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On convergence of an implicit iterative algorithm for non self asymptotically non expansive mappings

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Abstract

In this paper, we study an implicit iterative algorithm for two finite families of nonself asymptotically nonexpansive mappings. We prove some weak and strong convergence theorems for this iterative algorithm. Our results improve and extend the corresponding results of Soltuz [4], Xu and Ori [5], Khan et al. [6].

Keywords: Implicit iteration process, Nonself asymptotically nonexpansive mappings, Common fixed point, Strong convergence, Weak convergence.

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1. Introduction

Let E be a real normed linear space and K be a nonempty subset of E. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$. A mapping $T : K \to K$ is called uniformly L-Lipschitzian if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in K$ and $n \geq 1$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. They proved that if K is a nonempty closed bounded subset of a real uniformly convex

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Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Implicit iterative algorithms for asymptotically nonexpansive self-mapping in Banach spaces have been studied extensively by various authors; see for example [12, 13, 14, 15]. However, if the domain of T , $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then these iterative algorithms may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that, if a map P is a retraction, then $Py = y$ for all y in the range of P.

In 2003, Chidume, Ofoedu and Zegeye [2] further generalized the concept of asymptotically nonexpansive self-mapping, and proposed the concept of nonself asymptotically nonexpansive mapping, which is defined as follows:

1.1. Definition. [2] Let K be a nonempty subset of a real normed space E and $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ *is called asymptotically nonexpansive if there exists a sequence* $\{k_n\} \subset [1, \infty)$ *with* $\lim_{n\to\infty} k_n = 1$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\| \tag{1.2}$$

for all $x, y \in K$ *and* $n \geq 1$.

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ *such that*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

for all $x, y \in K$ *and* $n \geq 1$.

Note that if P is an identity mapping, then the above definitions reduce to those of a self-mapping T . By using the following iterative algorithm:

$$x_{n+1} = P((1 - \alpha_n)\, x_n + \alpha_n T\, (PT)^{n-1}\, x_n), \;\; x_1 \in K, \;\; n \geq 1,$$

Chidume et al. [2] gave some strong and weak convergence theorems for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces. They also established a demiclosedness principle. Convergence problems of an iterative algorithm to a common fixed point for nonself asymptotically nonexpansive mappings in uniformly convex Banach spaces have been considered by several authors (see, for example, [16, 17, 18, 19, 20, 21, 22, 23] and the references therein).

1.2. Remark. [3] If $T : K \to E$ is asymptotically nonexpansive and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \geq 1$, we have

$$\begin{aligned} \|(PT)^n x - (PT)^n y\| &= \left\|PT(PT)^{n-1}x - PT(PT)^{n-1}y\right\| \\ &\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\ &\leq k_n \|x - y\| . \end{aligned}$$

However, the converse may not be true.

Keeping in view the above fact, Zhou et al. [3] introduced the following generalized definition in 2007.

1.3. Definition. [3] Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive with respect to P if there exists sequences $\{k_n\} \in [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$(1.3) \quad \|(PT)^n x - (PT)^n y\| \le k_n \|x - y\|, \quad \forall x, y \in K, \ n \ge 1.$$

T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$(1.4) \quad \|(PT)^n x - (PT)^n y\| \le L \|x - y\|, \quad \forall x, y \in K, \ n \ge 1.$$

Throughout this paper, $J = \{1, 2, ..., N\}$ denotes the set of first N natural numbers. In what follows we fix $x_0 \in K$ as a starting point of an algorithm unless stated otherwise, and take $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ sequences in $(0, 1)$.

In 2001, Soltuz [4] introduced the Mann-type implicit algorithm for a nonexpansive mapping as follows:

$$(1.5) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T x_n, \ n \ge 1.$$

Xu and Ori [5] introduced the following implicit iterative algorithm for a finite family of nonexpansive mappings $\{T_i : i \in J\}$.

$$(1.6) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \ n \ge 1,$$

where $T_n = T_{n(modN)}$ and the $modN$ function takes values in J.

More recently, Khan, Yildirim and Ozdemir [6] introduced an implicit iterative algorithm for two finite families $\{S_i : i \in J\}$ and $\{T_i : i \in J\}$ of nonexpansive mappings as follows:

$$(1.7) \quad x_n = \alpha_n x_{n-1} + \beta_n S_n x_n + \gamma_n T_n x_n$$

where $T_n = T_{n(modN)}$, $S_n = S_{n(modN)}$ and $\alpha_n + \beta_n + \gamma_n = 1$.

In this paper, we modify the implicit iterative algorithm of Khan, Yildirim and Ozdemir [6] for two finite families of nonself asymptotically nonexpansive mappings as follows:

Let E be a real Banach space, and K be a nonempty closed convex subset of E which is also a nonexpansive retract of E with a retraction P. Let $S_i, T_i : K \to E$ $(i = 1, 2, ..., N)$ be two families of nonself asymptotictally nonexpansive mappings with respect to P. For arbitrarily chosen $x_0 \in K$,

$$(1.8) \quad x_n = \alpha_n x_{n-1} + \beta_n (PS_n)^n x_n + \gamma_n (PT_n)^n x_n$$

where $T_n = T_{n(modN)}$ and $S_n = S_{n(modN)}$.

In other words, if $n = (k-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $k = k(n) \ge 1$ is a positive integer and $k(n) \to \infty$, as $n \to \infty$, then we set $S_n = S_i$, $T_n = T_i$ and (1.8) can be expressed in the following form:

$$(1.9) \quad x_n = \alpha_n x_{n-1} + \beta_n \left(PS_{i(n)}\right)^{k(n)} x_n + \gamma_n \left(PT_{i(n)}\right)^{k(n)} x_n, \ n \ge 1.$$

If we take $T_i = S_i$ for all $i \in J$, then (1.8) reduces to modified Mann-type implicit iteration as follows:

$$(1.10) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) (PT_n)^n x_n, \ n \ge 1.$$

In addition, if we take $T_i = S_i = T$ for all $i \in J$ and T is a nonexpansive self-mapping, then (1.8) reduces to (1.5). Note that (1.8) reduces to (1.6) when $S_i =$

T_i for all $i \in J$ are nonexpansive self-mappings. Also (1.8) reduces to (1.7) when S_i and T_i are two nonexpansive self-mappings for all $i \in J$.

The purpose of this paper is to study the weak and strong convergence of the implicit iterative algorithm (1.8) for approximating common fixed points of the two finite families $\{S_i : i \in J\}$ and $\{T_i : i \in J\}$ of nonself asymptotically nonexpansive mappings with respect to P in Banach spaces. The results presented in this paper extend and improve the corresponding results of Soltuz [4], Xu and Ori [5], Khan et al. [6].

2. Preliminaries

In this section, we review definitions and lemmas used for the rest of the paper as follow:

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of E is the function $\delta_E(\varepsilon) : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = 1, \ \ \|y\| = 1, \ \varepsilon = \|x - y\| \right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. The space E said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$.

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the inward set of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \to E$ is called weakly inward if $Tx \in cl[I_K(x)]$ for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $K \subset E$ be a closed convex and P a mapping of E onto K. Then P is said to be sunny [7] if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p, if for each sequence $\{x_n\}$ in $D(T)$, the conditions $x_n \to x_0$ weakly and $Tx_n \to p$ strongly imply $Tx_0 = p$.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{Tx_{n_j}\}$ converges to some element of $R(T)$.

A mapping $T : K \to K$ is said to demi-compact if any sequence $\{x_n\}$ in K satisfying $x_n - Tx_n \to 0$ as $n \to \infty$ has a convergent subsequence.

Two mappings $T, S : K \to K$ are said to satisfy condition (A') [8] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all

$t \in (0, \infty)$ such that

$$\frac{1}{2}\left(\|x - T_1 x\| + \|x - T_2 x\|\right) \geq f\left(d\left(x, F\right)\right)$$

for all $x \in K$, where $d\left(x, F\right) = \inf\left\{\|x - p\| : p \in F := F(T) \cap F(S)\right\}$.

We modify this definition for two finite families of nonself asymptotically non-expansive mappings as follows: Let E be a Banach space, K a nonempty closed convex subset of E with nonempty fixed point set F and P a sunny nonexpansive retraction. Let $\{T_i : i \in J\}$ and $\{S_i : i \in J\}$ be two finite families of nonself asymptotically nonexpansive mappings of K with respect to P . These families are said to satisfy condition (B') if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f\left(0\right) = 0$, $f\left(t\right) > 0$ for all $t \in (0, \infty)$ such that

$$\max_{i \in J}\left\{\frac{1}{2}\left(\|x - PT_i x\| + \|x - PS_i x\|\right)\right\} \geq f\left(d(x, F)\right)$$

for all $x \in K$ where $F := \left(\cap_{i=1}^{N} F\left(S_i\right)\right) \cap \left(\cap_{i=1}^{N} F\left(T_i\right)\right)$.

In what follows, we shall make use of the following lemmas.

2.1. Lemma. [9] *If $\{a_n\}, \{b_n\}$ are two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq \left(1 + b_n\right) a_n, \;\; n \geq 1$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n \to \infty} a_n$ exists.

2.2. Lemma. [10] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \to \infty} \|x_n\| \leq d, \;\; \limsup_{n \to \infty} \|y_n\| \leq d \text{ and } \lim_{n \to \infty} \|\left(1 - t_n\right) x_n + t_n y_n\| = d$$

hold for some $d \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

2.3. Lemma. [11] *Let E be real smooth Banach space, let K be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

2.4. Lemma. [3] *Let E be a real smooth and uniformly convex Banach space, K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with the sequence $k_n \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3. Main results

Let E be a Banach space and K a nonempty closed convex subset of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $\lim_{n \to \infty} k_n = \lim_{n \to \infty} l_n = 1$. Let $\{x_n\}$ be defined by (1.8). Then $x_1 = \alpha_1 x_0 + \beta_1 \left(PS_1\right)^n x_1 + \gamma_1 \left(PT_1\right)^n x_1$. Define a mapping $W_1 : K \to K$ by $W_1 =$

$\alpha_1 x_0 + \beta_1 (PS_1)^n x + \gamma_1 (PT_1)^n x$ for all $x \in K$ where $\alpha_n + \beta_n + \gamma_n = 1$. Existence of x_1 is guaranteed if W_1 has a fixed point. Thus for any $x, y \in K$, we have

$$\begin{aligned}
\|W_1 x - W_1 y\| &= \|\alpha_1 x_0 + \beta_1 (PS_1)^n x + \gamma_1 (PT_1)^n x \\
&\quad -\alpha_1 x_0 - \beta_1 (PS_1)^n y - \gamma_1 (PT_1)^n y\| \\
&\leq \beta_1 \|(PS_1)^n x - (PS_1)^n y\| + \gamma_1 \|(PT_1)^n x - (PT_1)^n y\| \\
&\leq \beta_1 k_1 \|x - y\| + \gamma_1 l_1 \|x - y\| \\
&\leq (\beta_1 k_1 + \gamma_1 l_1) \|x - y\|
\end{aligned}$$

If $\beta_1 k_1 + \gamma_1 l_1 < 1$, then W_1 is a contraction. By Banach Contraction Principle, W_1 has a unique fixed point. Thus the existence of x_1 is established. Similarly, the existence of $x_2, x_{3,}...$ is established. Thus the implicit iterative algorithm (1.8) is well defined when $\beta_n k_n + \gamma_n l_n < 1$. Therefore, the implicit iterative algorithm (1.8) can be employed for the approximation of common fixed points of two families of nonself asymptotically nonexpansive mappings $\{S_i : j \in J\}$ and $\{T_i : i \in J\}$.

From now on, we denote the set of common fixed points of the two families $S_i, T_i : K \to E$ $(i \in J)$ by $F := \left(\cap_{i=1}^N F(S_i)\right) \cap \left(\cap_{i=1}^N F(T_i)\right)$.

3.1. Convergence Theorems in Real Banach Spaces. In this section, we prove the strong convergence of the iterative algorithm (1.8) to a common fixed point of two families of nonself asymptotically nonexpansive mappings with respect to P in real Banach spaces.

First, we prove the following lemma.

3.1. Lemma. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$. Suppose that $\{x_n\}$ defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0, 1)$.

If $F \neq \varnothing$, then

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each $p \in F$.*
(2) *there exists a constant $M > 0$ such that $\|x_{n+m} - p\| \leq M \|x_n - p\|$ for all $m, n \geq 1$ and $p \in F$.*

Proof. (1) Let $p \in F$. Set $k_n = 1 + u_n$, $l_n = 1 + v_n$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Using (1.8), we have

$$\begin{aligned}
\|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|(PS_n)^n x_n - p\| + \gamma_n \|(PT_n)^n x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n k_n \|x_n - p\| + \gamma_n l_n \|x_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (\beta_n k_n + \gamma_n l_n) \|x_n - p\| \qquad (3.1)
\end{aligned}$$

which leads to

$$[1 - \beta_n k_n - \gamma_n l_n] \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\|.$$

Since $\beta_n k_n + \gamma_n l_n < 1$, then $1 - \beta_n k_n - \gamma_n l_n > 0$, that is,

$$1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n) > 0$$

for all $n \geq 1$. Thus, from (3.1) we have

$$\begin{aligned}
\|x_n - p\| &\leq \frac{\alpha_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)} \|x_{n-1} - p\| \\
&= \frac{1 - \beta_n - \gamma_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)} \|x_{n-1} - p\| \\
&= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{1 - \beta_n (1 + u_n) - \gamma_n (1 + v_n)}\right] \|x_{n-1} - p\| \\
&= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{1 - \beta_n - \gamma_n - \beta_n u_n - \gamma_n v_n}\right] \|x_{n-1} - p\| \\
&= \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n}\right] \|x_{n-1} - p\| . \qquad (3.2)
\end{aligned}$$

Since $s \leq \beta_n, \gamma_n \leq 1 - s$ for some $s \in (0, 1)$, $\lim_{n\to\infty} \beta_n u_n = \lim_{n\to\infty} \gamma_n v_n = 0$. Thus for any given $\frac{\epsilon_o}{2}, \frac{\epsilon_1}{2} \in (0, \frac{s}{2})$, $\epsilon = \max\{\epsilon_0, \epsilon_1\}$, there exists positive integer n_0 such that

$$s - \epsilon < \alpha_n - \beta_n u_n - \gamma_n v_n, \qquad (3.3)$$

as $n \geq n_0$. By (3.2) and (3.3), we get

$$\|x_n - p\| \leq \left[1 + \frac{u_n}{s - \epsilon} + \frac{v_n}{s - \epsilon}\right] \|x_{n-1} - p\| . \qquad (3.4)$$

Since $\sum\limits_{n=1}^{\infty} u_n < \infty$ and $\sum\limits_{n=1}^{\infty} v_n < \infty$, we obtain $\sum\limits_{n=1}^{\infty} \left(\frac{u_n}{s-\epsilon} + \frac{v_n}{s-\epsilon}\right) < \infty$. It now follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$.

(2) As $1 + t \leq e^t$ for all $t > 0$, from (3.4), we obtain

$$\begin{aligned}
\|x_{n+m} - p\| &\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right)} \|x_{n+m-1} - p\| \\
&\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right)} \left[e^{\left(\frac{u_{n+m-1}}{s-\epsilon} + \frac{v_{n+m-1}}{s-\epsilon}\right)} \|x_{n+m-2} - p\|\right] \\
&\leq e^{\left(\frac{u_{n+m}}{s-\epsilon} + \frac{v_{n+m}}{s-\epsilon}\right) + \left(\frac{u_{n+m-1}}{s-\epsilon} + \frac{v_{n+m-1}}{s-\epsilon}\right)} \|x_{n+m-2} - p\| \\
&\quad \vdots \qquad (3.5) \\
&\leq e^{\sum_{j=n+1}^{n+m} \left(\frac{u_j}{s-\epsilon} + \frac{v_j}{s-\epsilon}\right)} \|x_n - p\| \\
&\leq M \|x_n - p\|
\end{aligned}$$

where $M = \sum_{j=n+1}^{n+m} \left(\frac{u_j}{s-\epsilon} + \frac{v_j}{s-\epsilon}\right)$. That is $\|x_{n+m} - p\| \leq M \ \|x_n - p\|$ for all $m, n \geq 1$ and $p \in F$. □

3.2. Theorem. *Let E be a real Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $S_i, T_i : K \to E$ $(i \in J)$ be two families of nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\} \subset [1, \infty)$ such that $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$,*

$\sum\limits_{n=1}^{\infty}(l_n-1)<\infty$. *Suppose that* $F\neq\varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1$, $\beta_n k_n+\gamma_n l_n<1$ *for each integer* $n\geq 1$.
(ii) $s\leq\alpha_n,\beta_n,\gamma_n\leq 1-s$ *for some* $s\in(0,1)$.

Then $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i\in J)$ *if and only if* $\liminf_{n\to\infty}d(x_n,F)=0$.

Proof. The necessity of the conditions is obvious. Let us prove the sufficiency of the theorem.

Taking infimum over all $p\in F$ in (3.4), we get

$$d\left(x_n,F\right)\leq\left[1+\frac{u_n}{s-\epsilon}+\frac{v_n}{s-\epsilon}\right]d\left(x_{n-1},F\right).$$

Thus, we obtain from Lemma 2.1 that $\lim_{n\to\infty}d\left(x_n,F\right)$ exists. But by hypothesis $\liminf_{n\to\infty}d(x_n,F)=0$, therefore we must have $\lim_{n\to\infty}d(x_n,F)=0$.

Next, we first show that $\{x_n\}$ is a Cauchy sequence in E. Since $\lim_{n\to\infty}d(x_n,F)=0$, given $\epsilon>0$, there exists a constant n_0 such that for all $n\geq n_0$, we have

$$d(x_n,F)<\frac{\epsilon}{1+M} \tag{3.6}$$

where $M>0$ is the constant in Lemma 3.1(2). So we can find $p\in F$ such that

$$\|x_n-p\|<\frac{\epsilon}{1+M}. \tag{3.7}$$

From Lemma 3.1(2) we get for all $n\geq n_0$ and $m\geq 1$ that

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq \|x_{n+m}-p\|+\|x_n-p\|\\ &\leq M\|x_n-p\|+\|x_n-p\|\\ &= (1+M)\|x_n-p\|\\ &< \epsilon.\end{aligned} \tag{3.8}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and so it must converge to a point q in K. Now, $\lim_{n\to\infty}d(x_n,F)=0$ gives that $d(q,F)=0$. Since F is closed, so we have $q\in F$. □

3.2. Convergence Theorems in Real Uniformly Convex Banach Spaces. In this section, we prove some strong and weak convergence of algorithm (1.8) to a common fixed point of two families of nonself asymptotically nonexpansive mappings with respect to P in real uniformly convex and smooth Banach space.

3.3. Lemma. *Let* E *be a real uniformly convex Banach space and* K *a nonempty closed convex subset of* E *which is also a nonexpansive retract of* E. *Let* $S_i,T_i:K\to E$ $(i\in J)$ *be two families of nonself asymptotically nonexpansive mappings with respect to* P *with common sequences* $\{k_n\},\{l_n\}\subset[1,\infty)$ *such that* $\sum\limits_{n=1}^{\infty}(k_n-1)<\infty$, $\sum\limits_{n=1}^{\infty}(l_n-1)<\infty$. *Suppose that* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1$, $\beta_n k_n+\gamma_n l_n<1$ *for each integer* $n\geq 1$.

(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0,1)$.

If $F \neq \varnothing$, *then* $\lim_{n\to\infty} \|x_n - (PT_i)\, x_n\| = \lim_{n\to\infty} \|x_n - (PS_i)\, x_n\| = 0$ $(i \in J)$.

Proof. From Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. We suppose that $\lim\limits_{n\to\infty} \|x_n - p\| = d$. Then

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n (x_{n-1} - p) + \beta_n ((PS_n)^n x_n - p) \\
&\quad + \gamma_n ((PT_n)^n x_n - p)\| \\
&= \lim_{n\to\infty} \left\| (1-\gamma_n) \left[\frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) \right] \right. \\
&\quad \left. + \gamma_n ((PT_n)^n x_n - p) \right\| \\
&= d \qquad (3.9)
\end{aligned}$$

Since T_i $(i \in J)$ are asymptotically nonexpansive mappings and $F \neq \varnothing$, we have $\|(PT_n)^n x_n - p\| \leq l_n \|x_n - p\|$ for each $p \in F$. Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \|(PT_n)^n x_n - p\| \leq \limsup_{n\to\infty} l_n \|x_n - p\| = d. \qquad (3.10)$$

Using (3.2) and $\sum\limits_{n=1}^{\infty} u_n < \infty$, $\sum\limits_{n=1}^{\infty} v_n < \infty$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| \frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) \right\| \\
\leq\ &\limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\gamma_n} \|x_{n-1} - p\| + \frac{\beta_n}{1-\gamma_n} k_n \|x_n - p\| \right] \\
\leq\ &\limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n}{1-\gamma_n} k_n \left[1 + \frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right] \|x_{n-1} - p\| \right] \\
\leq\ &\limsup_{n\to\infty} \left[\frac{k_n (\alpha_n + \beta_n)}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n k_n}{1-\gamma_n} \left(\frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right) \|x_{n-1} - p\| \right] \\
\leq\ &\limsup_{n\to\infty} \left[\frac{k_n (1-\gamma_n)}{1-\gamma_n} \|x_{n-1} - p\| \right. \\
&\left. + \frac{\beta_n k_n}{1-\gamma_n} \left(\frac{\beta_n u_n + \gamma_n v_n}{\alpha_n - \beta_n u_n - \gamma_n v_n} \right) \|x_{n-1} - p\| \right] \\
=\ &d. \qquad (3.11)
\end{aligned}$$

Now considering (3.9), (3.10) and (3.11) and applying Lemma 2.2, we obtain

$$\lim_{n\to\infty} \left\| \frac{\alpha_n}{1-\gamma_n} (x_{n-1} - p) + \frac{\beta_n}{1-\gamma_n} ((PS_n)^n x_n - p) - ((PT_n)^n x_n - p) \right\| = 0$$

which means that

$$\begin{aligned}
& \lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1}+\frac{\beta_n}{1-\gamma_n}\left(PS_n\right)^n x_n-\left(PT_n\right)^n x_n\right\| \\
= & \lim_{n\to\infty}\left(\frac{1}{1-\gamma_n}\right)\left\|\alpha_n x_{n-1}+\beta_n\left(PS_n\right)^n x_n-\left(1-\gamma_n\right)\left(PT_n\right)^n x_n\right\|=0.
\end{aligned}$$

Since $s\leq\gamma_n\leq 1-s$, we have $1/\left(1-s\right)\leq 1/\left(1-\gamma_n\right)\leq 1/s$. Therefore,

$$\lim_{n\to\infty}\|x_n-(PT_n)^n x_n\|=0. \tag{3.12}$$

In the same manner, we can prove that

$$\lim_{n\to\infty}\|x_n-(PS_n)^n x_n\|=0. \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
\|x_n-x_{n-1}\| &= \left\|\alpha_n x_{n-1}+\beta_n\left(PS_n\right)^n x_n+\gamma_n\left(PT_n\right)^n x_n-x_{n-1}\right\| \\
&= \left\|\left(\beta_n+\gamma_n\right)\left(x_n-x_{n-1}\right)-\beta_n\left(x_n-\left(PS_n\right)^n x_n\right)\right. \\
&\quad \left.-\gamma_n\left(x_n-\left(PT_n\right)^n x_n\right)\right\| \\
&\leq \left(\beta_n+\gamma_n\right)\|x_n-x_{n-1}\|+\beta_n\left\|x_n-\left(PS_n\right)^n x_n\right\| \\
&\quad +\gamma_n\left\|x_n-\left(PT_n\right)^n x_n\right\|.
\end{aligned}$$

And then from (3.12) and (3.13),

$$\begin{aligned}
\|x_n-x_{n-1}\| &\leq \frac{\beta_n}{\left(1-\beta_n-\gamma_n\right)}\left\|x_n-\left(PS_n\right)^n x_n\right\| \\
&\quad +\frac{\gamma_n}{\left(1-\beta_n-\gamma_n\right)}\left\|x_n-\left(PT_n\right)^n x_n\right\| \\
&\to 0, \quad \left(n\to\infty\right).
\end{aligned} \tag{3.14}$$

Since an asymptotically nonexpansive mapping with respect to P must be uniformly L-Lipschitzian with respect to P , then we have

$$\begin{aligned}
\|x_n-\left(PT_n\right)x_n\| &\leq \|x_n-(PT_n)^n x_n\|+\|(PT_n)^n x_n-\left(PT_n\right)x_n\| \\
&\leq \|x_n-(PT_n)^n x_n\|+L\left\|x_n-(PT_n)^{n-1}x_n\right\| \\
&\leq \|x_n-(PT_n)^n x_n\|+L\left\|(x_n-x_{n-1})+(x_{n-1}-(PT_n)^{n-1}x_{n-1})\right. \\
&\quad \left.+((PT_n)^{n-1}x_{n-1}-(PT_n)^{n-1}x_n)\right\| \\
&\leq \|x_n-(PT_n)^n x_n\|+L\left\|x_{n-1}-(PT_n)^{n-1}x_{n-1}\right\| \\
&\quad +L(L+1)\|x_n-x_{n-1}\|.
\end{aligned}$$

This together with (3.12) and (3.14) implies that $\lim_{n\to\infty}\|x_n-\left(PT_n\right)x_n\|=0$. Similarly we can prove that $\lim_{n\to\infty}\|x_n-\left(PS_n\right)x_n\|=0$. This completes the proof. □

We now prove the following strong convergence theorem by making use of Lemma 2.3.

3.4. Theorem. *Let K be a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $S_i, T_i : K\to E$ $(i\in J)$ be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to P with common sequences $\{k_n\}$, $\{l_n\}\subset$*

$[1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. *Suppose that* $F \neq \varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0,1)$.

If one of $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$ *is completely continuous, then* $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. Assume that S_k is completely continuous for some $k \in J$. Since P is nonexpansive, then there exists subsequence $\{PS_k x_{n_j}\} \subset \{PS_k x_n\}$ such that $PS_k x_{n_j} \to q$ as $j \to \infty$. Now $\|x_{n_j} - q\| \leq \|x_{n_j} - PS_k x_{n_j}\| + \|PS_k x_{n_j} - q\| \to 0$ by Lemma 3.3. Also $\|q - PS_k q\| = \lim_{n\to\infty} \|x_{n_j} - PS_k x_{n_j}\| = 0$, we get $q = PS_k q$. Similarly, $q = PT_k q$ and so $q \in F$. Since S_k is chosen arbitrarily, it follows from Lemma 2.3 that $\{x_n\}$ converges strongly to some common fixed point of $\{S_i\}$ and $\{T_i\}(i \in J)$. This completes the proof. □

Our next strong convergence theorem is an application of Theorem 3.2.

3.5. Theorem. *Let* K *be a nonempty closed convex subset of a real uniformly convex and smooth Banach space* E *with* P *as a sunny nonexpansive retraction. Let* $S_i, T_i : K \to E$ $(i \in J)$ *be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to* P *with common sequences* $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. *Suppose that* $F \neq \varnothing$ *and* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0,1)$.

If $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$ *satisfy Condition* $(B^{'})$, *then* $\{x_n\}$ *converges strongly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$, and so $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. Because $\{S_i\}$ and $\{T_i\}$ satisfy Condition $(B^{'})$, we have

$$f(d(x_n, F)) \leq \max_{i\in J}\left\{\frac{1}{2}\left(\|x_n - PT_i x_n\| + \|x_n - PS_i x_n\|\right)\right\}.$$

Applying Lemma 3.3, we get $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Now applying Theorem 3.2, we obtain the result. This completes the proof. □

Finally, we give our weak convergence theorem.

3.6. Theorem. *Let* K *be a nonempty closed convex subset of a real uniformly convex and smooth Banach space* E *satisfying Opial's condition with* P *as a sunny nonexpansive retraction. Let* $S_i, T_i : K \to E$ $(i \in J)$ *be two families of weakly inward and nonself asymptotically nonexpansive mappings with respect to* P *with* *common sequences* $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. *Suppose that* $\{x_n\}$ *defined by* (1.8) *satisfies the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n k_n + \gamma_n l_n < 1$ *for each integer* $n \geq 1$.
(ii) $s \leq \alpha_n, \beta_n, \gamma_n \leq 1 - s$ *for some* $s \in (0,1)$.

If $F \neq \varnothing$, *then* $\{x_n\}$ *converges weakly to some common fixed point of* $\{S_i\}$ *and* $\{T_i\}$ $(i \in J)$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and so $\{x_n\}$ is bounded. Note that PT_i and PS_i are self-mappings from K into itself. We now prove that $\{x_n\}$ has a unique weak subsequential limit in F. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to p_1 and p_2, respectively. By Lemma 3.3, we have $\lim_{n\to\infty} \|x_{n_k} - (PS_i)\, x_{n_k}\| = 0$ and $\lim_{n\to\infty} \|x_{n_k} - (PT_i)\, x_{n_k}\|$, $(i \in J)$. Lemma 2.4 guarantees that $(I - PS_i)\, p_1 = 0$, i.e., $(PS_i)\, p_1 = p_1$. Similary, $(PT_i)\, p_1 = p_1$. By Lemma 2.3, we get $p_1 \in F$. Again in the same way, we can prove that $p_2 \in F$. For uniqueness, assume $p_1 \neq p_2$, then by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{k\to\infty} \|x_{n_k} - p_1\| \\
&< \lim_{k\to\infty} \|x_{n_k} - p_2\| \\
&= \lim_{j\to\infty} \left\|x_{n_j} - p_2\right\| \\
&< \lim_{j\to\infty} \left\|x_{n_j} - p_1\right\| \\
&= \lim_{n\to\infty} \|x_n - p_1\|,
\end{aligned}$$

which is a contradiction and hence $p_1 = p_2$. Consequently, $\{x_n\}$ converges weakly to a common fixed point *of* $\{S_i\}$ and $\{T_i\}$ $(i \in J)$. □

3.7. Remark. (1) If the error terms are added in (1.8) and assumed to be bounded, then the results of this paper still hold.

(2) Since (1.5), (1.6), (1.7) and (1.10) are special cases of (1.8), the results proved using these algorithms follow as a special case to our above results.

References

[1] Goebel, K., Kirk, W. A. *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**, 171-174, 1972.

[2] Chidume, C. E., Ofoedu, E U. and Zegeye, H. *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, Math. Anal. and Appl., **280**, 364-374, 2003.

[3] Zhou, H. Y., Cho, Y. J. and Kang, S. M. *A New iterative algorithm for approximating common fixed points for asymptotically nonexpansive mappings*, Fixed Point Theory and Applications, vol. **2007**, Article ID 64874, 10 pages, 2007.

[4] Soltuz, S M. *The backward Mann iteration*, Octogon Mathematics Magazine, **9** (2), 797-800, 2001.

[5] Xu, H. K. and Ori, R. G. *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim., **22**, 767-773, 2001.

[6] Khan, S. H., Yildirim, I. and Ozdemir, M. *Convergence of an implicit algorithm for two families of nonexpansive mappings*, Computers and Mathematics with Applications, **59**, 3084–3091, 2010.

[7] Bruck, R.E., *Nonexpansive retracts of Banach spaces*, Bull. Am. Math. Soc., **76**, 384–386, 1970.

[8] Khan, S. H. and Fukharuddin, H. *Weak and strong convergence of a scheme with errors for two nonexpansive mappings*, Nonlinear Anal., **61**, 1295–1301, 2005.
[9] Tan, K. K. and Xu, H. K. *Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process*, J. Math. Anal. Appl., **178** (2), 301-308, 1993.
[10] Schu, J. *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43**, 153-159, 1991.
[11] Takahashi, W. *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, (Yokohama Publishers, Yokohama, Japan, 2000).
[12] Yildirim, I. and Khan, S. H. *A new one-step implicit iterative process for common fixed points of two asymptotically nonexpansive mappings in Banach spaces*, Expo. Math., **29**, no.2, 240–251, 2011.
[13] Chang, S. S., Tan, K. K., Lee, H. W. J. and Chan, C. K. *On the convergece of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **313**, 273-283, 2006.
[14] Zhou, Y. and Chang, S. S. *Convergece of implicit iteration process for a finite family of asymptotically nonexpansive mappings*, Numer. Funct. Anal. Optim., vol. **23**, no. 7-8, pp. 911-921, 2002.
[15] Cho, S. Y., Kang, S. M. and Qin, X. *Weak and strong convergence of an implicit iteration process for two families of asymptotically nonexpansive mappings*, Thai J. Math., Vol. **8**, no.3, 521-533, 2010.
[16] Yang, L. and Xie, X. *Weak and strong convergence theorems of three step iteration process with errors for nonself-asymptotically nonexpansive mappings*, Math. Comput. Modelling, **52**, 772-780, 2010.
[17] Yang, L., Xie, X. and Peng, S. *A new composite implicit iterative algorithm for nonself asymptotically nonexpansive mappings*, J. Appl. Math. Comput., **31**, 81-95, 2009.
[18] Khan, S. H. and Hussain, N. *Convergence theorems for nonself asymptotically nonexpansive mappings*, Computers and Mathematics with Applications, **55**, 2544-2553, 2008.
[19] Wang, L. *Explicit iteration method for common fixed points of a finite family of nonself asymptotically nonexpansive mappings*, Computers and Mathematics with Applications, **53**, 1012-1019, 2007.
[20] Thianwan, S. *Common fixed points of new iterations for two asymptotically nonexpansive nonselfmappings in a Banach space*, Journal of Computational and Applied Mathematics, vol. **224**, no. 2, 688–695, 2009.
[21] Chidume, C. and Ali, B. *Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **326**, 960-973, 2007.
[22] Khan, S.: *An iteration process for common fixed points of two nonself asymptotically nonexpansive mappings*, Bulletin of Mathematical Analysis and Applications, Vol. **3** (3), 165-176, 2011.
[23] Ozdemir, M., Akbulut, S. and Kiziltunc, H. *Convergence theorems on a new iteration process for two asymptotically nonexpansive nonself mappings with errors in Banach spaces*, Discrete Dyn. Nat. Soc., Volume **2010**, Article ID 307245, 19 pages.