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Bayesian estimation of Marshall–Olkin extended exponential parameters under various approximation techniques

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Abstract

In this paper, we propse Bayes estimators of the parameters of Marshall Olkin extended exponential distribution (MOEED) introduced by Marshall-Olkin [2] for complete sample under squared error loss function (SELF). We have used different approximation techniques to obtain the Bayes estimate of the parameters. A Monte Carlo simulation study is carried out to compare the performance of proposed estimators with the corresponding maximum likelihood estimator (MLE's) on the basis of their simulated risk. A real data set has been considered for illustrative purpose of the study.

Keywords: Bayes estimator, Squared error loss function, Lindley's approximation method, T-K approximation, MCMC method.

2000 AMS Classification: 62F15, 62C10

1. Introduction

Due to simple, elegant and closed form of distribution function, Exponential distribution is most popular distribution for life time data analysis. Further Borlow and Proschan [22] have discussed the justification regarding the use of exponential distribution as the failure law of complex equipment. However its uses are restricted to constant hazard rate, which is difficult to justify in many real situations. Thus one can think to develop alternative model which has non-constant hazard rate. In the literature, various methods may be used to generalise exponential distributions and these generalized models have the property of non-constant hazard rate like Weibull, gamma and exponentiated exponential distribution etc. These generalized models are frequently used to analyse the life time data. In addition Marshall and Olkin [2] introduced a method of adding a new parameter to a specified distribution. The resulting distribution is known as Marshall Olkin extended distribution. The general methodology regarding the introducing a new

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parameters is as follows:
Let $\bar{F}(x)$ be the survival function of existing or specified distribution then, the survival function of new distribution can be obtained by using following relation

$$\bar{S}(x) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}; \quad -\infty < x < \infty, \alpha > 0$$

where $\bar{\alpha} = 1-\alpha$ and $\bar{S}(x)$ is the survival function of new distribution. Note that, when $\alpha = 1$, $\bar{S}(x) = \bar{F}(x)$. Thus, the form of density corresponding to the survival function $\bar{S}(x)$ is obtained as,

$$f(x,\alpha) = \frac{\alpha f(x)}{\left\{1-\bar{\alpha}\bar{F}(x)\right\}^2}$$

Further more, Marshall and Olkin derived a distribution by introducing the survival function of exponential distribution say ($\bar{F}(x) = e^{-\lambda x}$). The resulting distribution is known as Marshall Olkin extended exponential distribution (MOEED) with increasing and decreasing failure rate functions see [2]. The probability density function (pdf) and cumulative distribution function (cdf) of this distribution are given as:

$$f(x,\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}}{(1-\bar{\alpha}e^{-\lambda x})^2}; \quad x,\alpha,\lambda \geq 0 \tag{1.1}$$

$$F(x,\alpha,\lambda) = \frac{1-e^{-\lambda x}}{1-\bar{\alpha}e^{-\lambda x}}; \quad x,\alpha,\lambda \geq 0 \tag{1.2}$$

respectively. The considered distribution is very useful in life testing problem and it may be used as a good alternative to the gamma, Weibull and other exponentiated family of distributions. The basic properties related to this distribution have been discussed in [2]. The density function (1) has increasing failure rate for $\alpha \geq 1$, decreasing failure rate for $\alpha \leq 1$ and constant failure rate for $\alpha = 1$ similar to one parameter exponential distribution. G. Srinivasa Rao et al [3] used this distribution for making reliability test plan with sampling point of view. Shape of this distribution is presented bellow see figure 1. for different choices of shape and scale parameter.

In this paper, we mainly consider both the informative and non-informative priors under squared error loss function to compute the Bayes estimators of parameters. It has been noticed that the Bayes estimators of the parameters cannot be expressed in a nice closed form. Thus the different numerical approximation procedures are used to obtain Bayes estimator. Here we use the Lindley's, Tierney and Kadane (T-K) approximation methods and Markov Chain Monte Carlo (MCMC) technique to compute the Bayes estimators of the parameters.

The rest of the paper is organized as follows: In section 2.1, we describe the classical estimation with maximum likelihood estimator (MLE) of parameters. In section 2.2, we compute Bayes estimator of parameters with gamma prior and in section 2.2.1, 2.2.2 and 2.2.3 we describe different Bayesian approaches like Lindley Approximation, Tierney and Kadane approximation and Monte Carlo Markov

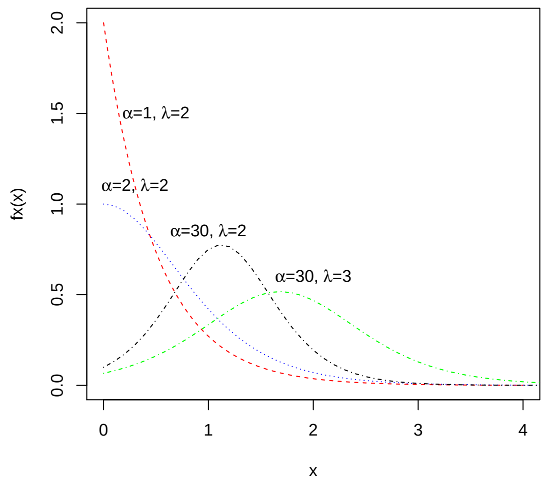

FIGURE 1. Density plot with different choice of α and λ

chain (MCMC) method for estimating the unknown parameters respectively. Section 3 provides the simulation and numerical result and one real data set has been analysed in section 4. Finally conclusion of the paper is provided in section 5.

2. Estimation of the parameters

2.1. Maximum likelihood estimators. Suppose $\{x_1, x_2, ..., x_n\}$ be a independently identically distributed (iid) random sample of size n from Marshall Olkin extended exponential distribution (MOEED) defined in (1). Thus the likelihood function of α and λ for the samples is,

$$L(x|\alpha,\lambda) = \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - \bar{\alpha} e^{-\lambda x_i})^{-2}; \quad x, \alpha, \lambda \geq 0 \tag{2.1}$$

The maximum likelihood estimators of the parameters have obtained by differentiating the log of likelihood function w.r.t.to parameters and equating to zero. Thus two normal equations have been obtained as,

$$\frac{n}{\alpha} - 2\sum_{i=1}^{n} e^{-\lambda x_i}(1 - \bar{\alpha} e^{-\lambda x_i})^{-1} = 0 \tag{2.2}$$

and

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \bar{\alpha} x_i e^{-\lambda x_i}(1 - \bar{\alpha} e^{-\lambda x_i})^{-1} = 0 \tag{2.3}$$

Above normal equation of α and λ form an implicit system and does not exist an unique root for above system of equations, so they can not be solved analytically.

Thus maximum likelihood estimators (MLE) have been obtained By using Newton-Raphson (N-R) method.

2.2. Bayesian Estimation of the parameters. The Bayesian estimation procedure of the parameters related to various life time models has been extensively discussed the literature (see in[5],[6],[8] and so on). It may be mentioned here, that most of the discussions on Bayes estimator are confined to quadratic loss function because this loss function is most widely used as symmetrical loss function which has been justified in classical method on the ground of minimum variance unbiased estimation procedure and associates equal importance to the losses for overestimation and underestimation of equal magnitudes. This may be defined as,

$$L(\hat{\theta}, \theta) \propto (\hat{\theta} - \theta)^2$$

where $\hat{\theta}$ is the estimate of the parameter θ.

Under the above mentioned loss function, Bayes estimators are the posterior mean of the distributions. In Bayesian analysis, parameters of the models are considered to be a random variable and following certain distribution. This distribution is called prior distribution. If prior information available to us which may be used for selection of prior distribution. But in many real situation it is very difficult to select a prior distribution. Therefore selection of prior distribution plays an important role in estimation of the parameters. A natural choice for the prior of α and λ would be two independent gamma distributions i.e. $gamma(a, b)$ and $gamma(c, d)$ respectively . It is important to mention that Gamma prior has flexible nature as a non-informative prior in particular when the values of hyper parameters are considered to be zero. Thus the proposed prior for α and λ may be considered as,

$$\nu_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha} \qquad and \qquad \nu_2(\lambda) \propto \lambda^{c-1} e^{-d\lambda}$$

respectively. Where a, b, c and d are the hyper-parameters of the prior distributions. Thus, the joint prior of α and λ may be taken as;

$$\nu(\alpha, \lambda) \propto \alpha^{a-1} \lambda^{c-1} e^{-d\lambda - b\alpha} \quad ; \quad \alpha, \lambda, a, b, c, d \geq 0 \tag{2.4}$$

Substituting $L(x|\alpha, \lambda)$ and $\nu(\alpha, \lambda)$ form equation no. (3) and (6) respectively then we can find the posterior distribution of α and λ i.e.$p(\alpha, \lambda|\underline{x})$ is given as,

$$p(\alpha, \lambda|\underline{x}) = K \alpha^{n+a-1} \lambda^{n+c-1} e^{-d\lambda - b\alpha - \lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - \bar{\alpha} e^{-\lambda x_i})^{-2} \tag{2.5}$$

where,

$$K^{-1} = \int_{\alpha} \int_{\lambda} \alpha^{n+a-1} \lambda^{n+c-1} e^{-d\lambda - b\alpha - \lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - \bar{\alpha} e^{-\lambda x_i})^{-2} d\alpha d\lambda \tag{2.6}$$

Here, we see that the posterior distribution involves an integral in the denominator which is not solvable and consequently the Bayes estimators of the parameters are the ratio of the integral, which are not in explicit form. Hence the determination of posterior expectation for obtaining the Bayes estimator of α and λ will be tedious. There are several methods available in literature to solve such type of integration problem. Among the entire methods we consider T-K, Lindley's and Monte Carlo Markov Chain (MCMC) approximation method, which approach the ratio of the integrals as a whole and produce a single numerical result. These methods are described bellow:

2.2.1. *Bayes estimator using Lindley's Approximation.* We consider the Lindley's approximation method to obtain the Bayes estimates of the parameters, which includes the posterior expectation is expressible in the form of ratio of integral as follow:

$$I(x) = E(\alpha, \lambda|\underline{x}) = \frac{\int u(\alpha,\lambda)e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)}{\int e^{L(\alpha,\lambda)+G(\alpha,\lambda)}d(\alpha,\lambda)} \tag{2.7}$$

where,
$u(\alpha, \lambda)$= is a function of α and λ only
$L(\alpha, \lambda)$= Log- likelihood function
$G(\alpha, \lambda)$= Log of joint prior density
According to D. V. Lindley [1], if ML estimates of the parameters are available and n is sufficiently large then the above ratio of the integral can be approximated as:

$$\begin{aligned} I(x) &= u(\hat\alpha, \hat\lambda) + 0.5[(\hat u_{\lambda\lambda} + 2\hat u_\lambda \hat\tau_\lambda)\hat\sigma_{\lambda\lambda} + (\hat u_{\alpha\lambda} + 2\hat u_\alpha \hat\tau_\lambda)\hat\sigma_{\alpha\lambda} + (\hat u_{\lambda\alpha} + 2\hat u_\lambda \hat\tau_\alpha)\hat\sigma_{\lambda\alpha} + \\ &(\hat u_{\alpha\alpha} + 2\hat u_\alpha \hat\tau_\alpha)\hat\sigma_{\alpha\alpha}] + \frac{1}{2}[(\hat u_\lambda \hat\sigma_{\lambda\lambda} + \hat u_\alpha \hat\sigma_{\lambda\alpha})(\hat L_{\lambda\lambda\lambda}\hat\sigma_{\lambda\lambda} + \hat L_{\lambda\alpha\lambda}\hat\sigma_{\lambda\alpha} + \hat L_{\alpha\lambda\lambda}\hat\sigma_{\alpha\lambda} + \\ &\hat L_{\alpha\alpha\lambda}\hat\sigma_{\alpha\alpha}) + (\hat u_\lambda \hat\sigma_{\alpha\lambda} + \hat u_\alpha \hat\sigma_{\alpha\alpha})(\hat L_{\alpha\lambda\lambda}\hat\sigma_{\lambda\lambda} + \hat L_{\lambda\alpha\alpha}\hat\sigma_{\lambda\alpha} + \hat L_{\alpha\lambda\alpha}\hat\sigma_{\alpha\lambda} + \hat L_{\alpha\alpha\alpha}\hat\sigma_{\alpha\alpha})] \end{aligned}$$

where $\hat\alpha$ and $\hat\lambda$ is the MLE of α and λ respectively, and

$$\begin{aligned} &\hat u_\alpha = \frac{\partial u(\hat\alpha,\hat\lambda)}{\partial\hat\alpha}, \hat u_\lambda = \frac{\partial u(\hat\alpha,\hat\lambda)}{\partial\hat\lambda}, \hat u_{\alpha\lambda} = \frac{\partial u(\hat\alpha,\hat\lambda)}{\partial\hat\alpha\partial\hat\lambda}, \hat u_{\lambda\alpha} = \frac{\partial u(\hat\alpha,\hat\lambda)}{\partial\hat\lambda\partial\hat\alpha}, \hat u_{\alpha\alpha} = \frac{\partial^2 u(\hat\alpha,\hat\lambda)}{\partial\hat\alpha^2}, \\ &\hat u_{\lambda\lambda} = \frac{\partial^2 u(\hat\alpha,\hat\lambda)}{\partial\hat\lambda^2}, \hat L_{\alpha\alpha} = \frac{\partial^2 L(\hat\alpha,\hat\lambda)}{\partial\hat\alpha^2}, \hat L_{\lambda\lambda} = \frac{\partial^2 L(\hat\alpha,\hat\lambda)}{\partial\hat\lambda^2}, \hat L_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\alpha^3}, \\ &\hat L_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\alpha\partial\hat\alpha\partial\hat\lambda}, \hat L_{\lambda\lambda\alpha} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\lambda\partial\hat\lambda\partial\hat\alpha}, \hat L_{\lambda\alpha\lambda} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\lambda\partial\hat\alpha\partial\hat\lambda}, \hat L_{\alpha\alpha\lambda} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\alpha\partial\hat\alpha\partial\hat\lambda}, \\ &\hat L_{\alpha\lambda\lambda} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\alpha\partial\hat\lambda\partial\hat\lambda}, \hat L_{\lambda\alpha\alpha} = \frac{\partial^3 L(\hat\alpha,\hat\lambda)}{\partial\hat\lambda\partial\hat\alpha\partial\hat\alpha}, \hat p_\alpha = \frac{\partial G(\hat\alpha,\hat\lambda)}{\partial\hat\alpha}, \hat p_\lambda = \frac{\partial G(\hat\alpha,\hat\lambda)}{\partial\hat\lambda} \end{aligned}$$

After substitution of $p(\alpha, \lambda|\underline{x})$ from (7) in above equation (9) then this integral must be reduces like Lindley's integral, where:

$$\begin{aligned} &u(\alpha, \lambda) = \alpha \\ &L(\alpha, \lambda) = n \ln\alpha + n\ln\lambda - \lambda\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\ln(1 - \bar\alpha e^{-\lambda x_i}) \quad \text{and} \\ &G(\alpha, \lambda) = (a-1)\ln\alpha + (c-1)\ln\lambda - (b\alpha + d\lambda) \end{aligned}$$

it may verified that,

$$u_\alpha = 1, \quad u_{\alpha\alpha} = u_{\lambda\lambda} = u_{\alpha\lambda} = u_{\lambda\alpha} = 0, \quad p_\alpha = \frac{a-1}{\alpha} - b, \ p_\lambda = \frac{c-1}{\lambda} - d$$

$$L_\alpha = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{e^{-\lambda x_i}}{(1-\bar\alpha e^{-\lambda x_i})}, \qquad L_{\alpha\alpha} = \frac{-n}{\alpha^2} + 2\sum_{i=1}^{n}\frac{e^{-2\lambda x_i}}{(1-\bar\alpha e^{-\lambda x_i})^2},$$

$$L_{\alpha\alpha\alpha} = \frac{2n}{\alpha^3} - 4\sum_{i=1}^{n}\frac{e^{-3\lambda x_i}}{(1-\bar\alpha e^{-\lambda x_i})^3}, \qquad L_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\frac{x_i\bar\alpha e^{-\lambda x_i}}{(1-\bar\alpha e^{-\lambda x_i})},$$

$$L_{\lambda\lambda}=\frac{-n}{\lambda^2}+2\sum_{i=1}^{n}\frac{x_i^2\bar{\alpha}e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})}+2\sum_{i=1}^{n}\frac{x_i^2\bar{\alpha}^2e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2},$$

$$L_{\lambda\lambda\lambda}=\frac{2n}{\lambda^3}-2\sum_{i=1}^{n}\frac{x_i^3\bar{\alpha}e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})}-6\sum_{i=1}^{n}\frac{x_i^3\bar{\alpha}^2e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2}-4\sum_{i=1}^{n}\frac{x_i^3\bar{\alpha}^3e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3},$$

$$L_{\alpha\alpha\lambda}=L_{\lambda\alpha\alpha}=-4\sum_{i=1}^{n}\frac{x_ie^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2}-4\sum_{i=1}^{n}\frac{x_i\bar{\alpha}e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3},$$

$$L_{\alpha\lambda\lambda}=L_{\lambda\lambda\alpha}=-2\sum_{i=1}^{n}\frac{x_i^2e^{-\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})}-6\sum_{i=1}^{n}\frac{x_i^2\bar{\alpha}e^{-2\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^2}-4\sum_{i=1}^{n}\frac{x_i^2\bar{\alpha}^2e^{-3\lambda x_i}}{(1-\bar{\alpha}e^{-\lambda x_i})^3}$$

If α and λ are orthogonal then $\sigma_{ij}=0$ for $i\neq j$ and $\sigma_{ij}=\left(-\frac{1}{L_{ij}}\right)$ for $i=j$ After evaluation of all U-terms, L-terms, and p- terms at the point $(\hat{\alpha},\hat{\lambda})$ and using the above expression, the approximate Bayes estimator of α under SELF is,

$$\hat{\alpha}_S^L=\hat{\alpha}+\hat{u}_\alpha\hat{p_\alpha}\hat{\sigma}_{\alpha\alpha}+0.5\left(\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\lambda\lambda}\hat{L}_{\alpha\lambda\lambda}+\hat{u}_\alpha\hat{\sigma}_{\alpha\alpha}^2\hat{L}_{\alpha\alpha\alpha}\right)\tag{2.8}$$

and similarly the Bayes estimate for λ under SELF is,
$u_\lambda=1,\quad u_{\alpha\alpha}=u_{\lambda\lambda}=u_{\alpha\lambda}=u_{\lambda\alpha}=0$ and remaining L-terms and -terms will be same as above thus we have,

$$\hat{\lambda}_S^L=\hat{\lambda}+\hat{u}_\lambda\hat{p_\lambda}\hat{\sigma}_{\lambda\lambda}+0.5\left(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda}^2\hat{L}_{\lambda\lambda\lambda}+\hat{u}_\lambda\hat{\sigma}_{\alpha\alpha}\hat{\sigma}_{\lambda\lambda}\hat{L}_{\alpha\alpha\lambda}\right)\tag{2.9}$$

2.2.2. *Bayes estimators using Tierney and Kadane's (T-K) Approximation.* Lindley's method of solving integral is accurate enough but one of the problems of this method is that it requires evaluation of third order partial derivatives and in p-parameters case the total number of derivatives is $\frac{p(p+1)(p+2)}{6}$ then this approximation will be quite complicated. thus one can think about T-K approximation method and this method may be used as an alternative to Lindley's method. According to the Tierney and Kadane's approximation any ratio of the integral of the form,

$$\hat{u}(\alpha,\lambda)=E_{p(\alpha,\lambda|\underline{\mathrm{x}})}[u(\alpha,\lambda|\underline{\mathrm{x}})]=\frac{\int_{\alpha,\lambda}e^{nL_*(\alpha,\lambda)}d(\alpha,\lambda)}{\int_{\alpha,\lambda}e^{nL_0(\alpha,\lambda)}d(\alpha,\lambda)}\tag{2.10}$$

where,

$$L_0(\alpha,\lambda)=\frac{1}{n}[L(\alpha,\lambda)+\ln\nu(\alpha,\lambda)]\quad and\quad L_*(\alpha,\lambda)=L_0(\alpha,\lambda)+\frac{1}{n}\ln u(\alpha,\lambda)\tag{2.11}$$

Thus estimate can be obtained as,

$$\hat{u}(\alpha,\lambda)=\sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}}e^{[n\{L_*(\alpha_*,\lambda_*)-L_0(\alpha_0,\lambda_0)\}]}\tag{2.12}$$

where (α_*,λ_*) and (α_0,λ_0) maximize $L_*(\alpha,\lambda)$ and $L_0(\alpha,\lambda)$ respectively, and Σ_* and Σ_0 are the negative of the inverse of the matrices of second derivatives of $L_*(\alpha,\lambda)$ and $L_0(\alpha,\lambda)$ at the point (α_*,λ_*) and (α_0,λ_0) respectively. In our study, based on (14) the function $L_0(\alpha,\lambda)$ is given as,

(2.13)

$$L_0(\alpha,\lambda)=\frac{1}{n}[(n+a-1)\ln\alpha-b\alpha+(n+c-1)\ln\lambda-\lambda(d+\sum_{i=1}^{n}x_i)-2\sum_{i=1}^{n}\ln(1-\bar{\alpha}e^{-\lambda x_i})]$$

and thus for the Bayes estimator of α and λ under SELF using this approximation (17) can be written as,

$$\hat{\alpha}_S^{T-K}(\alpha,\lambda)=\sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}}e^{[n\{L_*^{\alpha}(\alpha_*,\lambda_*)-L_0(\alpha_0,\lambda_0)\}]} \tag{2.14}$$

$$\hat{\lambda}_S^{T-K}(\alpha,\lambda)=\sqrt{\frac{|\Sigma_*|}{|\Sigma_0|}}e^{\left[n\left\{L_*^{\lambda}(\alpha_*,\lambda_*)-L_0(\alpha_0,\lambda_0)\right\}\right]} \tag{2.15}$$

where

$$L_*^{\alpha}(\alpha,\lambda)=L_0^{\alpha}(\alpha,\lambda)+\frac{1}{n}\ln\alpha \quad \textit{and} \quad L_*^{\lambda}(\alpha,\lambda)=L_0^{\lambda}(\alpha,\lambda)+\frac{1}{n}\ln\lambda$$

2.2.3. *Bayes estimator using Monte Carlo Markov Chain (MCMC) method.* In this section, we propose Monte Carlo Markov Chain (MCMC) method for obtaining the Bayes estimates of the parameters. Thus we consider the MCMC technique namely Gibbs sampler and Metropolis-Hastings algorithm to generate sample from the posterior distribution and then compute the Bayes estimate. The Gibbs sampler is best applied on problems where the marginal distributions of the parameters of interest are difficult to calculate, but the conditional distributions of each parameter given all the other parameters and the data have nice forms. If the conditional distributions of the parameters have standard forms, then they can be simulated easily. But generating samples from full conditionals corresponding to joint posterior is not easily manageable. Therefore we considered the Metropolis-Hastings algorithm. Metropolis step is used to extract samples from some of the full conditional to complete a cycle in Gibbs chain . For more detail about MCMC method see for example Gelfand and Smith [23], Upadhya and Gupta [24] . Thus utilizing the concept of Gibbs sampling procedure as mentioned above, generates sample from the posterior density function (7) under the assumption that parameters α and λ have independent Gamma density function with hyper parameters a, b and c, d respectively. To incorporate this technique we consider full conditional posterior densities of α and λ are written as ,

$$\pi(\alpha|\lambda,\underline{\mathrm{x}})\propto\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^{n}(1-\bar{\alpha}e^{-\lambda x_i})^{-2} \tag{2.16}$$

$$\pi(\lambda|\alpha,\underline{\mathrm{x}})\propto\lambda^{n+c-1}e^{-\lambda(d+\sum_{i=1}^{n}x_i)}\prod_{i=1}^{n}(1-\bar{\alpha}e^{-\lambda x_i})^{-2} \tag{2.17}$$

The Gibbs algorithm consist the following steps

- Start with k=1 and initial values (α^0,λ^0)
- Using M-H algorithm generate posterior sample for α and λ from (18) and (19) respectively, where asymptotic normal distribution of full conditional densities are considered as the proposal.
- Repeat step 2, for all $k=1,2,3,\ldots,M$ and obtain $(\alpha_1,\lambda_1),(\alpha_2,\lambda_2),...(\alpha_M,\lambda_M)$
- After obtaining the posterior sample the Bayes estimates of α and λ with respect to the SELF are as follows:

$$(2.18) \quad \hat{\alpha}^{MC} = [E_\pi(\alpha|\underline{\mathrm{x}})] \approx \left(\frac{1}{M - M_0} \sum_{i=1}^{M-M_0} \alpha_i \right)$$

$$(2.19) \quad \hat{\lambda}^{MC} = [E_\pi(\lambda|\underline{\mathrm{x}})] \approx \left(\frac{1}{M - M_0} \sum_{i=1}^{M-M_0} \lambda_i \right)$$

Where, M_0 is the burn-in-period of Markov Chain.

3. Simulation Study

This section, consists of simulation study to compare the performance of the various estimation techniques described in the previous section 2. Comparison of the estimators have been made on the basis of simulated risk (average loss over whole sample space). It is not easy to obtain the risk of the estimators directly. Therefore the risk of the estimators are obtained on the basis of simulated sample. For this purpose, we generate 1000 samples of size n (small sample size n = 20, moderate sample size n = 30, and large sample size n = 50) from Mrshall-Olkin Extended exponential distribution. In order to consider MCMC method for obtaining the Bayes estimate of the parameters, we generate 20000 deviates for the parameters α and λ using algorithm discussed in section 2.2.3. First five hundred MCMC iterations (Burn-in period) have discarded from the generated sequence. We have also checked the convergence of the sequences of α and λ for their stationary distributions through different starting values. It was observed that all the Markov chains reached to the stationary condition very quickly. Further, in Bayes estimation choice of hyper-parameters have great importance. Therefore the values of hyper- parameters have been considered as follows:

- The values of hyper parameters are assumed in such a way that prior mean is equal to the guess value of the parameters when prior variances are taken as small (see Table 1), large (see Table 2) along with variation of sample size and for fixed value of parameters.
- The value of hyper parameters are assumed to be zero (i.e. non-informative case) along with variation of sample sizes and for fixed value of parameters (see Tables 3).

Here, we know that the Gamma prior provides flexible approach to handle estimation procedure in both scenarios i.e. informative and non-informative. The case of non-informative prior has been obtained by assuming the values of hyper parameters as zero i.e.$a = b = c = d = 0$. For informative prior, we take prior mean (say,μ) to be equal to the guess value of the parameter with varying prior variance (say,ν). The prior variance indicates the confidence of our prior guess. A large prior variance shows less confidence in prior guess and resulting prior distribution is relatively flat. On the other hand, small prior variance indicates greater confidence in prior guess. Several variations of sample size and hyper-parameters have been obtained and due to similar patterns some of them are presented below. In Table 1 the variation of various sample sizes has been observed through fixing the value of shape and scale parameter i.e $\alpha = \lambda = 2$ and choice of hyper-parameter is assumed as a=4, b=2 and c=4, d=2, such that, prior mean is 2 and prior variance is small (say 1). Table 2 shows the same patterns described as above for different choice of hyper-parameters which is assumed as a=0.4, b=0.2 and c=0.4, d=0.2, such that prior mean is 2 but prior variance is very large (say 10). Table 3 exhibits similar results under consideration of non-informative prior scenario. It is also observed that the risks of all the estimators decrease as sample size increases in all the considered cases. As we expected, it is also observed that when we consider informative prior, the proposed Bayes estimators behave better than the classical maximum likelihood estimators. But in case of non-informative prior, their behaviour are almost same as MLE, which may be seen in the following connected tables (see Table 1,2 and 3).

4. Real Illustration

In this section; we analyze a real data set from A. Wood [21] to illustrate our estimation procedure. The data is based on the failure times of the release of software given in terms of hours with average life time be 1000 hours from the starting of the execution of the software. This data can be regarded as an ordered sample of size 16 are given as,

0.519	0.968	1.430	1.893	2.490	3.058	3.625	4.442
5.218	5.823	6.539	7.083	7.485	7.846	8.205	8.564

Given data set have been already considered by Rao et al.[3] to construct a sampling plan only if the life time has Marshall-Olkin extended exponential distribution. To identify the validity of proposed model criterion of log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) have been discussed. It has been verified that the given data set provides better fit than other exponetiated family such as exponential, Generalized exponential and gamma distributions see Table (5) and empirical cumulative distribution function (ECDF) plot of this data is represented in figure (2).
To calculate the Bayes estimates of the parameters in absence of prior information, we consider the non-informative prior. Further we calculate the Maximum likelihood estimates of the parameter and also Bayes estimates of the parameters under different considered estimation methods which are presented in Table 4. The MCMC iterations of α and λ are plotted respectively. Density and Trace plots are indicating that the MCMC samples are well mixed and stationary achieved see figure 3.

5. Conclusion

In this paper, we have considered the classical as well as Bayesian estimation of the unknown parameters of the Marshall- Olkin extended exponential distribution under various approximation techniques. On the basis of extensive study we may conclude the followings:

- Under informative setup the performance of Bayes estimators of the parameters is better than the maximum likelihood estimators (MLE's) in all considered approximation techniques and also Lindley's approximation technique works quite well than rest of other methods such as T-K and MCMC.

- Under non-informative set up, we observed that T-K approximation method behaves like maximum likelihood estimators (MLE's) and performs well than Lindleys and MCMC approximation methods.

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Table 1. This table represents the estimates of the parameters obtained through various estimation techniques when prior mean is 2 and prior variance is 1 i.e.$\mu = 2$,$\nu = 1$ and also the quantity in second row exhibits the average expected loss over sample space i.e. risks of corresponding estimators.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^L$	$\hat{\lambda}_S^L$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23737	2.06445	1.77297	1.98913	2.28136	2.07794
	1.39995	0.39985	1.40010	0.39962	0.26508	0.26972	1.15025	0.30349
30	2.21472	2.04999	2.21469	2.04986	1.95451	2.00332	2.23035	2.05050
	1.19207	0.26914	1.19255	0.26902	0.24508	0.21117	0.96397	0.19768
50	2.26295	2.06143	2.26278	2.06138	2.11909	2.03343	2.25792	2.05326
	1.00657	0.19669	1.00657	0.19668	0.39802	0.16998	0.88427	0.16388

Table 2. This table represents the estimates of the parameters obtained through various estimation techniques when prior mean is 2 and prior variance is 10 i.e.$\mu = 2$,$\nu = 10$ and also the quantity in second row exhibits the average expected loss over sample space i.e. risks of corresponding estimators.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^L$	$\hat{\lambda}_S^L$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23834	2.06459	2.28362	2.00365	2.18074	2.00827
	1.39995	0.39985	1.40243	0.40017	1.32136	0.36720	1.40399	0.42792
30	2.21472	2.04999	2.21525	2.05007	2.24926	2.00974	2.14724	1.99070
	1.19207	0.26914	1.19239	0.26918	1.14257	0.25448	1.18310	0.28731
50	2.26295	2.06143	2.26265	2.06135	2.28564	2.03720	2.21927	2.02727
	1.00657	0.19669	1.00658	0.19668	0.98506	0.18914	0.99635	0.20304

Table 3. Table represents the estimates of the parameters obtained through various estimation techniques and also the quantity in square bracketed exhibits the average expected loss over sample space i.e. risks under non-informative prior.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^L$	$\hat{\lambda}_S^L$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
20	2.23773	2.06460	2.23750	2.06448	2.34036	2.00526	2.15852	1.98538
	1.39995	0.39985	1.40114	0.40005	1.60352	0.37905	1.46574	0.47475
30	2.21472	2.04999	2.21514	2.05005	2.28201	2.01046	2.12848	1.97225
	1.19207	0.26914	1.19233	0.26916	1.30748	0.25955	1.23519	0.32177
50	2.26295	2.06143	2.26262	2.06137	2.30415	2.03762	2.21263	2.02207
	1.00657	0.19669	1.00659	0.19669	1.07019	0.19134	1.01519	0.21145

Table 4. This table represents the estimates of the parameters obtained by various methods of estimation for real data set under the assumption that prior information assume to be non-informative.

Size	MLE		T-K		Lindley's		MCMC	
n	$\hat{\alpha}_M$	$\hat{\lambda}_M$	$\hat{\alpha}_S^{T-K}$	$\hat{\lambda}_S^{T-K}$	$\hat{\alpha}_S^L$	$\hat{\lambda}_S^L$	$\hat{\alpha}_S^{MC}$	$\hat{\lambda}_S^{MC}$
16	8.62532	0.50074	8.62534	0.50074	9.12253	0.48963	8.62581	0.49910

Table 5. This Table represents the values of Log-likelihood, AIC and BIC for different models in real data set.

Distribution	-log L	Information Criterion	
		AIC	BIC
Exponential	40.762	83.524	84.296
Generalized Exponential	38.836	81.673	83.218
Gamma	38.629	81.258	82.803
MOEED	38.044	80.089	81.634

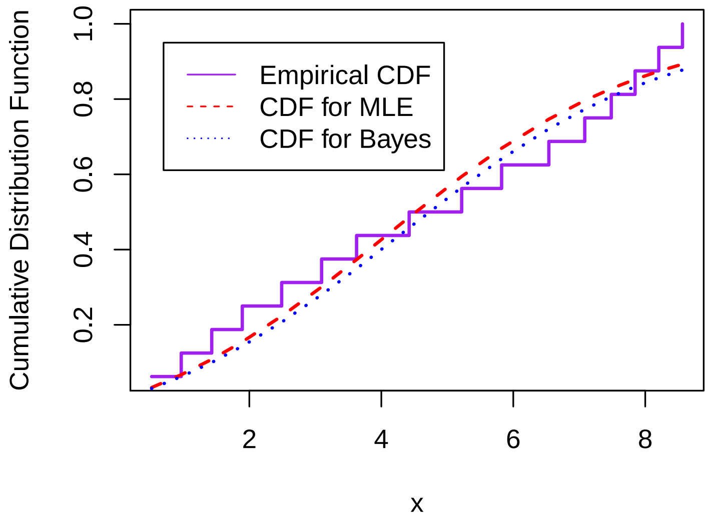

FIGURE 2. CDF plot for considered real data set

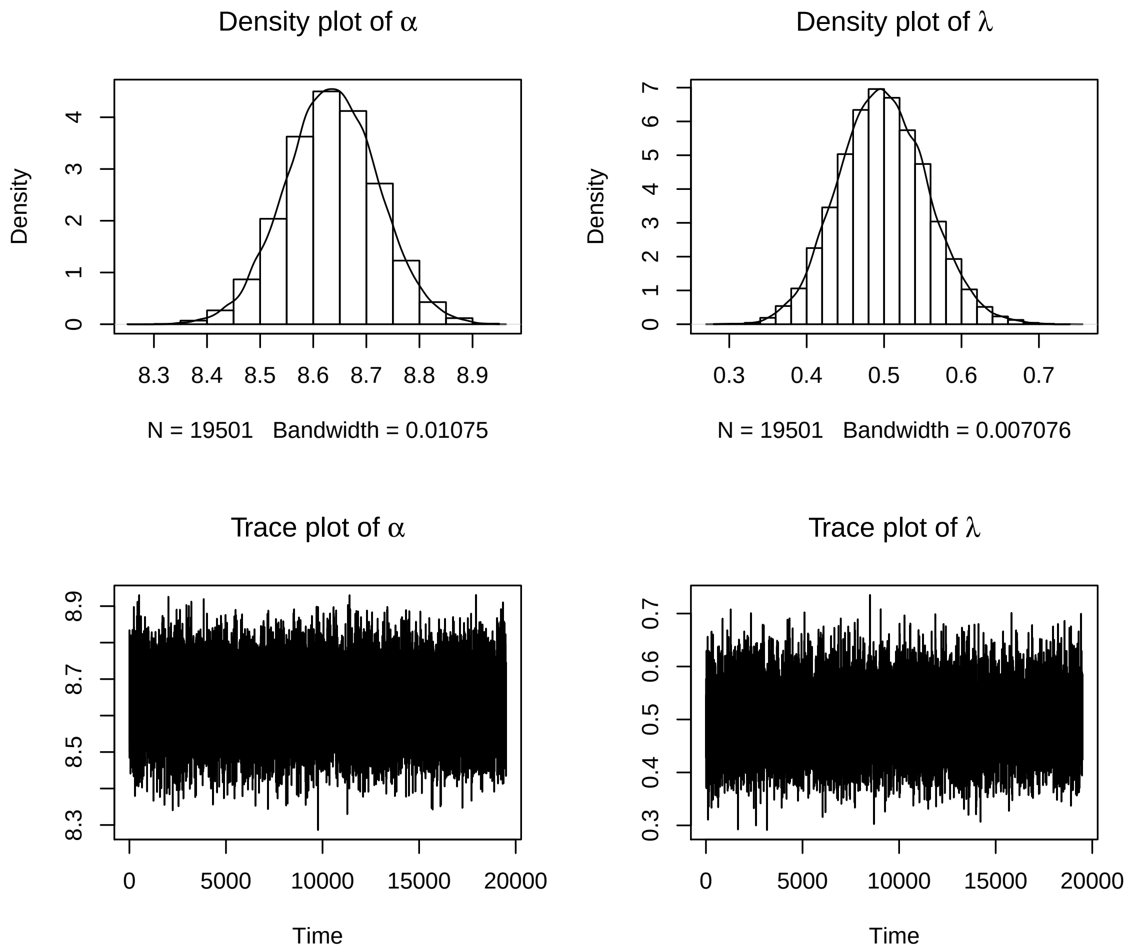

FIGURE 3. Posterior density and trace plot for considered real data set.