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On estimating population parameters in the presence of censored data: overview of available methods

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Abstract

This paper examines recent results presented on estimating population parameters in the presence of censored data with a single detection limit (DL). The occurrence of censored data due to less than detectable measurements is a common problem with environmental data such as quality and quantity monitoring applications of water, soil, and air samples. In this paper, we present an overview of possible statistical methods for handling non-detectable values, including maximum likelihood, simple substitution, corrected biased maximum likelihood, and EM algorithm methods. Simple substitution methods (e.g. substituting 0, $DL/2$, or DL for the non-detected values) are the most commonly used. It has been shown via simulation that if population parameters are estimated through simple substitution methods, this can cause significant bias in estimated parameters. Maximum likelihood estimators may produce dependable estimates of population parameters even when 90% of the data values are censored and can be performed using a computer program written in the R Language. A new substitution method of estimating population parameters from data contain values that are below a detection limit is presented and evaluated. Worked examples are given illustrating the use of these estimators utilizing computer program. Copies of source codes are available upon request.

Keywords: detection limits, censored data, normal and lognormal distributions, likelihood function, maximum likelihood estimators.

1. Introduction

Environmental data frequently contain values that are below detection limits. Values that are below DL are reported as being less than some reported limit of detection, rather than as actual values. A data set for which all observations may be identified and counted, with some observations falling into the restricted interval of measurements and the remaining observations being fully measured, is said to be censored. A situation where observations may be censored would

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be chemical measurements where some observations have a concentration below the detection limit of the analytical method. A sample for which some observations are known only to fall below a known detection limit, while the remaining observations falling above the detection limit are fully measured and reported is called left-singly censored or simply left censored. Detection limits are usually determined and justified in terms of the uncertainties that apply to a single routine measurement. Left-censored data commonly arise in environmental contexts. Left-censored observations (observations reported as $< DL$) can occur when the substance or attribute being measured is either absent or exists at such low concentrations that the substance is not present above the DL. In type I censoring, the detection limit is fixed a priori for all observations and the number of the censored observations varies. In type II censoring, the number of censored observations is fixed a priori, and the detection limit vary.

The estimation of the parameters of normal and lognormal populations in the presence of censored data has been studied by several authors in the context of environmental data. There has been a corresponding increase in the amount of attention devoted to the most proper analysis of data which have been collected in related to environmental issues such as monitoring water and air quality, and monitoring groundwater quality. The lognormal is frequently the parametric probability distribution of choice used in fitting environmental data Gilbert (1987). However, Shumway et al. (1989) examined transformations to normality from among the Box and Cox (1964) family of transformations: $Y = \frac{X^\lambda - 1}{\lambda}$ for $\lambda \neq 0$, and $Y = ln(X)$ for $\lambda = 0$. The transformed variable Y is assumed to be normally distributed with mean μ and standard deviation σ . Cohen (1959) used the method of maximum likelihood to derive estimators for the μ and σ parameters from left censored samples. Cohen (1959) also provided tables that are needed to report these maximum likelihood estimates ($MLEs$). Aboueissa and Stoline (2004) introduced a new algorithm for computing Cohen (1959) MLE estimators of normal population parameters from censored data with a single detection limit. Estimators obtained via this algorithm required no tables and more easily computed than the ($MLEs$) of Cohen (1959). Hass and Scheff (1990) compared methodologies for the estimation of the averages in truncated samples. Saw (1961) derived the first-order term in the bias of the Cohen (1959) MLE estimators for μ and σ, and proposed bias-corrected MLE estimators. Based on the bias-corrected tables in Saw (1961b), Schneider (1984,1986) performed a least-squares fit to produce computational formulas for normally distributed singly-censored data. Dempster et. al. (1977) proposed an iterative method, called the expectation maximization algorithm (EM algorithm), for obtaining the maximum likelihood estimates for these censored normal samples. The procedure consists of alternately estimating the censored observations from the current parameter estimates and estimating the parameters from the actual and estimated observations.

In practice, probably due to computational ease, simple substitution methods are commonly used in many environmental applications. One of the most commonly used replacement method is to substitute each left censored observation by half of the detection limit DL, Helsel et al. (1986) and Helsel et al. (1988). Two simple substitution methods were suggested by Gilliom and Helsel (1986). In one method, all left censored observations are replaced by zero. In the other method, all left censored observations are replaced by the detection limit DL. Aboueissa and Stoline (2004) developed closed form estimators for estimating normal population parameters from singly-left censored data based on a new replacement method. It has been shown that via simulation if left-censored observations are estimated through these substitution methods, this can cause significant bias in estimated parameters. In this article, a new substitution method, called weighted substitution method, is introduced and examined. This method is based on assigning different weights for each left-censored observation. These weights are estimated from the sample data prior to computing estimates of population parameters. It has been shown that via simulation if left-censored data are estimated through the weighted substitution method, this will reduce the bias in estimated parameters. Other suggested methods are discussed in Gibbons (1994), Gleit (1985), Schneider (1986), Gupta (1952), Stoline (1993), El-Shaarawi A. H. and Dolan D. M. (1989), El-Shaarawi and Esterby (1992), USEPA (1989), NCASI (1985, 1991), Gilliom and Helsel (1986), Helsel and Gilliom (1986), Helsel and Hirsch (1988), Schmee et. al. (1985), and Wolynetz (1979).

The objective of this article is to develop a new substitution method which yield reliable estimates of population parameters from left-censored data, and also to compare the performances of the various estimation procedures. In addition, a simple-to-use computer program is introduced and described for estimating the population parameters of normally or lognormally distributed left-censored data sets with a single detection limit using the eight parameter estimation methods described in this article. The authors of this article performed a simulation study to asses the performance of various estimate procedures in terms of bias and mean squared error (MSE). Several methods, including MLE, bias-corrected MLE ($UMLE$), and EM algorithm (EMA), have been considered.

2. Methods Used for Estimation

To simplify the presentation in this section, the method is described and illustrated by reference to the analysis of normally distributed data, though this condition occurs infrequently in typical environmental data analysis. However, it is frequently necessary to transform real environmental data before analysis; typically the logarithmic transformation of $x_i = log(y_i)$ is used, although other transformations are possible. When the logarithmic or other transformation is used prior to censored data set analysis, it is necessary to transform the analysis results back to the original scale of measurement following parameter estimation. Let $\underbrace{\overbrace{x_1, ..., x_{m_c}}^{m_c-observations}}_{left-censored}, \underbrace{\overbrace{x_{m_c+1}, ..., x_n}^{m-observations}}_{non-censored}$ be a random sample of n observations of which m_c are left-censored while $m = n - m_c$ are non-censored (or fully measured) from a normal population with mean μ and standard deviation σ . For censored observations, it is only known that $x_j < DL$ for $j = 1, ..., m_c$.

Let

$$(2.1) \qquad \bar{x}_m = \frac{1}{m}\sum_{i=m_c+1}^{n} x_i \ , \quad \textit{and} \quad s_m^2 = \frac{1}{m}\sum_{i=m_c+1}^{n}(x_i - \bar{x}_m)^2$$

be the sample mean and sample variance of the m non-censored observations $x_{m_c+1}, ..., x_n$.

2.1. MLE **Estimators of Cohen.** Cohen (1959) employed the method of maximum likelihood to the normally distributed left-censored samples, and developed the following MLE estimators for the mean and standard deviation in terms of a tabulated function of two arguments:

$$(2.2) \qquad \hat{\mu} = \bar{x}_m - \hat{\lambda}(\bar{x}_m - DL) \ ,$$

$$(2.3) \qquad \hat{\sigma} = \sqrt{s_m^2 + \hat{\lambda}(\bar{x}_m - DL)^2} \ ,$$

where

$$(2.4) \qquad \hat{\lambda} = \lambda(h, \gamma), \ h = \frac{m_c}{n} \ \textit{and} \ \gamma = \frac{s_m^2}{(\bar{x}_m - DL)^2}$$

Cohen (1959) provided tables of the function $\hat{\lambda} = \lambda(\gamma, h)$ restricted to values of $\gamma = 0.00(0.05)1.00$, and values of $h = 0.01(0.01)0.10(0.05)0.70(0.10)0.90$. The Cohen (1959) method requires use of these tables. Schneider (1986) extended these tables to include values of γ up to 1.48. Schmee et. al. (1985) extended these tables further to include values of $\gamma = 0.00(0.10)1.00(1.00)10.00$ and values of $h = 0.10(0.10)0.90$. However, interpolations for h and γ values are often required for most applications.

2.2. Aboueissa and Stoline Algorithm for Computing MLE **of Cohen.** Aboueissa and Stoline (2004) introduced an algorithm for computing the Cohen MLE estimators. This algorithm is based on solving the estimating equation

$$(2.5) \qquad \gamma = \frac{\left(1 - \frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)}\left(\frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)} - \xi\right)\right)}{\left(\frac{h}{1-h}\frac{\phi(\xi)}{\Phi(\xi)} - \xi\right)^2} \ ,$$

numerically for ξ ($say\ \hat{\xi}$). With $\hat{\xi}$ obtained via this algorithm, the exact value of the λ-parameter is then given by:

$$(2.6) \qquad \hat{\lambda}_{as} = \lambda(h, \hat{\xi}) = \frac{Y(h, \hat{\xi})}{Y(h, \hat{\xi}) - \hat{\xi}},$$

where

$$Y = Y(h, \xi) = \left(\frac{h}{1-h}\right) Z(\xi),$$

$$Z(\xi) = \frac{\phi(-\xi)}{1-\Phi(-\xi)} \ , \quad \textit{and} \quad h = \frac{m_c}{n} = CL = \textit{ censoring level} \ .$$

The functions $\phi(\xi)$ and $\Phi(\xi)$ are the *pdf* and *cdf* of the standard unit normal. with $\hat{\lambda}_{as}$ obtained from (2.6), the MLE estimators obtained via this algorithm are obtained from (2.2) and (2.3) as:

$$\text{(2.7)} \qquad \hat{\mu}_{as} = \bar{x}_m - \hat{\lambda}_{as}(\bar{x}_m - DL) \ ,$$

and

$$\text{(2.8)} \qquad \hat{\sigma}_{as} = \sqrt{s_m^2 + \hat{\lambda}_{as}(\bar{x}_m - DL)^2} \ .$$

MLE estimators obtained via this method are labeled the $ASAMLEOC$ method in this article. It should be noted that the $ASAMLEOC$ method can be used to obtain the MLE estimators of population parameters from censored samples for all values of h and γ without any restrictions, and for all censoring levels including censoring levels greater than 0.90. The $ASAMLEOC$ estimators $\hat{\mu}_{as}$ and $\hat{\sigma}_{as}$ given by (2.7) and (2.8) are essentially Cohen's (1959) MLE estimators, which are obtained without the use of any auxiliary tables. It should also be noted that Cohen's (1959) method can not be used to obtain the maximum likelihood estimates from censored samples that have a censoring level higher than 90% ($h > 0.90$).

2.3. Bias-Corrected MLE Estimators. Saw (1961) derived the first-order term in the bias of the MLE estimators of μ and σ and proposed bias-corrected MLE estimators. Based on the bias-corrected tables in Saw (1961), Schneider (1986) performed a least-squares fit to produce computational formulas for the unbiased MLE estimators of μ and σ from normally distributed singly-censored data. These formulas, for the singly left-censored samples can be written as

$$\text{(2.9)} \qquad \hat{\mu}_u = \hat{\mu} - \frac{\hat{\sigma}B_u}{n+1}, \quad \textit{and} \quad \hat{\sigma}_u = \hat{\sigma} - \frac{\hat{\sigma}B_\sigma}{n+1} \ ,$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the MLE estimators of Cohen (1959) or equivalently the $ASAMLE$ estimators $\hat{\mu}_{as}$ and $\hat{\sigma}_{as}$, and

$$\text{(2.10)} \quad B_u = -e^{2.692 - \frac{5.439m}{n+1}} \quad \textit{and} \quad B_\sigma = -\left(0.312 + \frac{0.859m}{n+1}\right)^{-2} \ .$$

This method will be referred to as the $UMLE$ method in this paper.

2.4. Haas and Scheff Estimators(1990). Haas and Scheff (1990)developed a power series expansion that fits the tabled values of the auxiliary function $\lambda(\gamma, h)$ to within 6% for Cohen's (1959) estimates. This power series expansion is given by:

(2.11)

$$\log\lambda = 0.182344 - \frac{0.3256}{\gamma+1} + 0.10017\gamma + 0.78079\omega - 0.00581\gamma^2 - 0.06642\omega^2$$
$$- 0.0234\gamma\omega + 0.000174\gamma^3 + 0.001663\gamma^2\omega - 0.00086\gamma\omega^2 - 0.00653\omega^3,$$
$$where \quad \omega = \log\left(\frac{h}{1-h}\right).$$

This method will be referred to as the HS method in this paper.

2.5. Expectation Maximization Algorithm. Dempster et. al. (1977) proposed an iterative method, called the expectation maximization algorithm, for obtaining the $MLE's$ for the mean μ and the standard deviation σ of the normal distribution from censored samples. The procedure used in expectation maximization algorithm is based on replacing the censored observations and their squares in the complete data likelihood function by their conditional expectations given the data and the current estimates of μ and σ. This method will be referred to as the EMA method here.

2.6. Substitution Methods. Replacement methods are easier to use and consist of calculating the usual estimates of the mean and standard deviation by assigning a constant value to observations that are less than the censoring limit. Two simple substitution methods were suggested by Gilliom and Helsel (1986). In one method, all censored observations are replaced by zero. This is the ZE method. In the other method, all censored observations are replaced by the detection limit (DL). This is the DL method. One of the most commonly used substitution method, suggested by Helsel et.al. (1988), is to substitute each censored observations by half of its detection limit ($\frac{DL}{2}$). This is the HDL method.

3. Weighted Substitution Method for Left-Censored Data

The common replacement methods are based on replacing censored observations that are less than DL by a single constant. Three existing substitution methods were discussed in Section 2 based on replacing all left-censored observations with a single value either 0, $DL/2$, *or* DL. To avoid tightly grouped replaced values in cases where there are several left-censored values that share a common detection limit, left-censored observations may be spaced from zero to the detection limit according to some specified weights assigned for these left-censored observations. In the suggested weighted substitution method left-censored observations that are less than DL are replaced by non-constant different values based on assigning a different weight for each left-censored observation.More details are now given in the proposed weighted substitution method yielding estimates for μ and σ. The following weights are assigned to the m_c left-censored observations $x_1, ..., x_{m_c}$:

$$(3.1) \qquad w_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(U \geq DL)\right)^{\ln(m+j-1)} \quad , \; for \;\; j = 1, 2, ..., m_c \; ,$$

where the probability $P(U \geq DL)$ is estimated from the sample data by:

$$(3.2) \qquad P(\widehat{U \geq D}L) = 1 - \Phi\left(\frac{DL - \bar{x}_m}{s_m}\right)$$

An extensive simulation study was conducted on several weights. The simulation results (shown in the appendix) indicate that the proposed estimators using (3.1) are superior to those using the other weights in the sense of mean square error (variance of the estimator plus the square of the bias) in addition to the ability to recover the true mean and standard deviation as well as the existing methods such as maximum likelihood and EM algorithm estimators.

Estimates of the weights given in (3.1) are given by:

$$(3.3) \qquad \widehat{w_j} = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(\widehat{U \geq D}L)\right)^{\ln(m+j-1)} .$$

where the distribution of U is approximated by a normal distribution with an estimated mean $\bar{x}_m$ and an estimated variance s_m^2.

These weights are selected on a trial and error basis by means of simulations to yield estimators of population parameters that perform nearly as well as estimators obtained via the existing methods such as MLE estimators and EMA method. Left-censored observations $x_1, x_2, ..., x_{m_c}$ are then replaced by the following weighted m_c observations:

$$(3.4) \qquad (x_1^w, x_2^w, ..., x_{m_c}^w) \equiv (\widehat{w_1}DL, \widehat{w_2}DL, ..., \widehat{w_{m_c}}DL)$$

Let

$$(3.5) \qquad \bar{x}_{m_c} = \frac{1}{m_c}\sum_{i=1}^{m_c} x_i^w \ , \quad and \quad s_{m_c}^2 = \frac{1}{m_c}\sum_{i=1}^{m_c}(x_i^w - \bar{x}_{m_c})^2$$

be the sample mean and sample variance of the weighted m_c observations $x_1^w, x_2^w, ..., x_{m_c}^w$. The corresponding weighted substitution method estimators $\hat{\mu}_w$ and $\hat{\sigma}_w$ of μ and σ are given by, respectively:

$$(3.6) \qquad \begin{aligned} \hat{\mu}_w &= \frac{1}{n}\left(\sum_{i=1}^{m_c} x_i^w \ + \sum_{i=m_c+1}^{n} x_i\right) \\ &= \bar{x}_m - \hat{\lambda}_{\mu_w}\ (\bar{x}_m - \bar{x}_{m_c})\ , \end{aligned}$$

and

$$(3.7) \qquad \begin{aligned} \hat{\sigma}_w &= \sqrt{\frac{1}{n}\left(\sum_{i=1}^{m_c}(x_i^w - \hat{\mu}_w)^2 \ + \sum_{i=m_c+1}^{n}(x_i - \hat{\mu}_w)^2\right)} \\ &= \sqrt{\frac{m\ s_m^2 + m_c\ s_{m_c}^2}{n} + \hat{\lambda}_{\sigma_w}\ (\bar{x}_m - \bar{x}_{m_c})^2}\ , \end{aligned}$$

where

$$(3.8) \qquad \hat{\lambda}_{\mu_w} = \frac{m_c}{n} \quad and \quad \hat{\lambda}_{\sigma_w} = \frac{m\, m_c}{n^2}\ .$$

It should be noted that $\hat{\mu}_w$ in (3.6) can be written as:

$$(3.9) \qquad \hat{\mu}_w = \frac{m\ \bar{x}_m + m_c\ \bar{x}_{m_c}}{n}\ ,$$

which is the weighted average of the sample means $\bar{x}_m$ and $\bar{x}_{m_c}$ of fully measured and weighted observations, respectively. It should also be observed that $\hat{\sigma}_w$ in (3.7) can be written as:

$$(3.10) \qquad \hat{\sigma}_w = \sqrt{s_w^2 + \hat{\lambda}_{\sigma_w}\ (\bar{x}_m - \bar{x}_{m_c})^2}$$

where $s_w^2 = \frac{m\ s_m^2 + m_c\ s_{m_c}^2}{n}$ is the weighted average of the sample variances s_m^2 and $s_{m_c}^2$ of fully measured and weighted observations, respectively. Extensive simulation results show that use of the WSM method leads to estimators that have the ability to recover the true population parameters as well as the maximum likelihood estimators, and are generally superior to the constant replacement methods. In environmental sciences such as applied medical and environmental studies most of the data sets include non-detected (or left-censored) data values. The use of statistical methods such as the proposed one allows estimates of population parameters from data under consideration.

Asymptotic Variances of Estimates: The asymptotic variance-covariance matrix of the maximum likelihood estimates $(\hat{\mu}, \hat{\sigma})$ is obtained by inverting the Fisher information matrix $\mathbf{I}$ with elements that are negatives of expected values of the second-order partial derivatives of the log-likelihood function with respect to the parameters evaluated at the estimates $\hat{\mu}$ and $\hat{\sigma}$. The asymptotic variance-covariance matrix showed by Cohen (1991, 1959), will be used to obtain the estimated asymptotic variances of both $\hat{\mu}$ and $\hat{\sigma}$. Cohen (1959) describes the estimated asymptotic variance-covariance matrix of $(\hat{\mu}, \hat{\sigma})$ by

$$Cov(\hat{\mu}, \hat{\sigma}) = \begin{pmatrix} (\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]})\frac{\hat{\varphi}_{22}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} & (\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]})\frac{-\hat{\varphi}_{12}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} \\ (\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]})\frac{-\hat{\varphi}_{12}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} & (\frac{\hat{\sigma}^2}{n[1-\Phi(\hat{\xi})]})\frac{\hat{\varphi}_{11}}{\hat{\varphi}_{11}\hat{\varphi}_{22}-\hat{\varphi}_{12}^2} \end{pmatrix}$$

where

$$\hat{\varphi}_{11} = \varphi_{11}(\hat{\xi}) = 1 + Z(\hat{\xi})[Z(-\hat{\xi}) + \hat{\xi}]$$

$$\hat{\varphi}_{12} = \varphi_{12}(\hat{\xi}) = Z(\hat{\xi})\left(1 + \hat{\xi}[Z(-\hat{\xi}) + \hat{\xi}]\right)$$

$$\hat{\varphi}_{22} = \varphi_{22}(\hat{\xi}) = 2 + \hat{\xi}\hat{\varphi}_{12}$$

For the $ASAMLEOC$ $\hat{\xi}$ is the solution of (2.5) as described in the previous section. For all other methods, without loss of generality, $\hat{\xi} = \frac{DL-\hat{\mu}}{\hat{\sigma}}$.

4. Computer Programs

To facilitate the application of parameter estimation methods described in this article, a computer programs is presented to automate parameters estimation from left-censored data sets that are normally or lognormally distributed. This computer program is called "$SingleLeft.Censored.Normal.Lognormal.estimates$", and is written in the R language. The EM Algorithm method has been programmed in the R language. The program is called "$EMA.Method$", and is presented as a part of the main computer program "$SingleLeft.Censored.Normal.Lognormal.estimates$". Copies of source codes are available upon request.

5. Worked Example

The guidance document Statistical Analysis of Ground-Water Monitoring Data at $RCRA$ Facilities, Interim Final Guidance ($USEPA$, 1989b) contains an example involving a set of sulfate concentrations (mg/L) in which three values are reported as ($< 1450 = DL$). The sulfate concentrations are assumed to come from a normal distribution. These 24 sulfate concentration values are:

$< 1,450$	$1,800$	$1,840$	$1,820$	$1,860$	$1,780$	$1,760$	$1,800$
$1,900$	$1,770$	$1,790$	$1,780$	$1,850$	$1,760$	$< 1,450$	$1,710$
$1,575$	$1,475$	$1,780$	$1,790$	$1,780$	$< 1,450$	$1,790$	$1,800$

For this sample $n = 24$, $m = 21$, $m_c = 3$, $h = \frac{3}{24}$. The sample mean and the sample variance of the non-censored sample values are $\bar{x}_m = 1771.905$ and $s_m^2 = 8184.467$.

WSM Method: From (3.3) and (3.4) we obtain the estimate weights and the weighted data as follows:

$$(\hat{w}_1\ ,\ \hat{w}_2\ ,\ \hat{w}_3) = (0.9348828\ ,\ 0.9430983\ ,\ 0.9680175),$$

and

$$(x_1^w\ ,\ x_2^w\ ,\ x_3^w) = (1355.580\ ,\ 1367.493\ ,\ 1403.625).$$

The updated data set (fully measured and weighted data) is given by:

$\mathbf{1,355.580}$	$1,800$	$1,840$	$1,820$	$1,860$	$1,780$	$1,760$	$1,800$
$1,900$	$1,770$	$1,790$	$1,780$	$1,850$	$1,760$	$\mathbf{1,367.493}$	$1,710$
$1,575$	$1,475$	$1,780$	$1,790$	$1,780$	$\mathbf{1,403.625}$	$1,790$	$1,800$

The sample mean $\bar{x}_{m_c}$ and sample variance $s_{m_c}^2$ of the weighted data $x_1^w\ ,\ x_2^w\ ,\ x_3^w$ are given by:

$$\bar{x}_{m_c} = 1375.566\ \ and\ \ s_{m_c}^2 = 417.3153$$

From (3.8) we obtain

$$\hat{\lambda}_{\mu_w} = \frac{m_c}{n} = \frac{3}{24} = 0.125\ \ and\ \ \hat{\lambda}_{\sigma_w} = \frac{m\ m_c}{n} = \frac{(21)(3)}{24^2} = 0.109375\ .$$

Accordingly, using estimators (3.6) - (3.7) we calculate the WSM method estimators $\hat{\mu}_w$ and $\hat{\sigma}_w$ as:

$$\hat{\mu}_w = 1771.905 - 0.125(1771.905 - 1375.566) = 1722.3626\ ,$$

TABLE 1. Estimates for μ and σ from Sulfate Data

Method of Estimation	$\hat{\mu}$	$\hat{\sigma}$
$ASAMLEOC$	1723.9951	153.6451
$UMLE$	1723.0543	159.3983
HS	1719.8363	157.9416
EMA	1723.9951	153.6451
ZE	1550.4167	592.0813
HDL	1641.0417	356.4231
DL	1731.6667	135.9968
WSM	**1722.3624**	**156.1880**

and

$$\hat{\sigma}_w = \sqrt{\frac{21(8184.467) + 3(417.3153)}{24} + 0.109375(1771.905 - 1375.566)^2} = 156.1880 \,.$$

Applying the computer program "$SingleLeft.Censored.Normal$" for these data as shown in the Appendix, yields estimates for μ and σ parameters via eight methods of estimation including the WSM method. The results are summarized in Table 1.

Discussion: An inspection of Table 1 reveals that the $ASAMLEOC$, $UMLE$, HS, EMA and WSM methods yield quite similar estimates for both μ and σ. The DL method estimate for μ is close to those obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The DL method estimate for σ seems to be underestimated comparing to those estimates obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The ZE and HDL methods yield estimates which are different from those produced by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. The estimates of σ obtained by the ZE and HDL methods are highly overestimated, while the estimates of μ are underestimated comparing to estimates obtained by $ASAMLEOC$, EMA, WSM, $UMLE$ and HS methods. Overall, the WSM method performs similar to $ASAMLEOC$, EMA, $UMLE$ and HS methods, and superior to the common substitution ZE, HDL and DL methods.

For more investigations of the performance of the parameter estimation methods described in section 2, the sulfate concentrations data are artificially censored at censoring levels (0.25 , 0.50 , 0.625 , 0.75 , 0.875 , 0.917) with a single detection limit of $1,450$. The corresponding number of left-censored observations for each of these censoring levels are 6, 12, 15, 18, 21 and 22, respectively. Then the estimates of μ and σ are computed using the computer program "$SingleLeft.Censored.Normal$". Results are summarized in Table 2. The following observations are made from an examination of the results reported in Table 2. The WSM estimates for μ and σ are similar to those reported by $ASAMLEOC$, EMA, $UMLE$ and HS for cases with censoring levels less than or equal to 0.75.

TABLE 2. Estimates for μ and σ from Sulfate Data with artificial censoring levels

	$m_c = 6,\ CL = 0.25$		$m_c = 12,\ CL = 0.50$		$m_c = 15,\ CL = 0.625$	
Method of Estimation	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
$ASAMLEOC$	1658.6581	205.1465	1497.0883	313.4529	1367.1360	361.7728
$UMLE$	1656.2454	214.6244	1483.4885	337.3514	1336.9885	399.2681
HS	1651.2191	210.7608	1484.2813	320.0817	1351.8316	368.3523
EMA	1658.6581	205.1465	1497.1077	313.4304	1367.8667	361.0457
ZE	1322.9166	768.2189	888.9583	890.6903	661.4583	855.3307
HDL	1504.1667	457.3376	1251.4583	529.3775	1114.5833	505.3091
DL	1685.4167	158.9478	1613.9583	173.1027	1567.7083	159.5957
WSM	**1647.8992**	**218.9194**	**1456.3776**	**341.5274**	**1314.6978**	**382.7615**
	$m_c = 18,\ CL = 0.75$		$m_c = 21,\ CL = 0.875$		$m_c = 22,\ CL = 0.917$	
$ASAMLEOC$	1191.5900	412.5770	815.9108	564.2837	623.1800	605.8733
$UMLE$	1125.5549	474.0433	642.4414	695.2905	391.6580	773.0710
HS	1170.5134	420.0236	801.0685	568.6071	633.4851	603.1457
EMA	1204.8421	401.5795	996.7565	443.5979	996.0501	381.4442
ZE	436.0417	756.5147	222.5000	588.7080	147.5000	489.2107
HDL	979.7917	443.4793	856.875	348.9562	812.0833	288.8372
DL	1523.5417	134.6947	1491.2500	109.2898	1476.6667	88.4904
WSM	**1149.1283**	**406.4031**	**968.9359**	**419.4892**	**897.6092**	**410.0749**

For cases with censoring levels above 0.75, the WSM and EMA methods yield similar results. For cases with censoring levels less than 0.75, μ is underestimated by both ZE and HDL Methods, while σ is overestimated comparing to estimates obtained by $ASAMLEOC$, EMA, $UMLE$ and HS methods. The DL method yield similar estimate for μ for cases with censoring levels less than 0.75, while σ is underestimated for all censoring levels via this method comparing to estimates obtained by $ASAMLEOC$ of Cohen, EMA, $UMLE$ and HS methods. Overall, the WSM method yields similar estimates to those obtained by $ASAMLEOC$, EMA, $UMLE$ and HS methods, and superior to the existing substitution methods ZE, HDL and DL for all censoring levels.

6. Comparison of Methods

In this section the estimation methods described above were compared by a simulation study. We shall assess the performance of estimators obtained via these methods in terms of the mean squared error MSE (variance of the estimator plus the square of the bias). The simulation study was performed with ten thousand repetitions ($N = 10000$) of samples from a normal distribution for each combination of n, μ, σ, and the censoring level $CL = h$. Simulations were conducted with censoring levels $0.15,\ 0.25,\ 0.50,\ 0.75,\ and\ 0.90$. The selected combinations of (n, μ, σ, CL) are:

$$
\begin{array}{c}
n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.15 \\
n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.25 \\
n = 10, 25, 50, 75, 100\ ,\ \mu = 25\ ,\ \ \sigma = 10, CL = 0.50 \\
n = 10, 25, 50, 75, 100\ ,\ \mu = 10\ ,\ \ \sigma = 5, CL = 0.75 \\
n = 10, 25, 50, 75, 100\ ,\ \mu = 10\ ,\ \ \sigma = 5, CL = 0.90
\end{array}
\tag{6.1}
$$

Given the censoring level CL, the detection limit is computed from the relation $DL = CL^{th}$ percentile. The data sets were then artificially censored at DL. Any value falling below DL was considered to be left-censored. These simulated data sets ($N = 10000$ for each combination of n, μ, σ and CL) were then utilized by these estimators to obtain estimates of μ and σ. The average of the $N = 10000$ estimates are reported as $\hat{\mu}$ and $\hat{\sigma}$ in Table 1 and 2. The MSE based on $N = 10000$ simulation runs are also reported in each table. The MSE of $\hat{\mu}$ is defined by:

(6.2) $$MSE(\hat{\mu}, \mu) = Var(\hat{\mu}) + (b(\hat{\mu}, \mu))^2 \ ,$$

where

(6.3) $$b(\hat{\mu}, \mu) = \hat{\mu} - \mu \ ,$$

is the bias of $\hat{\mu}$, where

(6.4) $$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N}\hat{\mu}_i \quad and \quad Var(\hat{\mu}) = \frac{1}{N-1}\sum_{i=1}^{N}(\hat{\mu}_i - \hat{\mu})^2.$$

The MSE of $\hat{\sigma}$ can be defined in a similar way.

Estimation Methods: The methods used for the estimation of the normal population parameters from singly-left-censored samples are:

$ASAMLEOC$**:** Aboueissa and Stoline Algorithm for Calculating MLE of Cohen,
$UMLE$**:** Bias-Corrected MLE Estimators,
HS**:** Haas and Scheff method,
EMA**:** Expectation Maximization algorithm method,
ZE**:** Replacing all left-censored data by zero method,
HDL**:** Replacing all left-censored data by half of the detection limit method,
DL**:** Replacing all left-censored data by the detection limit method,
WSM**:** The new Weighted Substitution Method.

Tables 4 and 5 are partitioned into 5 subgroups by increasing censoring level: $CL = 0.15, 0.25, 0.50$, 0.75 and 0.90. The simulation results within each subgroup are further partitioned by increasing sample size $n = 10, 25, 50$ and 75. Two simulation results are given for each method and for each combination of n, μ, σ and CL. These are the average value of the estimate and the MSE.

6.1. Comparison of Methods: μ Parameter.

WSM **to existing methods:** The following observations and conclusions are made from an examination of the simulation results reported for the mean μ.

For the $\mu = 25$ parameter value: the reported new WSM method estimates are all in the range $24.8722 - 25.2874$, the reported HDL method estimates are all in the range $23.7069 - 24.6108$, the reported EMA method estimates are all in the range $24.7206 - 25.002$ and the reported $ASAMLEOC$ method estimates are all in the range $24.5856 - 25.0355$ for cases with censoring level less than 50%. For cases with censoring levels less than 50%: (1) the MSE values for HDL method are larger than those reported by the new WSM method, and (2) the MSE values for WSM method are nearly equal to those reported by the new EMA and $ASAMLEOC$ methods. For cases with censoring level 50%: the reported new WSM method estimates are all in the range $23.4630 - 24.6600$, the reported HDL method estimates are all in the range $22.5559 - 22.9255$, the reported EMA method estimates are all in the range $24.8221 - 25.0050$ and the reported $ASAMLEOC$ method estimates are all in the range $25.0019 - 25.4548$. The MSE values for HDL method are larger than those reported by the new WSM method. The MSE values for the new WSM method are nearly equal to those reported by both EMA and $ASAMLEOC$ methods except for cases with sample sizes 50, 75, and 100.

For the $\mu = 10$ parameter value and for cases with censoring level greater than or equal to 75%: the reported new WSM method estimates are all in the range $8.5887 - 9.8231$, the reported HDL method estimates are all in the range $8.6633 - 9.2154$, the reported EMA method estimates are all in the range $10.1079 - 12.7350$ and the reported $ASAMLEOC$ method estimates are all in the range $9.8679 - 11.2427$. The MSE values for HDL and EMA methods are quite similar and smaller than those reported by EMA and $ASAMLEOC$ methods except for cases with sample sizes 75 and 100. For cases with censoring level 90% and sample size 10, it has been noted that estimates for the μ parameter are not available via EMA method.

Overall, the new WSM method appears to be superior to the existing methods for cases with censoring levels less than 50%, and superior to EMA and $ASAMLEOC$ methods for cases with censoring levels greater than or equal to 50% except for cases with sample sizes 75 and 100. The new WSM and HDL methods yield quite similar estimates for the μ parameter for cases with censoring levels greater than or equal to 50%.

6.2. Comparison of Methods: σ Parameter.

WSM to existing methods: The following observations and conclusions are made from an examination of the simulation results reported for the standard deviation σ.

For the $\sigma = 10$ parameter value: the reported new WSM method estimates are all in the range $9.2886 - 9.8007$, the reported HDL method estimates are all in the range $10.2680 - 10.7815$, the reported EMA method estimates are all in the range $9.6781 - 10.0459$ and the reported $ASAMLEOC$ method estimates are all in the range $9.5468 - 10.0068$ for cases with censoring level less than 50%. The MSE values for EMA and $ASAMLEOC$ methods are larger than those reported by the new WSM method for cases with censoring levels less than 50%. The MSE values reported by HDL and the new WSM methods are quite similar for cases with censoring levels less than 50%. For cases with censoring level 50%: the reported new WSM method estimates are all in the range $9.3601 - 10.5496$, the reported HDL method estimates are all in the range $10.7585 - 11.0397$, the reported EMA method estimates are all in the range $9.5578 - 9.9383$ and the reported $ASAMLEOC$ method estimates are all in the range $9.1672 - 9.8955$. The MSE values for HDL and the new WSM methods are quite similar, and smaller than those reported by both EMA and $ASAMLEOC$ methods except for cases with sample sizes 100.

For the $\sigma = 5$ parameter value and for cases with censoring level greater than or equal to 75%: the reported new WSM method estimates are all in the range $4.3496 - 5.0428$, the reported HDL method estimates are all in the range $3.0071 - 4.3463$, the reported EMA method estimates are all in the range $3.0289 - 4.8167$ and the reported $ASAMLEOC$ method estimates are all in the range $3.8187 - 4.9756$. The MSE values for EMA, EMA and $ASAMLEOC$ methods are larger than those reported by the new WSM method. For cases with censoring level 90% and sample size 10, it has been noted that estimates for the σ parameter are not available via EMA method. It should be noted that the $\sigma = 5$ parameter value for most cases is highly under estimated by EMA, EMA and $ASAMLEOC$ methods.

Overall, the new WSM method appears to be superior to HDL method for cases with censoring levels greater than or equal to 50%, and superior to EMA and $ASAMLEOC$ methods for all censoring cases. The HDL and the new WSM methods perform similarly for cases with censoring levels less than 50%.

In summary, the maximum likelihood estimators (ASAMLEOC), the new weighted substitution method estimators (WSM), and the EM algorithm estimators (EMA) perform similarly, and all are generally superior to the existing substitution method estimators.

6.3. Additional Simulation Results.

The following simulation results are obtained using the following combinations of n, μ, σ, and censoring level CL.

TABLE 3. Estimates for μ and σ from Sulfate Data

(n, μ, σ)	k	CL
$(k, 25, 10)$	$k = 10, 25, 50, 75, 100$	0.75 - 0.90
$(k, 10, 5)$	$k = 10, 25, 50, 75, 100$	0.15 - 0.50
$(k, 20, 3)$	$k = 10, 25, 50, 75, 100$	0.10 - 0.90

Tables 6, 7 and 8 are partitioned into two subgroups. Each subgroup has a different censoring level. The simulation results within each subgroup are given for both population mean μ and standard deviation σ. Two simulation results are given for each method and for each combination of n, μ, σ and CL. These simulation results are the average value of the estimate and the MSE.

TABLE 4. Simulation Estimates of the Mean μ from Normally Distributed Left-Censored Samples with a Single Detection Limit

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
				$CL = 0.15$					
$(10, 25, 10)$	$\hat{\mu}$	24.7206	24.5856	24.3367	24.4160	**25.0303**	22.4497	24.0517	25.6536
	MSE	12.9903	12.1390	12.4803	12.3479	**10.2139**	14.1331	10.3785	12.1721
$(25, 25, 10)$	$\hat{\mu}$	25.0022	25.0047	24.9302	24.8702	**25.2874**	23.3820	24.6056	25.8292
	MSE	4.0587	4.0515	4.0600	4.0765	**3.6776**	5.5471	3.5844	4.7208
$(50, 25, 10)$	$\hat{\mu}$	24.9873	24.9610	24.9221	24.8229	**25.2175**	23.4054	24.6045	25.8036
	MSE	1.9815	1.9524	1.9576	1.9836	**1.7494**	3.9807	1.8237	2.5930
$(75, 25, 10)$	$\hat{\mu}$	24.9569	24.9167	24.8892	24.7757	**25.1520**	23.3772	24.5654	25.7536
	MSE	1.3018	1.2937	1.2997	1.3415	**1.2036**	3.5491	1.2649	1.8417
$(100, 25, 10)$	$\hat{\mu}$	24.9303	24.9455	24.9308	24.8173	**25.1187**	23.5041	24.6108	25.7176
	MSE	1.0832	1.0840	1.0861	1.1177	**1.0118**	3.0278	1.0706	1.5900
				$CL = 0.25$					
$(10, 25, 10)$	$\hat{\mu}$	24.9314	24.7705	24.3606	24.5564	**25.1387**	20.8147	23.7069	26.5991
	MSE	11.1167	10.1121	10.6242	10.2893	**8.6504**	22.7221	8.7693	12.2048
$(25, 25, 10)$	$\hat{\mu}$	24.8567	25.0355	24.9278	24.8842	**25.0651**	21.9426	24.1728	26.4031
	MSE	4.7220	4.4562	4.4708	4.4865	**3.8785**	11.9771	4.0882	6.3499
$(50, 25, 10)$	$\hat{\mu}$	24.9379	24.9031	24.8405	24.7176	**24.9884**	21.6906	24.0783	26.4659
	MSE	2.3519	2.2546	2.2750	2.3375	**1.9278**	12.1852	2.4918	4.3222
$(75, 25, 10)$	$\hat{\mu}$	24.9199	24.9175	24.8798	24.7403	**24.8745**	21.7923	24.1097	26.4272
	MSE	1.3578	1.3316	1.3406	1.3983	**1.1832**	11.0518	1.7847	3.3294
$(100, 25, 10)$	$\hat{\mu}$	24.9792	25.0024	24.9770	24.8292	**24.8722**	21.9016	24.1936	26.4857
	MSE	1.0849	1.0738	1.0749	1.1061	**0.9745**	10.2353	1.4676	3.2606
				$CL = 0.50$					
$(10, 25, 10)$	$\hat{\mu}$	24.8221	25.1868	24.1506	24.9091	**24.6600**	16.2848	22.5559	28.8270
	MSE	18.5532	15.2003	17.3134	15.4780	**10.1436**	79.3801	12.9385	27.0316
$(25, 25, 10)$	$\hat{\mu}$	24.9778	25.4548	25.0936	25.2066	**23.8873**	16.9032	22.9255	28.9479
	MSE	6.7314	6.3569	6.3417	6.3287	**5.2874**	67.0511	7.2413	20.7299
$(50, 25, 10)$	$\hat{\mu}$	24.9382	25.0019	24.8038	24.7091	**23.8758**	16.4341	22.6824	28.9307
	MSE	3.2269	2.9649	3.0511	3.1225	**4.0171**	74.0480	6.7341	17.8859
$(75, 25, 10)$	$\hat{\mu}$	25.0050	25.1557	25.0237	24.8749	**23.76042**	16.6278	22.8010	28.9742
	MSE	1.9961	1.9971	1.9946	2.0344	**4.2894**	70.5353	5.7310	17.3938
$(100, 25, 10)$	$\hat{\mu}$	24.9496	24.9960	24.8980	24.7015	**23.4630**	16.4471	22.6953	28.9434
	MSE	1.4499	1.3884	1.4097	1.5089	**5.1825**	73.4808	5.9611	16.6976
				$CL = 0.75$					
$(10, 10, 5)$	$\hat{\mu}$	11.1600	10.9294	9.7694	10.7134	**9.8231**	4.6079	9.1331	13.6582
	MSE	4.9783	6.9497	8.5266	6.9843	**2.3121**	29.4568	2.5634	16.9481
$(25, 10, 5)$	$\hat{\mu}$	10.3701	10.7352	10.1216	10.4767	**9.2815**	4.4301	9.2023	13.9745
	MSE	3.2304	3.9538	4.0427	3.8897	**2.3455**	31.1690	2.3161	17.4969
$(50, 10, 5)$	$\hat{\mu}$	10.2091	10.2622	9.8291	9.9475	**9.1792**	4.1868	9.1008	14.0148
	MSE	1.6294	1.7626	1.9279	1.8441	**1.2663**	33.8541	1.1847	16.8751
$(75, 10, 5)$	$\hat{\mu}$	10.1761	10.0857	9.7393	9.7467	**9.0131**	4.1183	9.0916	14.0649
	MSE	1.0370	1.2607	1.4362	1.4306	**1.6401**	34.6373	1.0413	17.0802
$(100, 10, 5)$	$\hat{\mu}$	10.1079	9.9587	9.6655	9.6094	**8.9862**	4.0600	9.0399	14.0197
	MSE	0.8523	0.9697	1.1543	1.2111	**1.2560**	35.3154	1.0860	16.5823
				$CL = 0.90$					
$(10, 10, 5)$	$\hat{\mu}$	NAN	9.9275	6.4233	9.8992	**9.7546**	1.7684	8.6633	15.5582
	MSE	NAN	28.3201	78.0182	28.5537	**2.1788**	67.8434	3.4499	36.3779
$(25, 10, 5)$	$\hat{\mu}$	12.7350	11.2427	9.8571	10.8905	**9.0188**	2.1579	9.1766	16.1952
	MSE	11.2545	10.9418	13.8752	11.3257	**2.8197**	7.8420	1.3721	40.5871
$(50, 10, 5)$	$\hat{\mu}$	12.4702	9.8679	9.0350	9.4993	**9.0459**	1.8560	9.1220	16.3880
	MSE	8.8839	8.3572	11.1032	9.3857	**5.7839**	66.3434	2.1813	42.1725
$(75, 10, 5)$	$\hat{\mu}$	12.3973	10.5135	9.9385	10.1106	**9.1537**	1.9673	9.2154	16.4635
	MSE	7.6331	4.9507	5.4244	5.2242	**6.9610**	64.5361	4.8869	42.6704
$(100, 10, 5)$	$\hat{\mu}$	12.2278	9.8990	9.4768	9.4882	**8.5887**	1.8662	9.1851	16.5041
	MSE	6.3274	4.2162	4.9141	4.8985	**6.3863**	66.1671	3.8660	42.9826

Table 5. Simulation Estimates of the Standard Deviation σ from Normally Distributed Left-Censored Samples with a Single Detection Limit

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
$CL = 0.15$									
(10, 25, 10)	$\hat{\sigma}$	10.0459	9.6976	10.7021	9.8076	**9.4377**	13.0290	10.3384	8.0643
	MSE	6.3001	6.4146	8.1939	6.4824	**5.3325**	11.5996	4.9412	8.1432
(25, 25, 10)	$\hat{\sigma}$	9.8113	9.7730	10.1485	9.8612	**9.6837**	12.4614	10.2680	8.4608
	MSE	2.5130	2.5439	2.7096	2.5525	**2.2003**	7.0256	1.8620	4.2423
(50, 25, 10)	$\hat{\sigma}$	9.9661	10.0068	10.2000	10.0965	**9.7549**	12.5478	10.4218	8.6617
	MSE	1.3647	1.3279	1.4193	1.3572	**1.0455**	7.0106	0.9788	2.7903
(75, 25, 10)	$\hat{\sigma}$	9.9220	9.9845	10.1138	10.0765	**9.4640**	12.5021	10.4922	8.6389
	MSE	0.8380	0.8363	0.8708	0.8575	**0.7558**	6.5641	0.6781	2.4788
(100, 25, 10)	$\hat{\sigma}$	9.9273	9.9035	9.9950	9.9874	**9.7880**	12.3037	10.3035	8.6555
	MSE	0.6693	0.6685	0.6715	0.6704	**0.6262**	5.5621	0.5424	2.3117
$CL = 0.25$									
(10, 25, 10)	$\hat{\sigma}$	9.6782	9.7455	10.9469	9.8648	**9.2886**	14.7549	10.7656	7.2926
	MSE	7.6926	7.8342	10.6996	7.9515	**4.5755**	25.2537	3.3651	11.7217
(25, 25, 10)	$\hat{\sigma}$	9.7791	9.5468	9.9658	9.6299	**9.5273**	13.8585	10.4642	7.6557
	MSE	2.8727	2.7838	2.8110	2.7655	**2.0599**	15.9031	1.7138	7.1502
(50, 25, 10)	$\hat{\sigma}$	9.9417	9.9884	10.2161	10.0907	**9.5583**	14.2765	10.7815	7.8224
	MSE	1.5469	1.4527	1.5662	1.4891	**1.0459**	18.8102	1.1838	5.6361
(75, 25, 10)	$\hat{\sigma}$	9.9461	9.9494	10.0974	10.0468	**9.6981**	14.1626	10.7359	7.8518
	MSE	0.9648	0.9589	0.9944	0.9761	**0.6769**	17.6624	0.9304	5.2130
(100, 25, 10)	$\hat{\sigma}$	9.9602	9.9296	10.0385	10.0245	**9.8007**	14.1353	10.7264	7.8658
	MSE	0.7465	0.7489	0.7619	0.7601	**0.4713**	17.3686	0.8366	5.0213
$CL = 0.50$									
(10, 25, 10)	$\hat{\sigma}$	9.5578	9.1672	10.8553	9.2823	**9.3601**	16.7716	10.7585	5.3312
	MSE	15.6057	12.0724	16.6869	12.1633	**3.7688**	49.5103	3.1956	25.6978
(25, 25, 10)	$\hat{\sigma}$	9.7338	9.2809	9.9360	9.3811	**9.9686**	16.7956	10.8394	5.5985
	MSE	5.2199	5.2880	5.4728	5.2627	**1.6210**	47.6687	1.8370	21.1074
(50, 25, 10)	$\hat{\sigma}$	9.9095	9.8507	10.2101	9.9679	**10.2462**	16.9698	11.0231	5.7495
	MSE	2.6241	2.5035	2.7096	2.5395	**1.3816**	49.3191	1.6188	18.9206
(75, 25, 10)	$\hat{\sigma}$	9.9029	9.7586	9.9950	9.8706	**10.3847**	16.9536	11.0084	5.7604
	MSE	1.6039	1.5602	1.5756	1.5542	**1.4119**	48.8112	1.3613	18.4996
(100, 25, 10)	$\hat{\sigma}$	9.9383	9.8955	10.0758	10.0129	**10.5496**	16.9832	11.0397	5.7767
	MSE	1.2036	1.2086	1.2475	1.2261	**1.7183**	16.9832	1.3615	18.2478
$CL = 0.75$									
(10, 10, 5)	$\hat{\sigma}$	4.2618	3.8187	4.9812	3.8871	**4.3496**	7.1417	4.2411	1.5442
	MSE	5.2597	5.6734	7.2788	5.6663	**1.1665**	5.5348	1.5443	12.6638
(25, 10, 5)	$\hat{\sigma}$	4.5483	4.2664	4.8400	4.3438	**4.6328**	7.2150	4.3111	1.6639
	MSE	2.7109	2.7569	2.8810	2.7305	**0.4991**	5.3049	0.8201	11.4708
(50, 10, 5)	$\hat{\sigma}$	4.7104	4.6859	5.0415	4.7798	**4.8445**	7.1681	4.3241	1.7189
	MSE	1.3499	1.2885	1.3782	1.2854	**0.2456**	4.8861	0.5780	10.9302
(75, 10, 5)	$\hat{\sigma}$	4.8058	4.8796	5.1431	4.9806	**4.9818**	7.1746	4.3463	1.7528
	MSE	0.8648	0.8804	0.9823	0.9026	**0.1611**	4.8646	0.5148	10.6573
(100, 10, 5)	$\hat{\mu}$	4.8167	4.9337	5.1437	5.0375	**5.0428**	7.1359	4.3290	1.7547
	MSE	0.6827	0.6806	0.7556	0.7065	**0.1283**	4.6661	0.5177	10.6186
$CL = 0.90$									
(10, 10, 5)	$\hat{\sigma}$	NAN	4.2813	6.8391	4.2891	**4.3553**	5.3054	3.0071	0.7088
	MSE	NAN	13.5159	36.5542	13.5523	**1.5075**	1.463764	4.4102	18.7701
(25, 10, 5)	$\hat{\sigma}$	3.5159	4.0208	4.9838	4.1082	**4.4259**	5.8762	3.3071	0.8551
	MSE	5.3528	5.5123	6.9890	5.5572	**0.8011**	1.1049	3.0605	17.3969
(50, 10, 5)	$\hat{\sigma}$	3.4667	4.9175	5.5312	5.0063	**4.8617**	5.6002	3.2012	0.9177
	MSE	4.5614	3.7233	4.9842	3.8528	**0.4854**	0.5449	3.3510	16.7972
(75, 10, 5)	$\hat{\sigma}$	3.4707	4.5939	4.9854	4.6899	**4.8792**	5.7293	3.2513	0.9179
	MSE	3.8156	2.2375	2.4406	2.2567	**0.3493**	0.65004	3.1282	16.7499
(100, 10, 5)	$\hat{\sigma}$	3.5912	4.9756	5.2876	5.0733	**4.8105**	5.6338	3.2202	0.9437
	MSE	3.0289	1.7452	2.0531	1.8204	**0.2995**	0.4841	3.2185	16.5173

Table 6. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.75, 0.90$: $(k, 25, 10)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
					$CL = 0.75$				
(10, 25, 10)	$\hat{\mu}$	27.565	26.815	24.464	26.380	**27.349**	10.737	21.543	32.349
	MSE	21.380	34.594	42.150	34.912	**26.146**	205.140	19.480	72.169
(25, 25, 10)	$\hat{\mu}$	25.492	26.063	24.800	25.528	**25.797**	10.235	21.480	32.725
	MSE	11.982	14.436	16.016	14.680	**11.949**	218.58	14.862	65.793
(50, 25, 10)	$\hat{\mu}$	25.321	25.493	24.620	24.857	**25.407**	9.689	21.376	33.063
	MSE	6.356	6.912	7.578	7.245	**6.216**	234.71	14.493	68.410
(75, 25, 10)	$\hat{\mu}$	25.294	25.148	24.457	24.471	**25.221**	9.487	21.283	33.080
	MSE	4.253	5.072	5.826	5.801	**4.392**	240.82	14.641	67.417
(100, 25, 10)	$\hat{\mu}$	25.223	24.947	24.354	24.242	**25.086**	9.412	21.281	33.151
	MSE	3.434	3.905	4.632	4.850	**3.474**	243.12	14.450	68.053
(10, 25, 10)	$\hat{\sigma}$	8.464	7.742	10.099	7.880	**8.114**	16.589	9.635	3.133
	MSE	20.263	21.496	27.905	21.453	**18.758**	47.473	12.165	49.949
(25, 25, 10)	$\hat{\sigma}$	9.259	8.782	9.963	8.943	**9.040**	16.612	9.735	3.416
	MSE	11.333	11.516	12.918	11.534	**10.770**	45.296	7.948	44.854
(50, 25, 10)	$\hat{\sigma}$	9.548	9.451	10.167	9.640	**9.499**	16.532	9.734	3.469
	MSE	5.479	5.080	5.556	5.100	**5.083**	43.540	5.599	43.307
(75, 25, 10)	$\hat{\sigma}$	9.601	9.726	10.457	11.471	**9.664**	16.467	9.721	3.494
	MSE	3.588	3.637	5.826	5.801	**3.485**	42.344	2.422	42.789
(100, 25, 10)	$\hat{\sigma}$	9.748	9.970	10.394	10.179	**9.859**	16.483	9.756	3.543
	MSE	2.733	2.791	3.187	2.940	**2.666**	42.417	2.320	42.050
					$CL = 0.90$				
(10, 25, 10)	$\hat{\mu}$	NAN	24.304	16.853	24.244	**23.178**	4.080	20.178	36.276
	MSE		110.97	8.147	111.98	**29.204**	437.99	19.204	146.64
(25, 25, 10)	$\hat{\mu}$	30.247	27.918	25.236	27.246	**28.366**	4.916	21.211	37.506
	MSE	44.689	44.648	53.141	45.532	**42.684**	403.54	17.366	165.91
(50, 25, 10)	$\hat{\mu}$	29.900	24.606	22.926	23.866	**27.251**	4.214	20.983	37.751
	MSE	34.130	31.013	41.799	34.897	**21.481**	432.14	17.709	167.83
(75, 25, 10)	$\hat{\mu}$	29.811	25.967	24.815	25.164	**26.881**	4.465	21.179	37.894
	MSE	30.283	27.901	19.642	18.856	**18.244**	421.76	17.674	169.76
(100, 25, 10)	$\hat{\mu}$	29.637	24.944	24.116	24.138	**27.290**	4.213	21.053	37.892
	MSE	27.767	17.654	20.304	20.268	**15.268**	432.14	16.340	168.765
(10, 25, 10)	$\hat{\sigma}$	NAN	8.587	13.718	8.603	**11.177**	12.239	6.873	1.507
	MSE		52.124	141.74	52.263	**3.453**	8.032	11.582	73.601
(25, 25, 10)	$\hat{\sigma}$	7.251	7.784	9.649	7.951	**7.816**	13.367	7.390	1.654
	MSE	21.416	22.564	27.230	22.670	**19.561**	12.790	17.606	70.487
(50, 25, 10)	$\hat{\sigma}$	6.941	9.915	11.153	10.094	**8.475**	12.697	7.150	1.848
	MSE	17.354	13.288	18.132	13.751	**11.657**	7.945	8.526	66.917
(75, 25, 10)	$\hat{\sigma}$	6.935	9.207	9.991	9.398	**8.791**	12.984	7.257	1.842
	MSE	24.960	8.208	8.925	8.264	**9.549**	9.373	9.790	66.862
(100, 25, 10)	$\hat{\sigma}$	6.970	9.755	10.367	9.947	**8.363**	12.698	7.132	1.848
	MSE	13.718	7.583	8.631	7.832	**8.131**	7.606	8.736	66.734

The following observations and conclusions are made from an examination of the simulation results reported in Tables $6 - 8$. The new WSM method appears to be superior to existing substitution methods for all censoring cases, and yields quite similar estimates to EMA and $ASAMLEOC$ methods. The HDL and the new WSM methods perform similarly for cases with censoring levels less than 50%.

In summary, the maximum likelihood estimators (ASAMLEOC), the new weighted substitution method estimators (WSM), and the EM algorithm estimators (EMA)

TABLE 7. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.15, 0.50$: $(k, 10, 5)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
					$CL = 0.15$				
(10, 10, 5)	$\hat{\mu}$	10.078	10.103	9.888	9.928	**10.045**	9.405	9.976	10.547
	MSE	2.836	2.669	2.689	2.667	**2.437**	2.195	2.203	2.951
(25, 10, 5)	$\hat{\mu}$	10.046	10.047	10.010	9.980	**10.046**	9.626	10.043	10.460
	MSE	1.019	1.013	1.102	1.014	**1.011**	1.140	1.272	1.256
(50, 10, 5)	$\hat{\mu}$	9.977	9.964	9.945	9.894	**9.979**	9.580	9.978	10.380
	MSE	0.508	0.492	0.494	0.502	**0.498**	0.538	0.507	0.633
(75, 10, 5)	$\hat{\mu}$	9.959	9.939	9.925	9.868	**9.949**	9.585	9.973	10.362
	MSE	0.373	0.373	0.375	0.387	**0.371**	0.438	0.351	0.496
(100, 10, 5)	$\hat{\mu}$	9.993	9.999	9.991	9.934	**9.996**	9.643	10.213	10.382
	MSE	0.256	0.255	0.255	0.259	**0.254**	0.316	0.238	0.399
(10, 10, 5)	$\hat{\sigma}$	5.019	4.856	5.359	4.911	**5.027**	5.746	4.825	4.048
	MSE	1.524	1.703	2.178	1.723	**1.526**	1.727	1.852	2.194
(25, 10, 5)	$\hat{\sigma}$	4.917	4.886	5.073	4.929	**4.911**	5.520	4.819	4.230
	MSE	0.640	0.617	0.656	0.618	**0.596**	0.612	0.597	1.046
(50, 10, 5)	$\hat{\sigma}$	4.931	4.952	5.047	4.997	**4.956**	5.514	4.849	4.285
	MSE	0.337	0.328	0.341	0.332	**0.329**	0.398	0.321	0.755
(75, 10, 5)	$\hat{\sigma}$	5.015	5.045	5.111	5.092	**5.030**	5.553	4.913	4.366
	MSE	0.236	0.237	0.253	0.248	**0.234**	0.400	0.213	0.578
(100, 10, 5)	$\hat{\sigma}$	4.931	4.923	4.968	4.965	**4.927**	5.449	4.829	4.302
	MSE	0.120	0.161	0.159	0.159	**0.158**	0.267	0.175	0.605
					$CL = 0.50$				
(10, 10, 5)	$\hat{\mu}$	9.990	10.093	9.588	9.955	**10.061**	6.853	9.357	11.861
	MSE	3.691	3.433	3.878	3.485	**3.240**	10.724	2.066	6.368
(25, 10, 5)	$\hat{\mu}$	10.014	10.228	10.047	10.102	**10.121**	7.162	9.571	11.980
	MSE	1.548	1.533	1.531	1.528	**1.453**	8.392	1.183	5.131
(50, 10, 5)	$\hat{\mu}$	9.976	10.020	9.921	9.874	**9.983**	7.294	9.947	12.880
	MSE	0.754	0.711	0.728	0.743	**0.579**	5.436	0.441	4.692
(75, 10, 5)	$\hat{\mu}$	9.962	10.053	9.988	9.924	**10.007**	7.031	9.494	11.958
	MSE	0.571	0.522	0.525	0.538	**0.532**	8.934	0.490	4.252
(100, 10, 5)	$\hat{\mu}$	10.017	10.029	9.980	9.882	**10.022**	6.982	9.487	11.992
	MSE	0.329	0.339	0.340	0.359	**0.327**	9.196	0.432	4.265
(10, 10, 5)	$\hat{\sigma}$	4.626	4.464	5.287	4.522	**4.555**	7.115	4.736	2.590
	MSE	3.317	2.744	3.527	2.744	**2.785**	5.343	1.679	6.650
(25, 10, 5)	$\hat{\sigma}$	4.866	4.659	4.988	4.780	**4.763**	7.207	4.862	2.809
	MSE	1.316	1.280	1.333	1.272	**1.224**	5.217	1.097	5.230
(50, 10, 5)	$\hat{\sigma}$	4.975	4.934	5.114	4.992	**4.954**	7.294	4.947	2.880
	MSE	0.612	0.586	0.638	0.596	**0.579**	5.436	0.541	4.692
(75, 10, 5)	$\hat{\sigma}$	4.955	4.868	4.986	4.924	**4.917**	7.257	4.923	2.873
	MSE	0.431	0.427	0.430	0.425	**0.417**	5.217	0.302	4.666
(100, 10, 5)	$\hat{\sigma}$	4.936	4.924	5.024	4.983	**4.930**	7.290	4.924	2.873
	MSE	0.230	0.273	0.278	0.275	**0.265**	5.338	0.217	4.614

perform similarly, and all are generally superior to the existing substitution method estimators.

7. Conclusions and Recommendations

This article has dealt with the problem of estimating the mean and standard deviation of a normal and/or lognormal populations in the presence of left-censored data. To avoid clumping of replaced values in cases where there are several left-censored observations that share a common detection limit, a new replacement

TABLE 8. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.10, 0.90$: $(k, 20, 3)$, $(k = 10, 25, 50, 75, 100)$

		Methods Of Estimation							
			MLE			Replacement			
$(\mathbf{n}, \mu, \sigma)$		**EMA**	**ASAMLEOC**	**UMLE**	**HS**	**WSM**	**ZE**	**HDL**	**DL**
					$CL = 0.10$				
(10, 20, 3)	$\hat{\mu}$	19.984	20.026	19.982	20.004	**20.005**	18.482	19.323	20.164
	MSE	0.948	0.895	0.896	0.895	**0.915**	3.034	1.264	0.919
(25, 20, 3)	$\hat{\mu}$	19.962	19.948	19.929	19.914	**19.955**	18.151	19.137	20.123
	MSE	0.370	0.363	0.366	0.370	**0.365**	3.694	1.058	0.374
(50, 20, 3)	$\hat{\mu}$	19.973	19.980	19.973	19.952	**19.976**	18.496	19.309	20.122
	MSE	0.177	0.175	0.176	0.178	**0.176**	2.404	0.635	0.190
(75, 20, 3)	$\hat{\mu}$	19.900	19.983	19.977	19.952	**19.986**	18.405	19.271	20.137
	MSE	0.125	0.124	0.124	0.126	**0.124**	2.646	0.643	0.143
(100, 20, 3)	$\hat{\mu}$	19.990	19.992	19.989	19.964	**19.991**	18.513	19.324	19.991
	MSE	0.087	0.087	0.087	0.088	**0.087**	2.286	0.538	0.087
(10, 20, 3)	$\hat{\sigma}$	3.143	2.780	3.026	2.796	**3.068**	6.595	3.319	2.549
	MSE	0.537	0.554	0.600	0.552	**0.474**	13.047	1.948	0.629
(25, 20, 3)	$\hat{\sigma}$	2.988	2.967	3.075	2.992	**2.993**	7.090	4.644	2.666
	MSE	0.214	0.220	0.241	0.222	**0.211**	16.783	2.786	0.289
(50, 20, 3)	$\hat{\sigma}$	2.977	2.961	3.012	2.982	**2.970**	6.614	3.427	2.711
	MSE	0.104	0.102	104	0.102	**0.101**	13.083	2.080	0.168
(75, 20, 3)	$\hat{\sigma}$	2.988	2.999	3.035	3.022	**2.994**	6.789	4.526	2.728
	MSE	0.066	0.067	0.069	0.068	**0.065**	14.377	2.358	0.129
(100, 20, 3)	$\hat{\sigma}$	2.986	2.983	3.008	3.004	**2.985**	6.621	4.042	2.730
	MSE	0.053	0.052	0.053	0.052	**0.052**	13.129	2.103	0.116
					$CL = 0.90$				
(10, 20, 3)	$\hat{\mu}$	NAN	19.896	17.761	19.879	**18.866**	2.462	12.894	23.327
	MSE		11.399	32.365	11.512	**12.444**	307.61	51.034	12.850
(25, 20, 3)	$\hat{\mu}$	21.756	20.830	20.032	20.627	**21.385**	2.965	13.325	23.685
	MSE	4.333	4.200	5.098	4.317	**3.816**	290.20	44.812	14.411
(50, 20, 3)	$\hat{\mu}$	21.420	19.862	19.354	19.634	**20.631**	2.517	13.180	23.843
	MSE	2.985	3.364	4.479	3.796	**2.119**	305.67	46.661	15.274
(75, 20, 3)	$\hat{\mu}$	21.423	20.217	19.866	19.972	**20.716**	2.672	13.255	23.839
	MSE	2.733	1.646	1.881	1.787	**1.634**	300.27	45.576	15.018
(100, 20, 3)	$\hat{\mu}$	21.382	19.976	19.725	19.733	**20.678**	2.518	13.207	23.896
	MSE	2.393	1.468	1.696	1.691	**1.255**	305.62	46.219	15.425
(10, 20, 3)	$\hat{\sigma}$	NAN	2.609	4.167	2.613	**6.895**	7.387	3.909	0.432
	MSE		5.860	15.924	5.879	**6.173**	19.540	1.013	6.751
(25, 20, 3)	$\hat{\sigma}$	1.997	2.317	2.872	2.367	**2.320**	8.038	4.220	0.495
	MSE	2.061	2.027	2.411	2.034	**1.735**	25.496	1.546	6.353
(50, 20, 3)	$\hat{\sigma}$	2.128	3.002	3.377	3.057	**2.693**	7.560	4.012	0.561
	MSE	1.496	1.395	1.907	1.451	**1.086**	20.850	1.763	5.999
(75, 20, 3)	$\hat{\sigma}$	2.082	2.797	3.036	2.856	**2.444**	7.742	4.093	0.557
	MSE	1.404	0.811	0.908	0.824	**0.917**	22.529	1.220	5.999
(100, 20, 3)	$\hat{\sigma}$	2.120	2.954	3.139	3.011	**2.537**	7.563	4.008	0.559
	MSE	1.157	0.628	0.726	0.651	**0.665**	20.856	1.035	5.983

method called weighted substitution method is introduced. In this method left-censored observations are spaced from zero to the detection limit according to weights assigned to these non-detected data. To facilitate the application of estimation methods described in this article, a computer program is presented. The computer program "*SingleLeft.Censored.Normal*", written in the R language, is an easy-to-use computerized tool for obtaining estimates and their standard deviations of population parameters of singly left-censored data using either a normal or lognormal distribution. The simulation results presented in Tables 3-4 show that the new WSM and HDL methods perform similarly for cases where the censoring levels is less than 50%. The new WSM method perform better than EMA and $ASAMLEOC$ methods for cases where the censoring levels is less than 50%. For estimating the σ parameter the new WSM method perform better than the existing methods for cases where the censoring levels is greater than or equal to 75%. Taken together, the suggested new WSM method appear to work best for normally distributed censored samples, and lognormal versions of the estimator can be obtained simply by taking natural logarithm of the data and the detection limit.

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Appendix

The suggested weighted substitution method is based on replacing the left-censored observations that are less than the detection limit DL by non-constant different values based on assigning a different weight for each observation. Some of the choices of the weights that were examined are:

$$w1_j(=w_j) = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(U \geq DL)\right)^{\ln(m+j-1)}, \qquad (3.1 \ \ given \ \ above)$$

$$w2_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} [P(U \geq DL)]$$

$$w3_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} \left(P(U \geq DL)\right)^{m+j-1},$$

$$w4_j = \left(\frac{(m+j-1)}{n}\right)^{(\frac{j}{j+1})} \left(P(U \leq DL)\right)^{\ln(m+j-1)},$$

$$w5_j = \left(\frac{(m+j-1)}{n}\right)^{\frac{j}{j+1}} [P(U \leq DL)]^{(m+j-1)},$$

$$w6_j = \left(\frac{(m+j-1)}{n}\right) \left(P(U \geq DL)\right)^{\ln(m+j-1)},$$

$$w7_j = \left(\frac{(m+j-1)}{n}\right) \left(P(U \geq DL)\right),$$

$$for \ \ j = 1, 2, ..., m_c$$

where the probability $P(U \geq DL)$ is estimated from the sample data by:

$$P(\widehat{U \geq D}L) = 1 - \Phi\left(\frac{DL - \bar{x}_m}{s_m}\right)$$

An extensive simulation study was conducted on these weights in addition to other weights (not shown here). The simulation results indicate that the suggested weight in (3.1) leads to estimators that have the ability to recover the true mean and standard deviation as well as the existing methods such as maximum likelihood and EM algorithm estimators. More simulation results will be available in the web page of the author later on if needed.

Table 9. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.75, 0.90$: $(k, 25, 10)$, $(k = 10, 25, 50, 75, 100)$

$(\mathbf{n}, \mu, \sigma)$		Methods Of Estimation							
		MLE	$W1_j(= W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
					$CL = 0.75$				
(10, 25, 10)	$\hat{\mu}$	26.820	24.234	23.252	19.329	11.108	10.741	21.515	22.402
	MSE	34.258	12.401	16.047	51.433	194.777	205.259	21.974	15.600
(25, 25, 10)	$\hat{\mu}$	26.063	23.465	20.810	13.064	10.320	10.235	19.899	22.433
	MSE	14.436	6.016	22.861	148.885	216.079	218.583	30.475	9.668
(50, 25, 10)	$\hat{\mu}$	25.493	24.206	19.842	10.692	9.706	9.689	19.339	21.544
	MSE	6.912	5.299	30.481	206.662	234.184	234.705	35.300	7.859
(75, 25, 10)	$\hat{\mu}$	25.148	24.521	19.296	9.883	9.493	9.487	18.920	21.625
	MSE	5.072	5.233	35.658	229.296	240.630	240.824	40.013	8.410
(100, 25, 10)	$\hat{\mu}$	24.980	24.113	18.714	9.571	9.416	9.413	18.997	22.970
	MSE	3.789	4.305	42.200	238.248	242.978	243.065	38.541	6.051
(10, 25, 10)	$\hat{\sigma}$	7.490	8.921	8.536	11.143	16.354	16.586	10.111	3.023
	MSE	22.999	3.664	4.216	6.978	44.455	47.441	2.020	51.492
(25, 25, 10)	$\hat{\sigma}$	8.782	10.168	9.342	14.902	16.560	16.612	11.266	10.379
	MSE	11.516	1.413	5.342	27.034	44.618	45.296	5.969	7.088
(50, 25, 10)	$\hat{\sigma}$	9.451	9.745	11.873	15.960	16.522	16.532	12.475	11.987
	MSE	5.080	0.576	3.077	36.846	43.411	43.540	5.094	4.726
(75, 25, 10)	$\hat{\sigma}$	9.726	9.912	11.267	16.245	16.463	16.467	11.611	11.945
	MSE	3.637	0.314	2.292	39.594	42.298	42.344	4.972	5.237
(100, 25, 10)	$\hat{\sigma}$	9.950	10.486	11.732	16.397	16.484	16.485	12.464	10.997
	MSE	2.791	1.468	4.555	41.318	42.428	42.448	3.102	2.250
					$CL = 0.90$				
(10, 25, 10)	$\hat{\mu}$	24.891	24.215	22.568	7.982	5.174	4.045	20.009	20.099
	MSE	108.321	99.875	114.827	387.340	393.543	439.447	121.432	132.093
(25, 25, 10)	$\hat{\mu}$	27.918	23.327	20.050	12.317	5.044	4.918	18.748	20.927
	MSE	44.648	42.724	45.861	191.703	398.446	403.496	48.107	41.091
(50, 25, 10)	$\hat{\mu}$	24.606	23.938	18.526	8.797	4.245	4.214	17.937	20.844
	MSE	31.013	29.925	52.094	289.135	430.836	432.141	59.028	31.088
(75, 25, 10)	$\hat{\mu}$	25.967	23.983	16.584	5.541	4.485	4.465	15.983	20.569
	MSE	17.901	16.502	78.491	382.210	420.910	421.758	88.275	21.601
(100, 25, 10)	$\hat{\mu}$	24.944	23.896	16.544	5.135	4.222	4.213	16.188	21.690
	MSE	17.655	16.520	78.921	398.059	431.778	432.136	84.747	20.197
(10, 25, 10)	$\hat{\sigma}$	8.587	9.071	8.672	12.014	13.322	13.366	14.071	13.510
	MSE	52.124	50.071	55.982	67.803	84.007	58.602	55.341	52.762
(25, 25, 10)	$\hat{\sigma}$	7.784	9.771	9.585	11.906	16.560	16.612	12.647	12.993
	MSE	22.567	15.847	24.087	16.094	44.618	45.296	25.442	23.087
(50, 25, 10)	$\hat{\sigma}$	9.915	9.964	10.730	12.510	12.687	12.997	11.604	12.106
	MSE	13.288	10.487	11.522	13.951	14.890	13.944	15.604	12.106
(75, 25, 10)	$\hat{\sigma}$	9.207	10.156	11.415	12.656	12.977	12.984	10.938	11.048
	MSE	8.208	5.371	7.468	9.784	9.332	9.373	8.034	7.997
(100, 25, 10)	$\hat{\sigma}$	9.755	10.143	11.544	12.423	12.696	12.699	11.479	11.029
	MSE	7.583	5.264	6.514	7.486	8.591	8.606	6.479	8.029

TABLE 10. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.15, 0.50$: $(k, 10, 5)$, $(k = 10, 25, 50, 75, 100)$

$(\mathbf{n}, \mu, \sigma)$		Methods Of Estimation							
		MLE	$W1_j(=W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
					$CL = 0.15$				
(10, 10, 5)	$\hat{\mu}$	10.013	10.327	10.388	11.072	9.001	9.105	10.704	11.264
	MSE	2.669	2.617	2.988	2.899	3.120	3.195	3.560	3.626
(25, 10, 5)	$\hat{\mu}$	10.047	10.073	10.560	9.661	9.326	9.034	10.544	10.703
	MSE	1.013	1.008	1.788	1.843	1.848	1.901	1.196	1.934
(50, 10, 5)	$\hat{\mu}$	9.964	10.084	10.286	9.570	9.380	9.294	10.363	10.565
	MSE	0.492	0.490	0.553	0.804	0.638	0.701	0.781	0.739
(75, 10, 5)	$\hat{\mu}$	9.939	10.164	10.372	9.618	9.585	9.275	10.357	10.470
	MSE	0.373	0.367	0.426	0.621	0.438	0.509	0.470	0.478
(100, 10, 5)	$\hat{\mu}$	9.991	10.082	10.298	9.654	9.542	9.343	10.165	10.380
	MSE	0.255	0.270	0.334	0.375	0.416	0.493	0.273	0.326
(10, 10, 5)	$\hat{\sigma}$	4.856	4.609	4.241	4.065	5.974	6.746	4.221	4.244
	MSE	1.703	1.544	1.973	2.164	2.218	2.228	1.986	2.507
(25, 10, 5)	$\hat{\sigma}$	4.886	4.772	4.356	5.209	5.520	5.728	4.522	4.409
	MSE	0.617	0.690	0.842	0.973	0.908	0.937	0.690	0.798
(50, 10, 5)	$\hat{\sigma}$	4.952	4.885	4.404	5.456	5.513	5.743	4.270	4.417
	MSE	0.329	0.302	0.585	0.604	0.698	0.599	0.603	0.957
(75, 10, 5)	$\hat{\sigma}$	5.045	4.829	4.482	5.496	5.553	5.729	4.645	4.510
	MSE	0.237	0.293	0.434	0.450	0.500	0.564	0.375	0.609
(100, 10, 5)	$\hat{\sigma}$	4.923	4.772	4.412	5.431	5.449	5.793	4.589	4.430
	MSE	0.161	0.189	0.458	0.354	0.377	0.386	0.306	0.414
					$CL = 0.50$				
(10, 10, 5)	$\hat{\mu}$	10.093	10.114	10.490	9.245	6.897	6.853	9.706	10.655
	MSE	3.433	2.506	2.339	3.189	10.452	10.724	3.213	3.233
(25, 10, 5)	$\hat{\mu}$	10.228	9.975	10.593	7.873	7.169	7.162	9.560	10.445
	MSE	1.533	0.827	1.230	5.163	8.352	8.392	0.889	0.989
(50, 10, 5)	$\hat{\mu}$	10.020	9.969	10.493	7.200	6.980	6.979	9.620	10.464
	MSE	0.711	0.561	0.677	8.130	9.288	9.288	0.608	0.664
(75, 10, 5)	$\hat{\mu}$	10.054	9.779	10.573	7.078	7.031	7.957	9.483	10.950
	MSE	0.522	0.517	0.547	8.670	8.931	8.652	0.690	0.472
(100, 10, 5)	$\hat{\mu}$	10.029	9.827	10.469	6.996	6.982	6.982	9.465	10.482
	MSE	0.339	0.366	0.466	9.112	9.194	9.196	0.627	0.769
(10, 10, 5)	$\hat{\sigma}$	4.464	4.310	3.741	4.633	7.074	7.115	4.370	4.179
	MSE	2.744	1.864	3.279	2.321	5.166	5.343	2.190	2.320
(25, 10, 5)	$\hat{\sigma}$	4.659	4.599	3.968	6.502	7.200	7.207	4.705	10.534
	MSE	1.280	0.668	1.407	2.879	5.184	5.217	0.998	1.093
(50, 10, 5)	$\hat{\sigma}$	4.934	4.866	4.118	7.082	7.293	7.294	9.520	10.964
	MSE	0.586	0.267	0.933	4.565	5.565	5.436	0.703	0.864
(75, 10, 5)	$\hat{\sigma}$	4.868	4.868	4.116	7.211	7.257	7.946	4.658	4.242
	MSE	0.427	0.179	0.895	5.021	5.215	6.012	0.258	0.683
(100, 10, 5)	$\hat{\sigma}$	4.924	4.932	4.148	7.276	7.290	7.290	5.311	4.261
	MSE	0.273	0.118	0.800	5.275	5.337	5.338	0.216	0.617

TABLE 11. Simulation Estimates of the Mean μ and σ from Normally Distributed Left-Censored Samples with a Single Detection Limit and Censoring Levels $CL = 0.10, 0.90$: $(k, 20, 3)$, $(k = 10, 25, 50, 75, 100)$

$(\mathbf{n}, \mu, \sigma)$		MLE	$W1_j(=W_j)$	$W2_j$	$W3_j$	$W4_j$	$W5_j$	$W6_j$	$W7_j$
					$CL = 0.10$				
(10, 20, 3)	$\hat{\mu}$	20.026	20.008	19.881	19.067	18.488	18.482	19.752	19.375
	MSE	0.895	0.881	0.892	1.319	3.016	3.034	0.913	0.870
(25, 20, 3)	$\hat{\mu}$	19.948	19.937	19.762	18.863	18.151	18.151	19.474	19.683
	MSE	0.363	0.353	0.408	1.735	3.693	3.694	0.538	0.483
(50, 20, 3)	$\hat{\mu}$	19.980	19.984	19.799	18.736	18.496	17.968	19.672	19.754
	MSE	0.176	0.172	0.217	1.794	2.404	2.725	0.238	0.282
(75, 20, 3)	$\hat{\mu}$	19.983	19.990	19.781	18.531	18.405	17.998	19.667	19.873
	MSE	0.124	0.122	0.175	2.277	2.646	2.763	0.0.238	0.195
(100, 20, 3)	$\hat{\mu}$	19.992	19.999	19.783	18.559	18.513	18.092	19.761	19.072
	MSE	0.087	0.087	0.139	2.157	2.286	14.109	0.147	0.185
(10, 20, 3)	$\hat{\sigma}$	2.780	3.026	2.789	4.218	6.579	6.595	3.204	2.772
	MSE	0.554	0.432	0.455	2.324	12.932	13.047	0.543	0.493
(25, 20, 3)	$\hat{\sigma}$	2.967	2.953	3.274	5.311	7.090	7.390	3.367	3.245
	MSE	0.220	0.173	0.292	6.297	16.777	15.638	0.427	0.258
(50, 20, 3)	$\hat{\sigma}$	2.961	2.928	3.285	5.951	6.614	7.025	3.348	2.790
	MSE	0.102	0.081	0.187	9.016	13.083	12.573	0.218	0.276
(75, 20, 3)	$\hat{\sigma}$	2.999	2.960	3.359	6.447	6.789	6.993	3.408	3.209
	MSE	0.067	0.054	0.204	12.022	14.377	12.948	0.241	0.187
(100, 20, 3)	$\hat{\sigma}$	2.983	2.945	3.365	6.492	6.621	6.904	3.405	2.789
	MSE	0.052	0.045	0.194	12.238	13.129	12.839	0.225	0.098
					$CL = 0.90$				
(10, 20, 3)	$\hat{\mu}$	19.398	18.993	17.859	14.982	5.676	4.462	12.836	13.908
	MSE	15.410	16.107	17.703	30.444	282.441	307.606	51.858	47.054
(25, 20, 3)	$\hat{\mu}$	20.830	18.759	17.983	12.467	11.050	13.966	11.658	13.133
	MSE	4.200	6.295	8.054	77.596	83.965	92.837	72.274	57.946
(50, 20, 3)	$\hat{\mu}$	19.862	18.699	15.795	10.277	6.537	6.517	13.125	12.042
	MSE	3.364	3.109	5.895	8.973	12.948	56.666	10.102	41.033
(75, 20, 3)	$\hat{\mu}$	20.217	18.649	16.972	9.683	8.375	7.047	13.196	12.972
	MSE	1.646	2.017	3.896	11.874	13.874	15.266	11.551	16.801
(100, 20, 3)	$\hat{\mu}$	19.976	19.274	14.280	14.168	8.523	7.518	13.138	14.973
	MSE	3.706	4.003	8.604	16.173	21.403	23.619	33.287	49.818
(10, 20, 3)	$\hat{\sigma}$	2.609	3.546	4.013	5.546	7.470	7.387	4.975	4.998
	MSE	5.860	6.027	6.627	7.182	15.271	16.539	5.048	6.192
(25, 20, 3)	$\hat{\sigma}$	2.317	3.402	3.869	5.678	5.678	6.038	4.812	5.091
	MSE	2.027	2.377	3.094	4.289	7.286	11.494	3.750	10.700
(50, 20, 3)	$\hat{\sigma}$	2.936	3.078	4.948	5.826	6.553	6.560	5.329	4.958
	MSE	1.392	1.973	3.275	5.749	9.788	10.849	8.188	7.854
(75, 20, 3)	$\hat{\sigma}$	2.797	3.306	3.972	8.522	7.738	6.803	5.028	6.145
	MSE	2.811	2.913	4.870	10.611	17.492	15.529	10.722	9.321
(100, 20, 3)	$\hat{\sigma}$	2.594	3.514	6.014	7.378	7.562	7.563	6.235	6.663
	MSE	4.628	5.023	6.286	13.307	15.009	14.721	8.517	7.452