_1}(\gamma_2+\nu_2(s_m))^{m+\delta_2}.$$

The Bayes estimator under squared error loss is the mean of this posterior distribution which can not be computed analytically. Alternatively, using the approximate method of Lindley [13], it can be seen that the approximate Bayes estimator of R, say $\tilde{R}_B$, relative to squared error loss function is

$$\tilde{R}_B = \tilde{R}\left(1+\frac{(1-\tilde{R})^2}{n+\delta_1-1} - \frac{\tilde{R}(1-\tilde{R})}{m+\delta_2-1}\right) \tag{3.4}$$

where $\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1+\tilde{\theta}_2}$ and

$$\tilde{\theta}_1 = \left(\frac{n+\delta_1-1}{\gamma_1+\nu_1(r_n)}\right), \quad \tilde{\theta}_2 = \left(\frac{m+\delta_2-1}{\gamma_2+\nu_2(s_m)}\right)$$

are the mode of the posterior densitys θ_1 and θ_2 respectively. On the other hand, it follows from the posterior density θ_1 and θ_2 that $2(\gamma_1+\nu_1(r_n))(\theta_1|\underset{\sim}{r}) \sim \chi^2_{2(n+\delta_1)}$ and $2(\gamma_2+\nu_2(s_m))(\theta_2|\underset{\sim}{s}) \sim \chi^2_{2(m+\delta_2)}$. It follows that $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, the posterior distribution of R is equal to that of $(1+AW)^{-1}$, where $W \sim F_{2(m+\delta_2),2(n+\delta_1)}$ and $A = \frac{(m+\delta_2)(\gamma_1+\nu_1(r_n))}{(n+\delta_1)(\gamma_2+\nu_2(s_m))}$. Therefore a Bayesian $(1-\alpha)\%$ confidence interval for R is given by,

$$\left((AF_{1-\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1)^{-1}, (AF_{\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1)^{-1}\right). \tag{3.5}$$

The case of a noninformative prior can be treated similarly. We consider Jeffereys prior that say, $\pi(\theta_1) \propto \sqrt{|I(\theta_1)|}$ where $I(\theta_1)$ is the Fisher information. This suggest that prior densitys for θ_1 and θ_2 are proportional to $\frac{1}{\theta_1}$ and $\frac{1}{\theta_2}$ respectively. Using direct arguments one can show that $(\theta_1|\underset{\sim}{r}) \sim Gamma\,(n, \nu_1(r_n))$ and $(\theta_2|\underset{\sim}{s}) \sim Gamma\,(m, \nu_2(s_m))$. Therefore, it can be seen that the approximate Bayes estimator of R under the Jeffreys prior density, say $\tilde{R}_{JB}$, relative to squared error loss function is

$$\tilde{R}_{JB} = \tilde{R}\left(1 + \frac{(1-\tilde{R})^2}{n-1} - \frac{\tilde{R}(1-\tilde{R})}{m-1}\right) \tag{3.6}$$

where $\tilde{R} = \frac{\tilde{\theta}_1}{\tilde{\theta}_1+\tilde{\theta}_2}$ and

$$\tilde{\theta}_1 = \left(\frac{n-1}{\nu_1(r_n)}\right), \quad \tilde{\theta}_2 = \left(\frac{m-1}{\nu_2(s_m)}\right).$$

Furthermore, it follows that the posterior distribution of R is equal to that of $(1+\frac{m\nu_1(r_n)}{n\nu_2(s_m)}W)^{-1}$ where $W \sim F_{2m,2n}$. Therefore a Bayesian $(1-\alpha)\%$ confidence interval for R is given by,

$$\left(\left(\frac{m\nu_1(r_n)}{n\nu_2(s_m)}F_{1-\alpha/2,2m,2n}+1\right)^{-1}, \left(\frac{m\nu_1(r_n)}{n\nu_2(s_m)}F_{\alpha/2,2m,2n}+1\right)^{-1}\right). \tag{3.7}$$

Now consider the case when the parameters of prior distributions are themselves unknown. We consider the conjugate prior distributions for θ_1 and θ_2 above when the parameters γ_1 and γ_2 are unknown. In the empirical Bayes model, we must estimate them. In order to, we calculate the marginal distribution of lower records, with densitys

$$m(\underset{\sim}{r}\,|\gamma_1) = \int f_{\underset{\sim}{R}}(\underset{\sim}{r}\,|\theta_1)\pi(\theta_1|\gamma_1)d\theta_1, \quad 0 < r_n < ... < r_1 < \infty,$$

$$m(\underset{\sim}{s}\,|\gamma_2) = \int f_{\underset{\sim}{S}}(\underset{\sim}{s}\,|\theta_2)\pi(\theta_2|\gamma_2)d\theta_2, \quad 0 < s_m < ... < s_1 < \infty.$$

Using Equations (2.2) and (3.2), we obtain

$$m(\underset{\sim}{r}\,|\gamma_1) = \frac{\Gamma(n+\delta_1)2^n\gamma_1^{\delta_1}}{\Gamma(\delta_1)(\gamma_1+\nu_1(r_n))^{n+\delta_1}}\prod_{i=1}^{n}\left(\frac{r_i e^{-r_i^2}}{1-e^{-r_i^2}}\right),$$

$$m(\underset{\sim}{s}\,|\gamma_2) = \frac{\Gamma(m+\delta_2)2^m\gamma_2^{\delta_2}}{\Gamma(\delta_2)(\gamma_2+\nu_2(s_m))^{m+\delta_2}}\prod_{i=1}^{m}\left(\frac{s_i e^{-s_i^2}}{1-e^{-s_i^2}}\right). \tag{3.8}$$

It can be shown that the maximum likelihood estimators (MLE) of γ_1 and γ_2 based on the marginal distributions (3.8) are

$$\hat{\gamma}_1 = \frac{\delta_1\nu_1(r_n)}{n}, \quad \hat{\gamma}_2 = \frac{\delta_2\nu_2(s_m)}{m}. \tag{3.9}$$

Substituting $\hat{\gamma}_1$ and $\hat{\gamma}_2$ into Equation (3.4), the approximate empirical Bayes estimator of R, say $\tilde{R}_{EB}$, relative to squared error loss function is given by,

$$\tilde{R}_{EB} = \tilde{R}^*\left(1 + \frac{(1-\tilde{R}^*)^2}{n+\delta_1-1} - \frac{\tilde{R}^*(1-\tilde{R}^*)}{m+\delta_2-1}\right) \tag{3.10}$$

where $\tilde{R}^* = \frac{\tilde{\theta}_1^*}{\tilde{\theta}_1^*+\tilde{\theta}_2^*}$ and

$$\tilde{\theta}_1^* = \left(\frac{n+\delta_1-1}{\nu_1(r_n)\left(1+\frac{\delta_1}{n}\right)}\right), \quad \tilde{\theta}_2^* = \left(\frac{m+\delta_2-1}{\nu_2(s_m)\left(1+\frac{\delta_2}{m}\right)}\right)$$

Furthermore, it can be shown that $(\theta_1|\underset{\sim}{r},\hat{\gamma}_1) \sim Gamma(n+\delta_1,(1+\frac{\delta_1}{n})\nu_1(r_n))$ and $(\theta_2|\underset{\sim}{s},\hat{\gamma}_2) \sim Gamma(m+\delta_2,(1+\frac{\delta_2}{m})\nu_2(s_m))$. It follows that $\pi(R|\underset{\sim}{r},\hat{\gamma}_1,\underset{\sim}{s},\hat{\gamma}_2)$, the empirical posterior distribution of R is equal to that of $(1+\frac{m\nu_1(r_n)}{n\nu_2(s_m)}W)^{-1}$ where $W \sim F_{2(m+\delta_2),2(n+\delta_1)}$. Therefore a Bayesian $(1-\alpha)\%$ confidence interval for R is given by,

$$\left(\left(\frac{m\nu_1(r_n)}{n\nu_2(s_m)}F_{1-\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1\right)^{-1},\left(\frac{m\nu_1(r_n)}{n\nu_2(s_m)}F_{\alpha/2,2(m+\delta_2),2(n+\delta_1)}+1\right)^{-1}\right). \tag{3.11}$$

The construction of highest posterior density (HPD) regions requires finding the set $I=\{\theta:\pi(\theta|\underset{\sim}{r},\underset{\sim}{s})\geq k_\alpha\}$, where k_α is the largest constant such that $\Pr(\theta\in I)\geq 1-\alpha$. This often requires numerical optimization techniques. Chen and Shao [14] presented a simple Monte Carlo technique to approximate the HPD region.

4. Illustrative examples

In this section, two numerical examples are presented to illustrate the inferences discussed in the previous sections.

Example 4.1 (Real Data Set). We consider a data analysis for two data sets reported by Bennett and Filliben [15]. They have reported minority electron mobility for p-type $Ga_{1-x}Al_xAs$ with seven different values of mole fraction. We use two data sets related to the mole fractions 0.25 and 0.30. These data are given as follows:

Data Set 1 (belongs to mole fraction 0.25): 3.051, 2.779, 2.604, 2.371, 2.214, 2.045, 1.715, 1.525, 1.296, 1.154, 1.016, 0.7948, 0.7007, 0.6292, 0.6175, 0.6449, 0.8881, 1.115, 1.397, 1.506, 1.528.

Data Set 2 (belongs to mole fraction 0.30): 2.658, 2.434, 2.288, 2.092, 1.959, 1.814, 1.530, 1.366, 1.165, 1.041, 0.9198, 0.7241, 0.6403, 0.576, 0.5647, 0.5873, 0.8013, 1.002, 1.250, 1.347, 1.368.

We fit the Burr type X distribution to the two data sets separately. We used the Kolmogorov-Smirnov (K-S) tests for each data set to fit the Burr type X model. It is observed that for data sets 1 and 2, the K-S distances are 0.2453 and 0.2026 with the corresponding p-values 0.1395 and 0.3110, respectively. Therefore, it is clear that Burr type X model fits well to both the data sets. Moreover, we plot the empirical distribution functions and the fitted distribution functions in Figure 1. This figure show that the empirical and fitted models are very close for each data set.

For the above data, we observe that the first 15 values for both the data sets are the lower record values and the smallest records, r_n and s_m, are equal to 0.6175 and 0.5647, respectively. Therefore, we obtain the MLEs of θ_1 and θ_2 as, 13.0576 and 11.5551, respectively. Thus, the MLE of R becomes $\hat{R}=0.5305$. The corresponding 95% confidence interval based on Equation (2.7) is equal to (0.3527,0.7009). To obtain Bayes estimates, we assume $\delta_1=\delta_2=3$ and $\gamma_1=\gamma_2=2$ in Equation (3.4). We obtain $\tilde{\theta}_1=5.3990$, $\tilde{\theta}_2=5.15440$ and $\tilde{R}=0.5116$. Therefore, the approximate Bayes estimator of R becomes $\tilde{R}_B=0.5113$. The corresponding Bayesian 95% confidence interval based on Equation (3.5) is equal to (0.3504,0.6704). So, the approximate Bayes estimator of R based on Equation (3.6) becomes $\tilde{R}_{JB}=0.5294$ and the corresponding Bayesian 95% confidence interval based on Equation (3.7) is equal to (0.3527,0.7009). Finally, using Equation (3.10), we obtain $\tilde{\theta}_1^*=12.3322$, $\tilde{\theta}_2^*=10.9131$ and $\tilde{R}^*=0.5305$. Therefore, the approximate empirical Bayes estimator of R becomes $\tilde{R}_{EB}=0.5269$. The corresponding Bayesian 95% confidence interval based on Equation (3.11) is equal to (0.3678,0.6869).

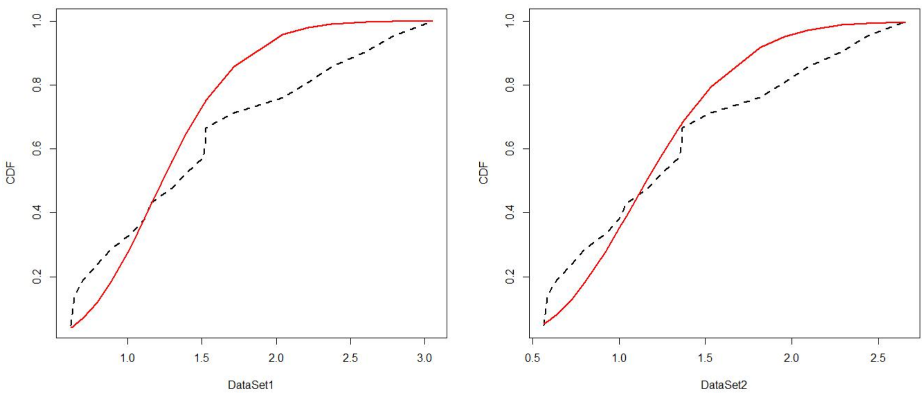

Figure 1. The empirical distribution function (dashed) and fitted distribution function for Data Sets 1 and 2.

Example 4.2 (Simulated Data). We simulate 6 lower record values from $Burr(1.5)$ and 8 lower record values from $Burr(2.5)$. Therefore, $R_{Exact} = 0.375$. The data has been truncated after four decimal places and it has been presented below. The $\underset{\sim}{r}$ lower record values are

$$1.2483, 1.0473, 0.6649, 0.2187, 0.1846, 0.0730,$$

and the corresponding $\underset{\sim}{s}$ lower record values are

$$1.4244, 0.5154, 0.4293, 0.3531, 0.2727, 0.2266, 0.1173, 0.0942.$$

Based on the above data, we obtain the MLEs of θ_1 and θ_2 as, 1.1456 and 1.6916, respectively. Therefore, the MLE of R becomes $\hat{R} = 0.4037$. The corresponding 95% confidence interval based on Equation (2.7) is equal to (0.1768,0.6617). Letting $\delta_1 = \delta_2 = 2$ and $\gamma_1 = \gamma_2 = 4$ in Equation (3.4), we obtain $\tilde{\theta}_1 = 0.7578$, $\tilde{\theta}_2 = 1.0310$ and $\tilde{R} = 0.4236$. Therefore, the approximate Bayes estimator of R becomes $\tilde{R}_B = 0.4321$. The corresponding Bayesian 95% confidence interval based on Equation (3.5) is equal to (0.2199,0.6582). So, the approximate Bayes estimator of R based on Equation (3.6) becomes $\tilde{R}_{JB} = 0.4077$ and the corresponding Bayesian 95% confidence interval based on Equation (3.7) is equal to (0.1768,0.6617). Finally, using Equation (3.10), we obtain $\tilde{\theta}_1^* = 1.0024$, $\tilde{\theta}_2^* = 1.5224$ and $\tilde{R}^* = 0.3970$. Therefore, the approximate empirical Bayes estimator of R becomes $\tilde{R}_{EB} = 0.4070$. The corresponding Bayesian 95% confidence interval based on Equation (3.11) is equal to (0.2016,0.6329).

5. A simulation study

In this section, a simulation study is conducted to investigate the performance of different types of estimators presented in this paper and to compare them with some bootstrap intervals. It is important here to note that all inference procedures in this paper depend only on the smallest records, r_n and s_m. In the simulation design we used all combinations of $n = 5, 10, 15$ and $m = 5, 10, 15$. We used $\theta_1 = 1$ and $R = 0.1, 0.25, 0.5$. The value of θ_2 is determined by the choice of θ_1 and R. In Bayesian simulation, we used

Table 1. Simulated biases and mean squared errors (in parentheses) of the estimators

n	m	R	ML	$Bayes$	$J.Bayes$	$E.Bayes$
5	5	0/1	0.0149(0.0050)	0.2724(0.0765)	0.0320(0.0063)	0.0247(0.0057)
5	5	0/25	0.0211(0.0162)	0.1633(0.0296)	0.0390(0.0157)	0.0313(0.0158)
5	5	0/5	0.0027(0.0234)	0.0013(0.0050)	0.0025(0.0192)	0.0026(0.0210)
5	10	0/1	0.0179(0.0040)	0.1858(0.0363)	0.0252(0.0044)	0.0234(0.0043)
5	10	0/25	0.0260(0.0129)	0.0990(0.0125)	0.0311(0.0120)	0.0303(0.0124)
5	10	0/5	0.0158(0.0191)	-0.0212(0.0056)	0.0057(0.0163)	0.0103(0.0173)
5	15	0/1	0.0176(0.0039)	0.1377(0.0205)	0.0218(0.0040)	0.0212(0.0040)
5	15	0/25	0.0238(0.0115)	0.0636(0.0065)	0.0245(0.0105)	0.0253(0.0109)
5	15	0/5	0.0173(0.0166)	-0.0326(0.0057)	0.0035(0.0143)	0.0094(0.0151)
10	5	0/1	0.0055(0.0025)	0.2931(0.0880)	0.0229(0.0035)	0.0156(0.0030)
10	5	0/25	0.0071(0.0104)	0.1844(0.0369)	0.0292(0.0108)	0.0201(0.0105)
10	5	0/5	-0.0076(0.0183)	0.0254(0.0056)	0.0020(0.0158)	-0.0024(0.0166)
10	10	0/1	0.0071(0.0019)	0.2028(0.0429)	0.0150(0.0022)	0.0130(0.0021)
10	10	0/25	0.0120(0.0069)	0.1193(0.0169)	0.0212(0.0070)	0.0189(0.0069)
10	10	0/5	-0.0012(0.0122)	-0.0007(0.0049)	-0.0012(0.0111)	-0.0012(0.0114)
10	15	0/1	0.0079(0.0018)	0.1531(0.0251)	0.0128(0.0019)	0.0120(0.0019)
10	15	0/25	0.0108(0.0061)	0.0831(0.0095)	0.0154(0.0060)	0.0148(0.0060)
10	15	0/5	0.0022(0.0097)	-0.0118(0.0045)	-0.0014(0.0090)	-0.0001(0.0091)
15	5	0/1	0.0035(0.0022)	0.3020(0.0932)	0.0210(0.0031)	0.0137(0.0027)
15	5	0/25	0.0024(0.0092)	0.1944(0.0405)	0.0260(0.0098)	0.0164(0.0094)
15	5	0/5	-0.0171(0.0165)	0.0333(0.0057)	-0.0033(0.0143)	-0.0092(0.0150)
15	10	0/1	0.0058(0.0015)	0.2115(0.0464)	0.0139(0.0018)	0.0119(0.0017)
15	10	0/25	0.0051(0.0061)	0.1267(0.0188)	0.0157(0.0062)	0.0130(0.0062)
15	10	0/5	-0.0023(0.0101)	0.0119(0.0047)	0.0013(0.0093)	0.0001(0.0095)
15	15	0/1	0.0053(0.0012)	0.1609(0.0273)	0.0105(0.0014)	0.0096(0.0014)
15	15	0/25	0.0067(0.0046)	0.0921(0.0109)	0.0128(0.0047)	0.0117(0.0047)
15	15	0/5	-0.0005(0.0081)	-0.0003(0.0042)	-0.0005(0.0076)	-0.0005(0.0076)

$\delta_1 = \delta_2 = 3$ and $\gamma_1 = \gamma_2 = 5$ where it is needed. All the results are based on 2000 replications.

First, we compare the performance of point estimators of R in terms of their biases and mean squared errors (MSEs). In order to, we compute the average biases and mean squared errors (MSEs) as

$$Bias = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{R}_i - R), \quad MSE = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{R}_i - R)^2$$

where $\hat{R}$ can be each of the maximum likelihood estimator and the approximate Bayes estimators based on Equations (3.4), (3.6) and (3.10). The results are reported in Table 1.

Next, a simulation study is conducted to investigate and compare the performance of the confidence intervals presented in this paper and some bootstrap intervals in terms of their coverage probability and expected length. There are several bootstrap based intervals discussed in the literature (Efron and Tibshirani [16]). Since all inferences in this paper depend only on the smallest records, therefore we shall use the parametric bootstrap based on the marginal distribution of R_n as given in Equation (2.4). In follows we describe the bootstrapping procedure:

1) Calculate $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{R}$, the maximum likelihood estimators of θ_1, θ_2 and R based on r_n and s_m.

2) Generate r_n^* from the distribution given in Equation (2.4) with θ_1 replaced by $\hat{\theta}_1$ and generate s_m^* similarly.

3) Calculate $\hat{\theta}_1^*$, $\hat{\theta}_2^*$ and $\hat{R}^*$ using the r_n^* and s_m^* obtained in step 2.

4) Repeat steps 2 and 3, B times to obtain $\hat{R}_1^*, ..., \hat{R}_B^*$.

Then we can calculate the following bootstrap intervals;

Normal Interval: The simplest $(1-\alpha)$ bootstrap interval is the Normal interval

$$(\hat{R} - z_{1-\alpha/2}\hat{se_{boot}}, \hat{R} + z_{1-\alpha/2}\hat{se_{boot}})$$

where $\hat{se_{boot}}$ is the bootstrap estimate of the standard error based on $\hat{R_1^*}, ..., \hat{R_B^*}$.

Basic Pivotal Interval: The $(1-\alpha)$ bootstrap basic pivotal confidence interval is

$$(2\hat{R} - \hat{r}^*_{(1-\alpha/2)B}, 2\hat{R} - \hat{r}^*_{(\alpha/2)B})$$

where $\hat{r}_\beta^*$ is the β quantile of $\hat{R_1^*}, ..., \hat{R_B^*}$.

Percentile Interval: The $(1-\alpha)$ bootstrap percentile interval is defined by

$$(\hat{r}^*_{(1-\alpha/2)B}, \hat{r}^*_{(1-\alpha/2)B})$$

that is, just use the $\alpha/2$ and $1-\alpha/2$ quantiles of the bootstrap sample.

Interested readers may refer to DiCiccio and Efron [17] and the references contained therein to observe more details.

For each generated pair of samples we calculated the following intervals;

1) ML: The interval based on the MLE given in Equation (2.7).

2) Bayes: The interval based on the Bayes estimator given in Equation (3.5).

3) J.B: The interval based on the Bayes estimator given in Equation (3.7).

4) E.B: The interval based on the empirical Bayes estimator given in Equation (3.11).

5) Norm: The normal interval.

6) Basic: The basic pivotal interval.

7) Perc: The percentile interval.

The empirical coverage probability and expected lengths of intervals are obtained by using the 2000 replications. For bootstrap intervals we used 1000 bootstrap samples. The results of our simulations for confidence level $(1-\alpha) = 0.95$ and 0.90 are given in Tables 2 and 3 respectively.

6. Conclusion and discussion

Based on simulation results in Table 1, we observe that the biases and the mean squared errors (MSEs) of the estimators are very close, especially for larger sample sizes. It appears that the performance of the MLE and the approximate Bayes estimators based on Equations (3.6) and (3.10) is almost the same in terms of their biases and mean squared errors (MSEs) but the MLE has the better performance for small values of R. Furthermore, the approximate Bayes estimators based on Equations (3.4) has the weak performance specially for small values of R. Hence, between the point estimators presented in this paper, we recommend to use the MLE.

Based on simulation results in Tables 2 and 3, it appears that the length of the intervals is maximized when $R = 0.5$ and gets shorter and shorter as we move away to the extremes. Increasing the sample size on either variable also results in shorter intervals. The performance of the both basic pivotal interval and percentile interval is similar in terms of expected length but in terms of coverage rate percentile interval has the better performance. The percentile interval appears to be the best among bootstrap intervals. The interval based on the MLE and the interval based on the Bayes estimator given in Equation (3.7) appears to perform almost as well as the percentile interval. The interval based on the Bayes estimator given in Equation (3.5) has the low coverage rate and the long expected length for small values of R since it is dependent on γ_1 and γ_2 values. Furthermore, the interval based on the empirical Bayes estimator has the shortest expected length between the other intervals but it has the low coverage rate. It appears that the intervals based on the MLE, the Bayes estimator given in Equation (3.7) and percentile interval simultaneously has the short expected length and very good coverage rate in comparison with the other intervals. Hence, we recommend to use this confidence intervals in all.

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Table 2. Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.95

n	m	R	ML	$Bayes$	$J.B$	$E.B$	$Norm$	$Basic$	$Perc$
5	5	0/1	0.275(0.947)	0.437(0.391)	0.276(0.951)	0.209(0.878)	0.282(0.937)	0.274(0.813)	0.274(0.947)
5	5	0/25	0.452(0.953)	0.449(0.901)	0.448(0.948)	0.353(0.872)	0.463(0.905)	0.451(0.802)	0.451(0.950)
5	5	0/5	0.538(0.952)	0.460(0.996)	0.538(0.952)	0.433(0.878)	0.551(0.891)	0.538(0.792)	0.538(0.949)
5	10	0/1	0.216(0.953)	0.344(0.408)	0.218(0.947)	0.179(0.882)	0.253(0.956)	0.246(0.835)	0.246(0.949)
5	10	0/25	0.388(0.952)	0.381(0.985)	0.388(0.945)	0.323(0.883)	0.423(0.933)	0.415(0.844)	0.415(0.946)
5	10	0/5	0.481(0.947)	0.415(0.995)	0.481(0.949)	0.399(0.890)	0.488(0.901)	0.479(0.828)	0.479(0.937)
5	15	0/1	0.204(0.939)	0.291(0.569)	0.202(0.957)	0.168(0.906)	0.249(0.963)	0.242(0.869)	0.242(0.927)
5	15	0/25	0.369(0.954)	0.344(0.966)	0.368(0.956)	0.308(0.906)	0.410(0.941)	0.403(0.859)	0.403(0.926)
5	15	0/5	0.461(0.948)	0.394(0.985)	0.460(0.952)	0.382(0.902)	0.465(0.906)	0.457(0.847)	0.457(0.931)
10	5	0/1	0.236(0.950)	0.405(0.403)	0.238(0.952)	0.182(0.900)	0.210(0.917)	0.206(0.827)	0.206(0.946)
10	5	0/25	0.405(0.955)	0.414(0.726)	0.403(0.949)	0.323(0.887)	0.381(0.905)	0.374(0.833)	0.374(0.935)
10	5	0/5	0.482(0.956)	0.416(0.994)	0.484(0.960)	0.402(0.909)	0.489(0.903)	0.480(0.834)	0.480(0.941)
10	10	0/1	0.178(0.942)	0.316(0.508)	0.177(0.953)	0.152(0.914)	0.180(0.937)	0.177(0.868)	0.177(0.940)
10	10	0/25	0.323(0.954)	0.344(0.864)	0.324(0.954)	0.284(0.912)	0.326(0.928)	0.322(0.865)	0.322(0.955)
10	10	0/5	0.405(0.942)	0.367(0.994)	0.406(0.950)	0.358(0.917)	0.409(0.909)	0.404(0.865)	0.404(0.938)
10	15	0/1	0.156(0.947)	0.263(0.560)	0.157(0.949)	0.139(0.927)	0.166(0.951)	0.163(0.871)	0.163(0.944)
10	15	0/25	0.297(0.946)	0.307(0.938)	0.296(0.955)	0.265(0.924)	0.308(0.932)	0.305(0.885)	0.305(0.940)
10	15	0/5	0.375(0.950)	0.342(0.990)	0.375(0.953)	0.336(0.925)	0.378(0.919)	0.374(0.881)	0.374(0.948)
15	5	0/1	0.223(0.946)	0.389(0.483)	0.224(0.958)	0.171(0.903)	0.189(0.913)	0.185(0.841)	0.185(0.942)
15	5	0/25	0.389(0.957)	0.396(0.750)	0.385(0.954)	0.308(0.891)	0.356(0.893)	0.349(0.844)	0.349(0.928)
15	5	0/5	0.460(0.952)	0.394(0.992)	0.460(0.955)	0.382(0.902)	0.466(0.915)	0.458(0.855)	0.458(0.939)
15	10	0/1	0.160(0.950)	0.299(0.510)	0.160(0.951)	0.140(0.927)	0.154(0.926)	0.152(0.861)	0.152(0.944)
15	10	0/25	0.298(0.952)	0.326(0.781)	0.301(0.946)	0.267(0.913)	0.292(0.920)	0.289(0.877)	0.289(0.944)
15	10	0/5	0.375(0.950)	0.341(0.985)	0.374(0.942)	0.335(0.906)	0.379(0.920)	0.375(0.884)	0.375(0.951)
15	15	0/1	0.140(0.946)	0.246(0.518)	0.139(0.952)	0.126(0.920)	0.142(0.943)	0.140(0.885)	0.140(0.946)
15	15	0/25	0.268(0.945)	0.286(0.854)	0.267(0.950)	0.243(0.924)	0.270(0.937)	0.267(0.886)	0.267(0.947)
15	15	0/5	0.339(0.952)	0.315(0.982)	0.339(0.951)	0.310(0.928)	0.341(0.919)	0.339(0.886)	0.339(0.950)

Table 3. Expected lengths and coverage rates (in parentheses) of the confidence intervals with (1-α)=0.90

n	m	R	ML	$Bayes$	$J.B$	$E.B$	$Norm$	$Basic$	$Perc$
5	5	0/1	0.225(0.888)	0.371(0.340)	0.226(0.905)	0.172(0.813)	0.236(0.899)	0.224(0.791)	0.224(0.889)
5	5	0/25	0.382(0.892)	0.382(0.716)	0.378(0.895)	0.297(0.802)	0.388(0.857)	0.382(0.769)	0.382(0.889)
5	5	0/5	0.461(0.898)	0.392(0.995)	0.461(0.910)	0.368(0.805)	0.462(0.831)	0.461(0.753)	0.461(0.895)
5	10	0/1	0.180(0.904)	0.291(0.430)	0.181(0.892)	0.149(0.814)	0.212(0.932)	0.200(0.811)	0.200(0.896)
5	10	0/25	0.329(0.909)	0.323(0.924)	0.329(0.898)	0.273(0.814)	0.355(0.897)	0.348(0.816)	0.348(0.900)
5	10	0/5	0.411(0.902)	0.353(0.980)	0.410(0.899)	0.339(0.818)	0.410(0.846)	0.409(0.781)	0.409(0.890)
5	15	0/1	0.170(0.891)	0.245(0.447)	0.168(0.915)	0.140(0.833)	0.209(0.930)	0.196(0.846)	0.196(0.881)
5	15	0/25	0.314(0.894)	0.291(0.974)	0.312(0.913)	0.261(0.836)	0.344(0.888)	0.338(0.820)	0.338(0.881)
5	15	0/5	0.392(0.905)	0.335(0.964)	0.392(0.911)	0.324(0.828)	0.390(0.852)	0.390(0.802)	0.390(0.886)
10	5	0/1	0.191(0.911)	0.343(0.363)	0.193(0.908)	0.149(0.831)	0.176(0.879)	0.170(0.798)	0.170(0.904)
10	5	0/25	0.339(0.897)	0.351(0.495)	0.337(0.899)	0.271(0.816)	0.320(0.856)	0.317(0.798)	0.317(0.892)
10	5	0/5	0.411(0.905)	0.354(0.988)	0.413(0.923)	0.341(0.845)	0.411(0.856)	0.410(0.792)	0.410(0.893)
10	10	0/1	0.147(0.890)	0.266(0.421)	0.146(0.903)	0.126(0.859)	0.151(0.902)	0.146(0.836)	0.146(0.892)
10	10	0/25	0.271(0.910)	0.291(0.715)	0.272(0.901)	0.238(0.848)	0.274(0.880)	0.271(0.827)	0.271(0.908)
10	10	0/5	0.343(0.896)	0.311(0.975)	0.344(0.907)	0.303(0.856)	0.343(0.860)	0.343(0.818)	0.343(0.898)
10	15	0/1	0.130(0.898)	0.221(0.419)	0.130(0.906)	0.116(0.864)	0.139(0.919)	0.135(0.844)	0.135(0.899)
10	15	0/25	0.250(0.893)	0.259(0.835)	0.250(0.904)	0.223(0.864)	0.259(0.886)	0.256(0.840)	0.256(0.886)
10	15	0/5	0.318(0.903)	0.289(0.970)	0.318(0.906)	0.284(0.859)	0.317(0.871)	0.317(0.840)	0.317(0.895)
15	5	0/1	0.180(0.905)	0.329(0.491)	0.181(0.909)	0.140(0.825)	0.158(0.881)	0.154(0.820)	0.154(0.900)
15	5	0/25	0.325(0.900)	0.336(0.543)	0.321(0.900)	0.257(0.825)	0.299(0.853)	0.297(0.810)	0.297(0.884)
15	5	0/5	0.392(0.909)	0.335(0.967)	0.392(0.911)	0.324(0.828)	0.391(0.862)	0.391(0.809)	0.391(0.895)
15	10	0/1	0.131(0.903)	0.251(0.501)	0.132(0.909)	0.115(0.863)	0.129(0.884)	0.126(0.834)	0.126(0.897)
15	10	0/25	0.250(0.902)	0.275(0.546)	0.252(0.896)	0.224(0.852)	0.245(0.883)	0.243(0.841)	0.243(0.901)
15	10	0/5	0.318(0.901)	0.289(0.959)	0.317(0.883)	0.283(0.844)	0.318(0.871)	0.318(0.841)	0.318(0.897)
15	15	0/1	0.116(0.893)	0.207(0.553)	0.115(0.893)	0.104(0.854)	0.119(0.900)	0.116(0.851)	0.116(0.890)
15	15	0/25	0.225(0.898)	0.241(0.726)	0.224(0.894)	0.204(0.864)	0.226(0.877)	0.225(0.840)	0.225(0.898)
15	15	0/5	0.287(0.892)	0.266(0.957)	0.287(0.900)	0.262(0.865)	0.286(0.865)	0.287(0.836)	0.287(0.892)