Hacettepe Journal of Mathematics and Statistics
Volume 43 (3) (2014), 355 – 363

On groups with relatively small normalizers of nonprimary subgroups

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Abstract

We consider the structure of a finite nonsolvable group G in which for any nonprimary subgroup A the index $|N_G(A) : A{\cdot}C_G(A)|$ is equal unit or a prime number.

Keywords: finite group, subgroup, normalizer, centralizer.

If A is an arbitrary subgroup of a group G, then $N(A) \geq A{\cdot}C(A)$, and the index $|N(A) : A{\cdot}C(A)|$ equals to the order of a subgroup of $Out(A)$, which is induced by elements of G. In this paper we consider the structure of finite groups G in which for any nonprimary subgroup A the index $|N(A) : A{\cdot}C(A)|$ is a divisor of a certain prime number, i.e., it is equal to 1 or a prime number. We'll call these groups NP-groups.

Note that any subgroup and factor-group of a NP-group is also a NP-group. The aim of this article is to describe the structure of nonsolvable NP-groups.

1.1. Lemma. *If a nonsolvable NP-group G is a central product of two subgroups G_1 and G_2, then one of the factors is abelian.*

Proof. Suppose that G_1 is nonabelian. Then ([1], Corollary of Lemma 2) there exists a subgroup A of G_1 such that $|N_{G_1}(A) : A \cdot C_{G_1}(A)|=p$ for a prime p. If A is nonprimary and B is an arbitrary subgroup of G_2, then from the fact that $|N(AB) : AB \cdot C(AB)|$ divides a prime number, it follows that $N_{G_2}(B) = B \cdot C_{G_2}(B)$. Then G_2 is abelian (see [1]). If A is primary and $|A| = q^n$ for a prime q, then the equality $N_{G_2}(B) = B{\cdot}C_{G_2}(B)$ holds for any q'-subgroup B of G_2. By Lemma 4 from [1], $G_2 = Q \leftthreetimes H$, where H is an abelian Hall q'-subgroup of G_2. i.e. G_2 is solvable. If G_2 is nonabelian, then for any q'-subgroup A of G_1, the equality $|N_{G_1}(A) : A{\cdot}C_{G_1}(A)|$ holds too. But then the group G_1 is also solvable, which is impossible. □

1.2. Lemma. *If Q is a Sylow q-subgroup of a NP-group G, $C(Q) \leq Q$ and $N(Q) = (Q \leftthreetimes \langle a\rangle) \leftthreetimes \langle b\rangle$, where $a \neq 1 \neq b$, then a and b are elements of prime orders, and if $N(Q) = Q \leftthreetimes \langle x\rangle$, then $|x|$ is the product of no more than two prime factors.*

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Proof. In the first case, if we let $A = Q \leftthreetimes \langle a \rangle$, we get that $|b| = |N(A) : A{\cdot}C(A)|$ is a prime. And supposing $A = Q \leftthreetimes \langle c \rangle$, where c is an element of prime order r from $\langle a \rangle$, then from the equality $|N(A) : A{\cdot}C(A)| = \frac{|a|}{r}|b|$ we get that $|a| = r$. In second case, it's sufficient to choose a subgroup $A = Q \leftthreetimes \langle y \rangle$, where y is an element of prime order from $\langle x \rangle$. □

Later on we will repeatedly use Frattini's argument ([7], theorem 1.3.7): if $H \lhd G$ and P is a Sylow p-subgroup of H, then $G = H \cdot N(P)$. In a solvable group all Hall π-subgroups are conjugate. Therefore a similar proposition is true in a case where P is a Hall π-subgroup of a solvable group H. We will call this Frattini's argument as well.

1.3. Theorem. *A finite nonabelian simple group G is a NP-group if and only if G satisfies one of the following conditions:*

1) $G \cong PSL(2, q^n)$, $\frac{q^n-1}{(2,q^n-1)}$ *is either a prime or a product of two primes;*

2) $G \cong PSU(3, 2^{2n})$, *and either $n = 2$ or each of the numbers $(2^n - 1)$ and $\frac{2^n+1}{3}$ are primes;*

3) $G \cong Sz(2^n)$, $n \in \{3, 5\}$.

Proof. Necessity. Let G be a finite nonabelian simple NP-group. It is known that any nonabelian simple group is either an alternating group, a Lie type group, or a sporadic simple group.

First, assume that $G \cong A_n$. If $n = 5$, then $G \cong PSL(2, 4)$, and if $n = 6$, then $G \cong PSL(2, 9)$. If, however, $n > 6$ then G contains a subgroup which is isomorphic to A_7. Let $G = A_7$, $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$, $c = (5\ 6\ 7)$, $x = (1\ 2)(5\ 6)$, $y = (1\ 2\ 3)$ and $A = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$. Then $C(A) = A$ and $N(A) = A \leftthreetimes (\langle y \rangle \leftthreetimes \langle x \rangle)$, i.e. $|N(A) : A{\cdot}C(A)| = 6$, which is impossible.

Now let G be a simple Lie type group over the Galois field $GF(q^n)$, where q is a prime. Suppose that the Lie rank l of G is more than 2. If J is a parabolic subgroup of G, corresponding to two nonadjoit nodes of the Dynkin diagram of G, then ([4], Proposition 2.17) $\overline{J} = J/O_q(J) = (\overline{Y}_1 \times \overline{Y}_2) \cdot \overline{H}$, where $\overline{Y}_1$ and $\overline{Y}_2$ are Lie type groups of Lie rank 1 over $GF(q^n)$ and H is a Cartan subgroup of G. By Lemma 1.1 each of $\overline{Y_i}$ is a solvable group. Since ([4], Theorem 2.13) solvable Lie type groups are either $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$ or ${}^2B_2(2)$, so $q^n \in \{2, 3\}$. Let $p_i \in \pi(\overline{Y}_i) \setminus \{q\}$, $\overline{A_1}$ and $\overline{A}_2$ be Sylow p_1- and p_2-subgroup from $\overline{Y}_1$ and $\overline{Y}_2$, respectively, then for the nonprimary subgroup $A = A_1 \cdot A_2$ the index $|N(A) : A{\cdot}C(A)|$ is divisible by q^2, which is impossible.

Therefore $l \leq 2$. Let $l = 2$, i.e., G is isomorphic to one of the groups $A_2(q^n)$, $B_2(q^n)$, ${}^2A_3(q^n)$, ${}^2A_4(q^n)$, ${}^3D_4(q^n)$, ${}^2F_4(2^{2n+1})$, $n > 0$, $({}^2F_4(2))'$.

First suppose that the Cartan subgroup H of the group G is trivial. The group $({}^2F_4(2))'$ contains a subgroup K isomorphic to $PSL(2, 25)$, which is not NP-group, because it has a Cartan subgroup of order 12, which contradicts Lemma 1.2. Because of this, G is a group of classical type over the field $GF(2)$, i.e., either $G \cong A_2(2) = PSL(3, 2)$, or $G \cong B_2(2) = PSp(4, 2)$. It's left to be noticed that $PSL(3, 2) \cong PSL(2, 7)$, and that the group $PSp(4, 2) \cong S_6$ is not simple.

Therefore $H \neq 1$. Let J be a proper parabolic subgroup of G. Then $\overline{J} = J/O_q(J) = \overline{Y} \cdot \overline{H}$, where $\overline{Y}$ is a Lie type group of Lie rank 1. If $G \cong {}^2F_4(2^{2n+1})$, $n > 0$, then subgroup J can be chosen so that $\overline{Y} \cong {}^2B_2(2^{2n+1})$, and if $\overline{A}$ is a subgroup of the order $2^{2n+1}+2^{n+1}+1$ from $\overline{Y}$, then A is nonprimary and $|N_Y(A) : A{\cdot}C_Y(A)| = 4$, which is imposible. If $G \cong {}^2A_4(q^n)$, then $\overline{Y} \cong PSL(2, q^{2n})$, and if $\overline{H}_1$ is a Cartan subgroup of $\overline{Y}$, then the index $|N(H_1) : H_1{\cdot}C(H_1)| = 2 \cdot |\overline{H}/\overline{H_1}|$ is not a prime.

In all the other cases, subgroup $\overline{J}$ may choosen in such a way that $\overline{Y} \cong A_1(q^n)$. If $q^n = 2$ and $\overline{A}$ is a subgroup of order 3 of $\overline{Y}$, then by Frattini's argument we assume that $\overline{H} \leq N(\overline{A})$ which also leads to a contradiction. However, if $q^n \neq 2$, then as $\overline{A}$ we can take a Cartan subgroup of $\overline{Y}$.

Therefore $l = 1$. If Q is a Sylow q-subgroup of G, then $C(Q) \leq Q$ and $N(Q) = Q \rtimes H$, where H is a Cartan subgroup of G. From the definition of an NP-group and the fact that H is abelian, one of the following is true: $|H| = 1$, $|H|$ is a prime number or $|H| = pr$ where p and r are primes. Since group $A_1(2)$ is solvable, then the first case is impossible.

First, suppose that G is a twisted group. Let $G \cong {}^2A_2(q^n) = PSU(3, q^{2n})$. Then $|H| = \frac{q^{2n}-1}{(3,q^n+1)} = (q^n - 1) \cdot \frac{q^n+1}{(3,q^n+1)}$. If $q > 2$ then $|H|$ is divisible by 8, which is impossible. Therefore $q = 2$ and all of the numbers $(2^n - 1)$ and $\frac{2^n+1}{(3,2^n+1)}$ are primes. The primarity of $(2^n - 1)$ implies that either $n = 2$ or n is an odd prime and then $(2^n + 1, 3)$=3, i.e., G is a group of type 2) from this Theorem.

The group ${}^2B_2(2^{2n+1})$ contains, as subgroups, the Frobenius groups of orders $(2^{2n+1} \pm 2^{n+1} + 1) \cdot 4$. Therefore each of the numbers $2^{2n+1} + 2^{n+1} + 1$ and $2^{2n+1} - 2^{n+1} + 1$ must be powers of the primes. Because their product is equal $(2^2)^{2n+1} + 1$ it is divisible by 5. But then either $2^{2n+1} + 2^{n+1} + 1 = 5^m$, or $2^{2n+1} - 2^{n+1} + 1 = 5^m$ for some number m.

Consider the first case. If $2^{2n+1}+2^{n+1}+1 = 5^m$, then either $n = 4t$ or $n = 4t-1$ for some $t > 0$. Since $2^7+2^4+1 = 145 \neq 5^m$, then $n \geq 4$ in any case. Let $m = 2^k r$, where r is an odd number. Then from

$$2^{n+1}(2^n + 1) = 5^m - 1 = 2^{k+2} \cdot \frac{5^r - 1}{4} \cdot \prod_{i=0}^{k-1} \frac{5^{2^i r} + 1}{2}$$

it follows that $k = n - 1 \geq 3$. But the inequality

$$\prod_{i=0}^{k-1} \frac{5^{2^i r} + 1}{2} > 2^{k+1} + 1 = 2^n + 1$$

is true for $k \geq 3$, which is impossible.

If, however, $2^{2n+1} - 2^{n+1} + 1 = 5^m$, then either $n = 4t + 1$ or $n = 4t + 2$ for some $t \geq 0$. The equality

$$2^{n+1}(2^n - 1) = 5^m - 1 = 2^{k+2} \cdot \frac{5^r - 1}{4} \cdot \prod_{i=0}^{k-1} \frac{5^{2^i r} + 1}{2}$$

implies $k = n - 1$. If $k > 1$ then from $k \in \{4t, 4t + 1\}$ it follows that $k \geq 4$ and we have the contradiction again. Therefore, $k \in \{0, 1\}$ and, consequently, $n \in \{1, 2\}$, i.e., G is a group of the type 3) from this Theorem.

Let $G \cong {}^2G_2(3^{2n+1})$. Since the group ${}^2G_2(3)$ is nonsimple, then $n > 0$. In this case (see [8]) G has a subgroup H such that $H = (V_4 \times D) \rtimes \langle b \rangle$, where

$|b| = 3$, $V_4 = \langle a_1 \rangle \times \langle a_2 \rangle$, $|a_i| = 2$, and D is isomorphic to the dihedral group of order $\frac{3^{2n+1}+1}{2}$. If a is an element of order $\frac{3^{2n+1}+1}{4}$ from D, then the subgroup $A = V_4 \times \langle a \rangle$ is nonprimary and $|N_H(A) : A \cdot C_H(A)| = 6$, which is impossible.

Now suppose that G is a classical nontwisted group of Lie type rank 1, i.e., $G \cong A_1(q^n) \cong PSL(2, q^n)$. In this case $|H| = \frac{q^n-1}{(2,q^n-1)}$. Because of this $\frac{q^n-1}{(2,q^n-1)}$ is either be a prime, or a product of two primes, i.e., G is a group of the type 1) from this Theorem.

Now using the survey [10] we can show that G cannot be a sporadic simple group. To demonstrate this, it's sufficient to show that any sporadic simple group contains a subgroup, which is not NP-group. Let G_p denote a Sylow p-subgroup of G for a prime p.

1) In the group M_{11} the subgroup G_3 is self-centralizing and its normaliser has a form $N(G_3) = G_3 \leftthreetimes K$, where K is isomorphic to the semi-dihedral group of order 16, again contrary with Lemma 1.2.

2) M_{12}, M_{23}, M_{24}, Co_3, Suz and McL contain M_{11}.

3) M_{22} and M_{24} contain A_7, F_{22} contains S_{10}, and F_{23} and F'_{24} contain S_{12}.

4) The group $O'N$ contains J_1, and in the group J_1 the subgroup $N(G_3)$ is a direct product of two dihedral groups of orders 6 and 10. If A is a subgroup from $N(G_3)$ of order 15, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

5) In the group J_2 we have $N(G_3) = G_3 \leftthreetimes \langle a \rangle$, where $C(G_3) = G_3$ and $|a| = 8$.

6) In the groups J_3 and He the subgroup $N(G_{17})$ is a Frobenius group of order $17 \cdot 8$; in J_4 and Co_2 the subgroup $N(G_{29})$ is a Frobenius group of order $29 \cdot 28$, again contrary to Lemma 1.2 and Co_1 and F_2 contain Co_2.

7) The group F_1 contain an involution τ such that $C(\tau)/O_2(C(\tau)) \cong Co_2$.

8) In the groups Ly and F_3 the subgroups $N(G_{37})$ and $N(G_{19})$ are Frobenius groups of orders $37 \cdot 18$ and $19 \cdot 18$, respectively.

9) The group F_5 contains HS, and in the group HS the subgroup $N(G_3)$ is isomorphic to $S_3 \times S_5$, and if $A_3 \times A_5 \cong A \leq N(G_3)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

10) The group Ru contains an involution τ such that $C(\tau) \cong V_4 \times Sz(8)$, and if $A \cong V_4 \times H$, where H is a subgroup of order 5 from $Sz(8)$, then $|N(A) : A \cdot C(A)|$ is divisible by 4.

Sufficiency. If A is a proper nonprimary subgroup of G, then $N(A) < G$. Therefore, it is sufficient to prove, that any maximal subgroup of G is a NP-group.

Suppose first that $G \cong PSL(2, q^n)$, where q is a prime. Since $\frac{q^n-1}{(2,q^n-1)}$ is either a prime or a product of two primes, then, it is not difficult to see, that either $n = 1$ or $q \in \{2, 3\}$ and n is either a prime or the square of a prime (odd, if $q = 3$).From Dickson's Theorem ([6], Theorem 2.8.27) it follows that the maximal subgroups of G are the groups from the following list: $N(Q) = Q \leftthreetimes \langle a \rangle$, where Q is a Sylow q-subgroup of G, $|a| = \frac{q^n-1}{(2,q^n-1)}$; the dihedral groups of the orders $2 \cdot \frac{q^n \pm 1}{(2,q^n-1)}$; S_4 for $q^n \equiv \pm 1(8)$, A_4 for $q^n \equiv \pm 3(8)$, A_5 for $q^n \equiv \pm 1(10)$; $PSL(2, q^p)$ for $n = p^2$. It's not difficult to check that all these groups are NP-groups.

If $G \cong PSU(3, 2^{2n})$, then since $(2^n - 1)$ is a prime, n is a prime too. From [5] it follows that the maximal subgroups of G are the groups of the following types: $N(Q) = Q \leftthreetimes \langle a\rangle$, where Q is a Sylow 2-subgroup of G, $|a| = \frac{2^{2n}-1}{(3,2^n+1)}$; $C(b) = \langle b\rangle \times B$, where $|b| = \frac{2^n+1}{(3,2^n+1)}$, $B \cong PSL(2,2^n)$; the Frobenius group $\langle a\rangle \leftthreetimes \langle b\rangle$, $|a| = \frac{2^{2n}-2^n+1}{(3,2^n+1)}$, $|b| = 3$; the Frobenius groups $(\langle a\rangle \times \langle b\rangle) \leftthreetimes C$, $|a| = 2^n+1$, $|b| = \frac{2^n+1}{(3,2^n+1)}$, $C \cong S_3$.

In the groups $Sz(2, 2^{2n+1})$ for a prime n, the maximal subgroups are the groups of the following types (see [9]): $N(Q) = Q \leftthreetimes \langle a\rangle$, Q is a Sylow 2-subgroup, $|a| = 2^n - 1$; the dihedral group of order $2 \cdot (2^n - 1)$; the Frobenius groups $\langle a\rangle \leftthreetimes \langle b\rangle$, $|a| = 2^n \pm 2^{\frac{n+1}{2}} + 1$, $|b| = 4$. □

Below F and F^* denote the Fitting subgroup and the generalized Fitting subgroup of G, respectively.

1.4. Theorem. *Let G be a nonsolvable nonsimple NP-group. Then one of the following holds:*

1) *subgroup $F = F^*$ is a nontrivial p-group for some prime p, and $G/F \cong PSL(2,4)$;*

2) *$G \cong \mathrm{Aut}(PSL(2,2^n))$, $n \in \{2,3\}$;*

3) *$G = Z(G) \cdot L$, $L \cong PSL(2,q^n)$ or $SL(2,q^n)$, the number $\frac{q^n-1}{(2,q^n-1)}$ is a prime, and if $n = 1$ then either $q \not\equiv \pm 1(8)$ or $Z(G)$ is a 2-group;*

4) *$G = Z(G) \times L$ and either $L \cong PSL(2,q^n)$, $\frac{q^n-1}{(2,q^n-1)}$ is a product of the two prime numbers and $Z(G)$ is a q-group, or $Z(G)$ is a 2-group and $L \cong PSU(2,2^{2n})$ is a group from Theorem 1.3;*

5) *$G = Z(G) \cdot L$, $Z(G)$ is a 3-group and L is isomorphic to the covering group for $PSL(2,9)$ with $|Z(L)| = 3$.*

Proof. Let G be a group satisfiy conditions of this Theorem. Let's assume first that $F = F^*$. Then $C(F) \leq F$. If F is a nonprimary group, then $|G : F|$ is a prime and G is a solvable group. Therefore, F is a p-group for some prime p. Moreover, if A/F is a p'-subgroup of G/F, then $|N(A) : A|$ divides a prime number.

Let G_1/F is a minimal normal subgroup of G/F. Then G_1 is a non-nilpotent group, and consequently, is nonprimary. Therefore $|G : G_1|$ is a divisor of a prime. Assume that $G = G_1$. Then G/F is a simple NP-group. i.e., a group from Theorem 1.3.

Let $G/F \cong PSU(3,2^{2n})$. If $p \neq 2$ and A/F is a Sylow 2-subgroup of G/F, then A is nonprimary, and $|N(A) : A \cdot C(A)| = \frac{2^{2n}-1}{(3,2^n+1)}$ is not a prime. Therefore $p = 2$. Then ([4], p.166), for subgroup H/F of order $\frac{2^{2n}-1}{(3,2^n+1)}$ from $N_{G/F}(A/F)$ the equality $C_{G/F}(H/F) = H/F \times L/F$, where $L/F \cong PSL(2,2^n)$, is true. Therefore, for the nonprimary subgroup H, the index $|N(H) : H \cdot C(H)|$ divides by $|L/F|$, which is impossible.

In the case $G/F \cong Sz(8)$, a Sylow 2-subgroup of G/F has the order 2^6. Hence $p = 2$. If A/F is a subgroup of order 5 from G/F, then $|N(A) : A| = 4$, which is impossible. If $G/F \cong Sz(2^5)$, then by analogy $p = 2$ and if A/F is a subgroup of order 25, then $|N(A) : A| = 4$.

Therefore, $G/F \cong PSL(2,q^n)$. If $q \neq p$ and Q/F is a Sylow q-subgroup of G/F, then Q is nonprimary and the primarity of the number $|N_{G/F}(Q/F) : Q/F|$

implies that $\frac{q^n-1}{(2,q^n-1)}$ is a prime. If aF is an element of order q from Q/F then the index $|N(\langle a,F\rangle):\langle a,F\rangle|$ divides a prime number and, therefore, $n \leq 2$.

If $n=2$ then from the primarity of $\frac{q^2-1}{(2,q^2-1)}$ we get that $q=2$, i.e. $G/F \cong PSL(2,4)$. Let $n=1$. Since the groups $PSL(2,2)$ and $PSL(2,3)$ are solvable, and $PSL(2,5) \cong PSL(2,4)$ then we can suppose that $q>5$. Let A/F is a subgroup of the prime order r, where r divides $\frac{q+1}{2}$. If $r \neq p$ then the primarity of $|N(A):A| = 2 \cdot \frac{q+1}{2r}$ implies $r = \frac{q+1}{2}$. But the numbers $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are primes at the same time only when $q=5$. Suppose now that $r=p$. Then by the arbitrariness of r, the equation $\frac{q+1}{2} = p^k$ is solvable. Since $q>5$ then the prime number $\frac{q-1}{2}$ is odd. But then $q+1$ is divisible by 4. i.e. $p=2$. Since one of the numbers, either k or $k+1$, is even, then the numbers $q = 2^{k+1}-1$ and $\frac{q-1}{2} = 2^k - 1$ cannot both be prime at the same time.

Assume now that $q=p$ and aF is an element of prime order from a subgroup of order $\frac{q^n \pm 1}{(2,q^n-1)}$ from G/F. Because $N_{G/F}(\langle aF\rangle)$ is isomorphic to the dihedral group of order $\frac{q^n\pm1}{(2,q^n-1)} \cdot 2$, and $|N(\langle a,F\rangle):\langle a,F\rangle|$ is a prime, then the numbers $\frac{q^n\pm1}{(2,q^n-1)}$ are primes. If q is odd, then $q^n=5$. But $PSL(2,5) \cong PSL(2,4)$. If $q=2$, then because (2^n-1) is a prime it follows that n is a prime. But then in the case $n>2$ the number 2^n+1 is not prime. Therefore, $G/F \cong PSL(2,4)$.

Suppose now that $G_1 < G$. Then, by using what's already been proved, $G_1/F \cong PSL(2,4)$ and $G/F = (G_1/F) \leftthreetimes \langle aF\rangle$, where aF is an automorphism of the group G_1/F. Let A/F be a subgroup of order 5 from G_1/F. By Frattini's argument we can assume that $aF \in N_{G/F}(A/F)$. But then $|N(A):A\cdot C(A)|$ is divisible by 4.

Therefor, if $F=F^*$, then by the theorem conditions, G is of type 1). Because of this, we'll further assume that $F<F^*$. Then $F^* = F\cdot L$, when L is the layer of the group G. By Lemma 1.1, the subgroup F is abelian and F^*/F is a simple group, i.e., a group from Theorem 1.3. Moreover, one of the following holds: $F=1$, $G=F^*$ or $1<F<F^*<G$.

In the first case F^* is a group from Theorem 1.3 and $F^* < G \leq \mathrm{Aut}(F^*)$. From the definition of the NP-group it follows that $|G/F^*|$ is a prime. The structure of the automorphism groups of Lie type groups (e.g. [4], theorem 4.238) implies that in our case $G = F^* \leftthreetimes \langle a\rangle$, a is a prime order automorphism of group F^*. Set $|a|=p$.

First assume that $F^* \cong PSL(2,q^n)$. Let Q be a Sylow q-subgroup of $F^*$and $B = Q \leftthreetimes H$ be a Borel subgroup of group F^*. By Frattini's argument we can assume that $a \in N(Q)$. But then $a \in N(N_{F^*}(Q)) = N(B)$. Since $C(Q) \leq Q$ and $|N(Q):Q| = |H|\cdot p$, then, by Lemma 1.2, the number $|H| = \frac{q^n-1}{(2,q^n-1)}$ must be a prime number. But then, as it was noted in the proof of Theorem 1.3, either $q \in \{2,3\}$, or $n=1$. By analogy, for a subgroup A of order $\frac{q^n+1}{(2,q^n-1)}$ from F^* the equality $|N(A):A\cdot C(A)| = 2p$ implies that subgroup A must be a primary group.

Let $q=2$. The primarity of the number (2^n-1) implies that n is a prime. If $n>2$, then 2^n+1 is divisible by 3 and, consequently, $2^n+1=3^k$ for a number k. Let $k>2$. If $k=2r$ is even, then $2^n = 3^k-1 = (3^r-1)(3^r+1)$, which is impossible. However, if $k=2r+1$, then $3^k-1 = 2(1+3+3^2+\cdots+3^{2r}) \neq 2^n$

where the second factor is odd. Therefor, if $q = 2$, then the group F^* is isomorphic to one of the groups $PSL(2, 4)$ or $PSL(2, 8)$.

If $q = 3$ then the primarity of the number $\frac{3^n-1}{2}$ implies that n is an odd prime. However, from that fact that $\frac{3^n+1}{2}$ is even and prime it follows that $\frac{3^n+1}{2} = 2^k$, i.e., $3^n = 2^{k+1} - 1$ for a number k. Since the number $\frac{3^n-1}{2} = 2^k - 1$ is prime, then k is an odd prime. But then $k + 1 = 2r$ and $3^n = (2^r - 1)(2^r + 1)$, which is impossible for $r > 1$. However if $r = 1$, then $k = 1$. But then $n = 1$ as well, which contradicts the primarity of the group F^*.

Finally, let q and $\frac{q-1}{2}$ be primes. If $q = 5$, then $F^* \cong PSL(2, 4)$. However if $q > 5$, then $\frac{q-1}{2}$ is odd. Because $\frac{q+1}{2}$ is primary, we obtain that $\frac{q+1}{2} = 2^k$, i.e. $q = 2^{k+1} - 1$. But then $\frac{q-1}{2} = 2^k - 1$. Since one of the numbers k, $k + 1$ is even, and $k > 2$, then the numbers $(2^k - 1)$ and $(2^{k+1} - 1)$ can't both be prime simultaneously.

Suppose now that $F^* \cong PSU(3, 2^{2n})$. If $p \neq 2$ and A is a Sylow 2-subgroup of F^*, then $|N(A) : A \cdot C(A)| = p \cdot (2^n - 1) \cdot \frac{2^n+1}{(3,2^n+1)}$, which is impossible. However, if $p = 2$ and H is a Cartan subgroup of F^*, then H is nonprimary and $|N(H) : H \cdot C(H)| = 4$.

If $F^* \cong Sz(2^3)$ or $Sz(2^5)$ and A is a subgroup of order 5 or 25 of F^*, respectively, then $|N(A) : A| = 4p$, which contradicts Lemma 1.2.

Therefore, if $F = 1$, then G is of a type 2) from this Theorem.

Consider the case when $G = F^*$, i.e., $G = F \cdot L$, where L is the layer of the group G. By Lemma 1.1, the subgroup F is abelian, i.e., $F = Z(G)$, and L is a quasi simple group. Since the group G isn't simple, then $F \neq 1$. If F is nonprimary, then the index $|N_L(A) : A \cdot C_L(A)|$ divides a prime for any subgroup $A \leq L$. By theorem 4 from [2] $L \cong PSL(2, q^n)$ or $SL(2, q^n)$, the number $\frac{q^n-1}{(2,q^n-1)}$ is a prime and if $n = 1$, then $q \not\equiv \pm 1(8)$, i.e., G is of type 3) from this Theorem.

Now suppose that F is a p-group for a prime p. Since the Schur multiplier of group $Sz(2^5)$ is trivial then either L is a group from Theorem 1.3 or L is isomorphic to a covering of group $PSL(2, q^n)$, $Sz(8)$ or $PSU(3, 2^{2n})$.

Let $L/Z(L) \cong Sz(8)$. Then $L/Z(L)$ contains the subgroups $A_1/Z(L)$ and $A_2/Z(L)$ of order 5 and 13, respectively, such that $|N_L(A_i) : A_i \cdot C(A_i)| = 4$. Since p isn't at least one of the numbers 5 or 13, then supposing $A = F \cdot A_i$, we get a contradiction with the definition of NP-group. If $L \cong Sz(2^5)$ then subgroups of order 25 and 41 should be taken as subgroups A_1 and A_2 in the group G.

Therefore, we can assume that $L/Z(L) \cong PSL(2, q^n)$ or $PSU(3, 2^{2n})$.

First, assume that $Z(L) = 1$, i.e., $G = Z(G) \times L$. If $L \cong PSL(2, q^n)$ and $p \neq q$, then the number $\frac{q^n-1}{(2,q^n-1)}$ should be prime. Moreover, if $n = 1$ and $q \equiv \pm 1(8)$, then $L/Z(L)$ contains a subgroup $H/Z(L) \cong S_4$. If $V/Z(L)$ is a four-group from $H/Z(L)$, then the equality $|N_{H/Z(L)}(V/Z(L)) : V/Z(L)| = 6$ implies that in this case subgroup V is primary, i.e., $p = 2$. However, if $p = q$, then the number $\frac{q^n-1}{(2,q^n-1)}$ could be the product of two primes. But, if $q^n \equiv \pm 1(8)$, then when checking a four-group $V/Z(L)$ again, we get that $p = 2$. But then $q^n = 2^n \not\equiv \pm 1(8)$. If, however $L/Z(L) \cong PSU(3, 2^{2n})$ and $p \neq 2$, then for a Sylow 2-subgroup A of L, the subgroup $A \cdot Z(L))$ is nonprimary and again we get a contradiction with the definition of NP-group.

Now suppose that $Z(L) \neq 1$. Since the Schur multiplier is trivial for groups $PSL(2, 2^n)$ when $n > 2$, we can assume that in the case of $L/Z(L) \cong PSL(2, q^n)$ the number q is odd. Then the order of the Schur multiplier is equal to 2 (i.e. $L \cong SL(2, q^n)$) for $q^n \neq 9$ and 6 for $q^n = 9$. Consider the second case. If $|Z(L)|$ is divisible by 2 and $Q/Z(L)$ is a Sylow 3-subgroup of the group $L/Z(L)$, then the subgroup Q is nonprimary and $|N(Q) : Q \cdot C(Q)| = 4$, which is impossible. Hence, when $q^n = 9$ the order of $Z(L)$ is equal to 3. In the case of $L/Z(L) \cong PSU(3, 2^{2n})$ the Schur multiplier order is equal to 3, and if $A/Z(L)$ is a Sylow 2-subgroup of $L/Z(L)$, then subgroup A is nonprimary and $|N_L(A) : A \cdot C_L(A)|$ is not a prime.

Therefore, if $G = F^*$ then G is a group of type 3) or 5) from this Theorem. Finally, consider the case when $1 < F < F^* < G$. Then, by using what's already been proved, F^* is a group of type 3) or 4), while G/F is a group of type 2) from this Theorem. Let $G = F^* \cdot \langle a \rangle$, $a^p \in F^*$. If A/F is a Sylow q-subgroup from F^*/F, then the fact that $|N(A) : A \cdot C(A)|$ is divisible by $p \cdot |H/F|$, where H/F is a Cartan subgroup of group F^*/F, implies that subgroup F is a q-group for a prime q. But then, for the nonprimary subgroup H, the index $|N(H) : H \cdot C(H)|$ is divisible by $2p$, which is impossible. □

1.5. Note. It isn't difficult to see that the groups type 2) and 5) of Theorem 1.4 are NP-groups. For type 1) groups, the proof of the sufficiency requires the fulfillment of a number of additional restrictions. Let's note some of them.

Let t be a p'-element from G, A be a t-invariant subgroup from F and $H = F \leftthreetimes \langle t \rangle$. Then the index $|N_H(A \leftthreetimes \langle t \rangle) : (A \leftthreetimes \langle t \rangle) \cdot C_H(A \leftthreetimes \langle t \rangle)|$ divides p. Looking at the intersections of these subgroups with F and taking into account that $N_F(A \leftthreetimes \langle t \rangle) = A \cdot (N_F(A) \cap C(t))$, we get that

$$|A \cdot (N_F(A) \cap C(t)) : A \cdot (C_H(A) \cap C(t))| = |N_F(A) \cap C(t) : (C_H(A) \cap C(t)) \cdot (A \cap C(t))|,$$

i.e., $|C_{N_F(A)}(t) : C_A(t) \cdot C_{C_F(A)}(t)|$ divides p.

Let $N_{G/F}(\langle tF \rangle) = \langle tF \rangle \leftthreetimes \langle hF \rangle$ and A be a $\langle t, h \rangle$-invariant subgroup from F. Since $h \in N(A \leftthreetimes \langle t \rangle)$, then in the same notation $N_H(A \leftthreetimes \langle t \rangle) = (A \leftthreetimes \langle t \rangle) \cdot C_H(t)$. But then $C_{N_F(A)}(t) = C_A(t) \cdot C_{C_F(A)}(t)$. Since the subgroup $N_F(A)$ is also $\langle t, h \rangle$-invariant, then

$$C_{N_F(N_F(A))}(t) = (N_F(A) \cap C(t)) \cdot C_{C_F(N_F(A))}(t) = C_A(t) \cdot C_{C_F(A)}(t).$$

Continuing this process and taking into account that F satisfies the normaliser conditions, we get the equality $C_F(t) = C_A(t) \cdot C_{C_F(A)}(t)$.

Supposing that in this equation $A = [F, a]$ and taking into account that $F = [F, a] \cdot C_F(a)$, we get that $C_F(a) = C_{[F,a]}(a) \cdot C_{C_F([F,a])}(a)$, i.e., $F = [F, a] \cdot C_F([F, a])$.

By analogy we can prove, that if $p \neq 2$ and $(\langle aF \rangle \times \langle bF \rangle) \leftthreetimes \langle cF \rangle$ is a subgroup of order 12 from G/F and subgroup $A \leq F$ is $\langle a, b, c \rangle$-invariant, then $C_F(\langle a, b \rangle) = C_A(\langle a, b \rangle) \cdot C_{C_F(A)}(\langle a, b \rangle)$ and $F = [F, \langle a, b \rangle] \cdot C_F([F, \langle a, b \rangle])$.

Note that all these properties hold if subgroup F is abelian, i.e., in this case G is a NP-group.

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