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SPECTRAL EXPANSION OF STURM-LIOUVILLE PROBLEMS WITH EIGENVALUE-DEPENDENT BOUNDARY CONDITIONS

NIHAL YOKUŞ AND ESRA KIR ARPAT

Abstract. In this paper, we consider the operator L generated in $L_2(\mathbb{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, x \in \mathbb{R}_+ := [0, \infty)$$

and the boundary condition

$$\frac{y'(0)}{y(0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2,$$

where q is a complex valued function and $\alpha_i \in \mathbb{C}$, $i = 0, 1, 2$ with $\alpha_2 \neq 0$. We have proved that spectral expansion of L in terms of the principal functions under the condition

$$q \in AC(\mathbb{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \sup_{x\in\mathbb{R}_+} [e^{\varepsilon\sqrt{x}}|q'(x)|] < \infty, \quad \varepsilon > 0$$

taking into account the spectral singularities. We have also proved the convergence of the spectral expansion.

1. INTRODUCTION

The spectral analysis of a non-selfadjoint differential operators with continuous and discrete spectrum was investigated by Naimark [1]. He showed the existence of spectral singularities in the continuous spectrum of the non-selfadjoint differential operator L_0, generated in $L_2(\mathbb{R}_+)$, by the differential expression

$$l_0(y) = -y'' + q(x)y, \ x \in \mathbb{R}_+ := [0, \infty) \tag{1.1}$$

with the boundary condition $y'(0) - hy(0) = 0$, where q is a complex valued function and $h \in \mathbb{C}$. If the following condition

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$$\int e^{\varepsilon x}|q(x)|dx < \infty, \quad \varepsilon > 0$$

satisfies, then L_0 has a finite number of eigenvalues and spectral singularities with finite multiplicities. Lyance investigated the effect of the spectral singularities in the spectral expansion in terms of the principal functions of L_0 [2]. The Laurent expansion of the resolvents of non-selfadjoint operators in neigbourhood of spectral singularities was investigated by Gasymov-Maksudov [3] and Maksudov-Allakhverdiev [4]. They also studied the effect of spectral singularities in the spectral analysis of these operators.

Using the boundary uniqueness theorems of analytic functions, the structure of the eigenvalues and the spectral singularities of a quadratic pencil of Schrödinger, Klein-Gordon, discrete Dirac and discrete Schrödinger operators was investigated in [5]-[10]. The effect of the spectral singularities in the spectral expansion of a quadratic pencil of Schrödinger operators was obtained in [9]. In [10] the spectral expansion of the discrete Dirac and Schrödinger operators with spectral singularities was derived using the generalized spectral function (in the sense of Marchenko [11]) and the analytical properties of the Weyl function.

Spectral analysis of the quadratic pencil of Schrödinger operators was done in [9]. Spectral expansion of a non-selfadjoint differential operator on the whole axis was studied in [12]. The other expansion of the non-selfadjoint Sturm-Liouville Operator with a singular potential was studied in [13].

Let us consider the operator L generated in $L_2(\mathbb{R}_+)$by the differential expression

$$l(y) = -y'' + q(x)y, \ x \in \mathbb{R}_+ \tag{1.2}$$

and the eigenvalue-dependent boundary condition

$$\frac{y'(0)}{y(0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda_2 \tag{1.3}$$

where q is a complex-valued function and $\alpha_i \in \mathbb{C}, \ i = 0, 1, 2$ with $\alpha_2 \neq 0$. In ([14]) it has been proved that the operator L has of a finite number and spectral singularities, each of them is of finite multiplicity under the conditions

$$q \in AC(\mathbb{R}_+), \ \underset{x\to\infty}{lim}\, q(x) = 0, \ \underset{x\in\mathbb{R}_+}{sup}\, [e^{\varepsilon\sqrt{x}}|q'(x)|] < \infty, \ \varepsilon > 0 \tag{1.4}$$

In this paper, which is a continuation of ([15]), we find a spectral expansion of L in terms of the principal functions under the conditions (1.4) taking into account the spectral singularities using a contour integral method, and the regularization of divergent integrals, using summability factors. We also investigate the convergence of the spectral expansion.

2. SPECIAL SOLUTIONS

Let us consider the equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ \tag{2.1}$$

We have previously considered in [15] that the only complex valued function, q is almost everywhere continuous in $\mathbb{R}_+$ and satisfies the following condition

$$\int_0^\infty x|q(x)|dx < \infty \tag{2.2}$$

Let $\varphi(x,\lambda)$ and e(x,λ) denote the solutions of (2.1) satisfying the conditions

$$\varphi(x,\lambda) = 1, \varphi'(x,\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2, \underset{x\to\infty}{lim} e(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}} \tag{2.3}$$

respectively. The solution $e(x,\lambda)$ is called Jost Solution of (2.1). Note that, under the condition (2.2), the solution $\varphi(x,\lambda)$ is an entire function of λ and the Jost Solution is an analytic function of λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, Im\lambda > 0\}$ and continuous in $\overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, Im\lambda \geqslant 0\}$ ([14]).

Moreover, Jost Solution has a representation ([11])

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}}_+ \tag{2.4}$$

where the kernel $K(x,t)$ satisfies

$$\begin{aligned} K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^\infty q(s)\,\mathrm{d}s + \frac{1}{2}\int_x^{\frac{x+t}{2}}\int_{t+x-s}^{t+s-x} q(s)K(s,u)\,\mathrm{d}u\mathrm{d}s \\ + \frac{1}{2}\int_{\frac{x+t}{2}}^\infty \int_s^{t+s-x} q(s)K(s,u)\,\mathrm{d}u\mathrm{d}s \end{aligned} \tag{2.5}$$

and $K(x,t)$ is continuously differentiable with respect to x and t.

$$|K(x,t)| \leqslant cw\Big(\frac{x+t}{2}\Big) \tag{2.6}$$

$$|K_x(x,t)|, |K_t(x,t)| \leqslant \frac{1}{4}\Big|q\Big(\frac{x+t}{2}\Big)\Big| + cw\Big(\frac{x+t}{2}\Big) \tag{2.7}$$

where $w(x) = \int_x^\infty |q(s)|\,\mathrm{d}s$ and $c > 0$ is a constant.

Let $e^{\pm}(x,\lambda)$ denote the solutions of (2.1) subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\lambda x} e^{\pm}(x,\lambda) = 1, \quad \lim_{x\to\infty} e^{\pm i\lambda x} e^{\pm}_x(x,\lambda) = \pm i\lambda, \quad \lambda \in \overline{\mathbb{C}}_\pm \tag{2.8}$$

Then

$$\begin{aligned} W[e(x,\lambda), e^{\pm}(x,\lambda)] = \pm 2i\lambda, \quad \lambda \in \mathbb{C}_\pm \\ W[e(x,\lambda), e^{\pm}(x,-\lambda)] = -2i\lambda, \quad \lambda \in \mathbb{R} = (-\infty,\infty) \end{aligned} \tag{2.9}$$

where $W[f_1, f_2]$ is the Wronskian of f_1 and f_2 ([14]).

We will denote the Wronskian of the solutions with $e(x,\lambda)$ and $e(x,-\lambda)$ by $E^{+}(\lambda)$ and $E^{-}(\lambda)$, respectively, where

$$E^{+}(\lambda) := e'(0,\lambda) - (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_+ \tag{2.10}$$

$$E^{-}(\lambda) := e'(0,-\lambda) - (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)e(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_- \tag{2.11}$$

Therefore, $E^{+}$and E^{-} are analytic with respect to λ in $\mathbb{C}_+ = \{\lambda : \lambda \in \mathbb{C},\ \mathrm{Im}\lambda > 0\}$ and $\mathbb{C}_- = \{\lambda : \lambda \in \mathbb{C},\ \mathrm{Im}\lambda < 0\}$, respectively, and continuous up to real axis, and

$$E^{+}(\lambda) = -\alpha_2\lambda^2 + \beta^{+}\lambda + \delta^{+} + o(1), \quad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \to \infty \tag{2.12}$$

$$E^{-}(\lambda) = -\alpha_2\lambda^2 + \beta^{-}\lambda + \delta^{-} + o(1), \quad \lambda \in \overline{\mathbb{C}}_-, \quad |\lambda| \to \infty \tag{2.13}$$

where

$$\begin{aligned} \beta^{+} &= i - \alpha_1 - i\alpha_2 K(0,0) \\ \delta^{+} &= -K(0,0) - \alpha_0 - i\alpha_1 K(0,0) + \alpha_2 K_t(0,0) \\ f^{+}(t) &= K_x(0,t) - \alpha_0 K(0,t) - i\alpha_1 K_t(0,t) + \alpha_2 K_{tt}(0,t) \end{aligned} \tag{2.14}$$

hold [14].

3. THE SPECTRUM OF L

We have previously shown ([14]) that

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda \in \mathbb{C}_+, \quad E^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_+, \quad E^{-}(\lambda) = 0\} \\ \sigma_{ss}(L) &= \{\lambda : \lambda \in \mathbb{R}^*, \quad E^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^*, \quad E^{-}(\lambda) = 0\} \end{aligned} \tag{3.1}$$

where $\mathbb{R}^* = \mathbb{R}\diagdown\{0\}$, by $\sigma_d(L)$ and $\sigma_{ss}(L)$ we denote the eigenvalues and spectral singularities of L, respectively.

Let

$$G(x,t,\lambda) = \begin{cases} G^{+}(x,t,\lambda), & \lambda \in \mathbb{C}_+ \\ G^{-}(x,t,\lambda), & \lambda \in \mathbb{C}_- \end{cases} \tag{3.2}$$

be the Green Function of L, where

$$G^{+}(x,t,\lambda) = \left\{ \begin{array}{ll} -\frac{\varphi(t,\lambda)e(x,\lambda)}{E^{+}(\lambda)}, & 0 \leqslant t \leqslant x \\ -\frac{\varphi(x,\lambda)e(t,\lambda)}{E^{+}(\lambda)}, & x \leqslant t < \infty \end{array} \right\} \tag{3.3}$$

$$G^{-}(x,t,\lambda) = \left\{ \begin{array}{ll} -\frac{\varphi(t,\lambda)e(x,-\lambda)}{E^{-}(\lambda)}, & 0 \leqslant t \leqslant x \\ -\frac{\varphi(x,\lambda)e(t,-\lambda)}{E^{-}(\lambda)}, & x \leqslant t < \infty \end{array} \right\} \tag{3.4}$$

Under the conditions (1.4), we know that L has a finite number of eigenvalues and spectral singularities, and each of them is finite multiplicity ([14]). Let $\lambda_1, \ldots, \lambda_j$ and $\lambda_{j+1}, \ldots, \lambda_k$ denote the zeros of E^{+}in $\mathbb{C}_+$and E^{-}in $\mathbb{C}_-$(which are the eigenvalues of L) with multiplicities $m_1, \ldots, m_j$ and $m_{j+1}, \ldots, m_k$, respectively.

We will also need the Hilbert Spaces

$$H_m = \left\{ f : \int_0^\infty (1+x)^{2m} |f(x)|^2 \, \mathrm{d}x < \infty \right\}, \ m = 0, 1, \dots$$
$$H_{-m} = \left\{ g : \int_0^\infty (1+x)^{-2m} |g(x)|^2 \, \mathrm{d}x < \infty \right\}, \ m = 0, 1, \dots$$

with

$$\|f\|_m^2 = \int_0^\infty (1+x)^{2m} |f(x)|^2 \, \mathrm{d}x, \quad \|g\|_m^2 = \int_0^\infty (1+x)^{-2m} |g(x)|^2 \, \mathrm{d}x$$

respectively. It is obvious that $H_0 = L_2(\mathbb{R}_+)$ and

$$H_{m+1} \subsetneqq H_m \subsetneqq L_2(\mathbb{R}_+) \subsetneqq H_{-m} \subsetneqq H_{-(m+1)}, \ m = 1, 2, \dots$$

and H_{-m} is isomorphic to the dual of $H_m : H_m' \sim H_{-m}$

We have previously shown that ([15]):

$$U_{n,p} \in L_2(\mathbb{R}_+),\ n = 0, 1, ..., m_p - l,\ p = 1, 2, ..., \alpha \tag{3.5}$$

$$U_{n,p} \in H_{-(n+1)},\ n = 0, 1, ..., m_p - l,\ p = \alpha + 1, ..., k \tag{3.6}$$

where

$$\begin{aligned} U_{n,p}(x) &= \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi^+(x,\lambda) \right\}_{\lambda = \lambda p} = \sum_{\beta=0}^{n} A_{n-\beta}(\lambda) \left\{ \frac{\partial^\beta}{\partial \lambda^\beta} e(x,\lambda) \right\}_{\lambda = \lambda p} \\ U_{n,p}(x) &= \left\{ \frac{\partial^n}{\partial \lambda^n} \varphi^-(x,\lambda) \right\}_{\lambda = \lambda p} = \sum_{\beta=0}^{n} A_{n-\beta}(\lambda) \left\{ \frac{\partial^\beta}{\partial \lambda^\beta} e(x,-\lambda) \right\}_{\lambda = \lambda p} \end{aligned} \tag{3.7}$$

The functions $U_{n,p}(x)$, $n = 0, 1, \dots, m_p - 1,\ p = 1, 2, \dots, \alpha$ and $p = \alpha + 1, \dots, k$ are the principal functions corresponding to the eigenvalues and the spectral singularities of L, respectively.

4. SPECTRAL EXPANSION

Let $C_0^\infty(\mathbb{R}_+)$ denote the set of infinitely differentiable functions in $\mathbb{R}_+$with compact support. Evidently,

$$\psi(x) = R(L)R^{-1}(L)\psi(x) = R(L)(L - \lambda^2 I)\psi(x)$$
$$\psi(x) = \int_0^\infty G(x,t,\lambda) \left[-\psi'' + q(t)\psi(t) - \lambda^2 \psi(t) \right] \, \mathrm{d}t$$

for each $\psi \in C_0^\infty(\mathbb{R}_+)$. We obtain

$$\frac{\psi(x)}{\lambda} = \frac{1}{\lambda} \int_0^\infty G(x,t,\lambda)\theta(t) \, \mathrm{d}t - D(x,\lambda) \tag{4.1}$$

where

$$\theta(t) = -\psi'' + q(t)\psi(t), \quad D(x,\lambda) = \int_0^\infty \lambda G(x,t,\lambda)\psi(t) \, \mathrm{d}t$$

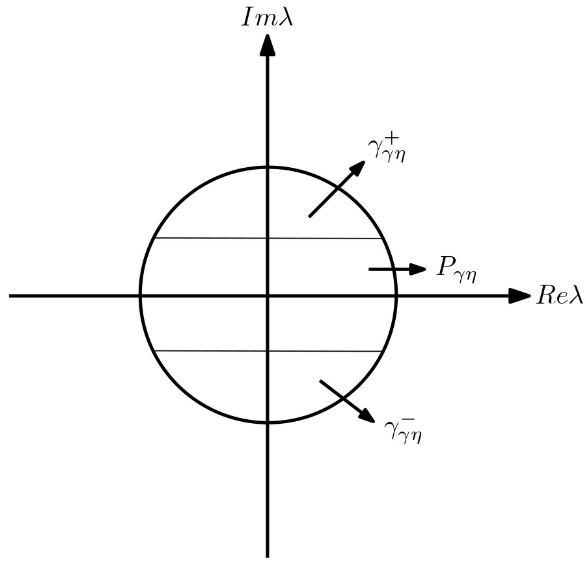

FIGURE 4.1

Let γ_r denote the contour with center at the origin having radius r; let $\partial\gamma_r$ be the boundary ofγ_r. r will be chosen so that all eigenvalues and spectral singularities of L are in γ_r. $P_{r\eta}$ denotes the part of γ_r lying in the strip $|Im\lambda| \leq \eta$ and $\gamma_{r\eta} = \gamma^+_{r\eta} \cup \gamma^-_{r\eta}$, where $\gamma^+_{r\eta}$ and $\gamma^-_{r\eta}$ are the parts of $\gamma_r \backslash P_{r\eta}$ in the upper and the lower half-planes, respectively (see Figure 4.1). We chose η so small that $P_{r\eta}$ does not contain any eigenvalues of L.

So we easily see that

$$\partial\gamma_r = \partial\gamma_{r\eta} \cup \partial P_{r\eta} \tag{4.2}$$

From (4.1) we get

$$\psi(x) = \frac{1}{2\pi i}\int_{\partial\gamma_r}\left\{\frac{1}{\lambda}\int_0^\infty G(x,t,\lambda)\theta(t)\,\mathrm{d}t\right\}\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\partial\gamma_r} D(x,\lambda)\,\mathrm{d}\lambda \tag{4.3}$$

Using (2.12), (2.13), (3.2) and Jordan's lemma, we see that the first term of the right hand side of (4.3) vanishes as $r \to \infty$. The same result holds for the second term. Then considering (4.2) we find

$$\psi(x) = -\lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int_{\partial\gamma_{r\eta}} D(x,\lambda)\,\mathrm{d}\lambda - \lim_{\substack{r\to\infty\\ \eta\to 0}}\frac{1}{2\pi i}\int_{\partial P_{r\eta}} D(x,\lambda)\,\mathrm{d}\lambda \tag{4.4}$$

FIGURE 4.2

We easily obtain that the first integral in (4.4) gives

$$\lim_{\substack{r\to\infty\\ \eta\to 0}} \int D(x,\lambda)\,\mathrm{d}\lambda = \sum_{i=1}^{\alpha} \underset{\lambda=\lambda_i^+}{Res} \left[D^+(x,\lambda)\right] + \sum_{i=1}^{\alpha} \underset{\lambda=\lambda_i^-}{Res} \left[D^-(x,\lambda)\right]$$

where

$$D^{\pm}(x,\lambda) = \int_0^{\infty} G^{\pm}(x,t,\lambda)\psi(t)\,\mathrm{d}t.$$

Let Γ be the contour which isolates the real zeros of E^+ by semicircles with centers at λ_i, $i = 1,2,\ldots,\nu$ having the same radius δ_0 in the upper-half plane. Similarly, let Γ_- be the corresponding contour for the real zeros of E^- in the lower half-plane. The radius of semicircles being chosen so small that their diameters are mutually disjoint and do not contain the point $\lambda = 0$ (see Figure 4.2).

From Figure 4.1, we obtain

$$\lim_{\substack{r\to\infty\\ \eta\to 0}} \frac{1}{2\pi i}\int_{\partial P_{r\eta}} D(x,\lambda)\,\mathrm{d}\lambda = \frac{1}{2\pi i}\int_{\Gamma_-} D^-(x,\lambda)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\Gamma_+} D^+(x,\lambda)\,\mathrm{d}\lambda$$

Therefore (4.4) can be written as

$$\begin{aligned} \psi(x) \quad = \quad & -\sum_{i=1}^{\alpha} \underset{\lambda=\lambda_i^+}{Res} \left[D^+(x,\lambda)\right] - \sum_{i=1}^{\alpha} \underset{\lambda=\lambda_i^-}{Res} \left[D^-(x,\lambda)\right] \\ & + \frac{1}{2\pi i}\int_{\Gamma_+} D^+(x,\lambda)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\Gamma_-} D^-(x,\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{4.5}$$

Theorem 1. *For every* $\psi \in C_0^{\infty}(\mathbb{R}_+)$

$$\begin{aligned} \psi(x) = & \sum_{i=1}^{\alpha} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^+-1} \left[a_i^+(\lambda)U(x,\lambda)\,U(\psi,\lambda)\right] \right\}_{\lambda=\lambda_i^+} \\ & + \sum_{i=1}^{\alpha} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^--1} \left[a_i^-(\lambda)U(x,\lambda)\,U(\psi,\lambda)\right] \right\}_{\lambda=\lambda_i^+} \\ & + \frac{1}{2\pi i}\int_{\Gamma_+} \frac{e_x^+(0,\lambda)}{E^+(\lambda)} U(x,\lambda)\,U(\psi,\lambda)\;\mathrm{d}\lambda \\ & - \frac{1}{2\pi i}\int_{\Gamma_-} \frac{e_x^-(0,\lambda)}{E^-(\lambda)} U(x,\lambda)\,U(\psi,\lambda)\;\mathrm{d}\lambda \end{aligned} \tag{4.6}$$

$$\begin{aligned}0 = & \sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{m_i^+-1}\left[b_i^+(\lambda)U(x,\lambda)\,U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_i^+}\\ & +\sum_{i=1}^{\alpha}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{m_i^--1}\left[b_i^-(\lambda)U(x,\lambda)\,U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_i^-}\\ & +\frac{1}{2\pi i}\int_{\Gamma_+}\frac{e_x^+(0,\lambda)}{E^+(\lambda)}U(x,\lambda)\,U(\psi,\lambda)\,\mathrm{d}\lambda\\ & -\frac{1}{2\pi i}\int_{\Gamma_-}\frac{e_x^-(0,\lambda)}{E^-(\lambda)}U(x,\lambda)\,U(\psi,\lambda)\,\mathrm{d}\lambda\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}a_i^+(\lambda) &= -\frac{(\lambda-\lambda_i^+)^{m_i}e_x^+(0,\lambda)}{(m_i-1)!E^+(\lambda)},\ i=1,...,\alpha\\ a_i^-(\lambda) &= \frac{(\lambda-\lambda_i^-)^{m_i}e_x^-(0,\lambda)}{(m_i-1)!E^-(\lambda)},\ i=1,...,k\end{aligned}\tag{4.8}$$

$$\begin{aligned}b_i^+(\lambda) &= -\frac{(\lambda-\lambda_i^+)^{m_i}e_x^+(0,\lambda)}{(m_i-1)!E^+(\lambda)},\ i=1,...,\alpha\\ b_i^-(\lambda) &= -\frac{(\lambda-\lambda_i^-)^{m_i}e_x^-(0,\lambda)}{(m_i-1)!E^-(\lambda)},\ i=1,...,k\end{aligned}\tag{4.9}$$

and

$$U(\psi,\lambda)=\int_0^\infty \psi(t)U(x,\lambda)\,\mathrm{d}t.$$

Proof. Let $B(x,\lambda)$ be the solution of (2.1) subject to the initial conditions

$$B(0,\lambda)=1,\quad B'(0,\lambda)=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2.$$

Then

$$G^{\pm}(x,t,\lambda)=\frac{e_x^{\pm}(0,\lambda)}{E^{\pm}(\lambda)}U(x,\lambda)\,U(t,\lambda)+a(x,t,\lambda)\tag{4.10}$$

where

$$a(x,t,\lambda)=\begin{cases}B(x,\lambda)U(t,\lambda),\ 0<t\leqslant x\\ B(t,\lambda)U(x,\lambda),\ x\leq t<\infty\end{cases}$$

and $a(x,t,\lambda)$ is an entire function of λ. From (4.5) and (4.10) we obtain (4.6). Writing (4.1) as

$$\frac{\psi(x)}{\lambda^2}=\frac{1}{\lambda^2}\int_0^\infty G(x,t,\lambda)\theta(t)\,\mathrm{d}t-D(x,\lambda)$$

and repeating the calculation as we done for (4.1), we have (4.7). $\square$

Since the contour Γ_+ and Γ_- in (4.6) and (4.7) do not coincide with the continuous spectrum of L, these formulae contains non-spectral objects. The aim of this article is to transform (4.6) and (4.7) into two-fold spectral expansion with respect to the principal functions of L.

Theorem 2. *For any $\psi \in C_0^\infty(\mathbb{R}_+)$ there exists a constant $c > 0$ so that*

$$\int_{-\infty}^{\infty} |\lambda U(\psi, \lambda)|^2 \, \mathrm{d}\lambda \leq c \int_0^\infty |\psi(x)|^2 \, \mathrm{d}x \tag{4.11}$$

Proof. From (3.7) we get

$$U(\psi, \lambda) = \sum_{\beta=0}^{n} M_{n-\beta}(\lambda_p) \frac{1}{\beta!} \left\{ \frac{\partial^\beta}{\partial \lambda^\beta} e^{\pm}(\psi, \lambda) \right\}_{\lambda=\lambda_p} \tag{4.12}$$

where

$$e^{\pm}(\psi, \lambda) = \int_0^\infty \psi(x) e^{\pm}(x, \lambda) \, \mathrm{d}x.$$

Using (2.4), we obtain

$$\begin{aligned} e^{\pm}(\psi, \lambda) &= \int_0^\infty \left\{ e^{i\lambda x} + \int_x^\infty K(x,t) e^{i\lambda t} \, \mathrm{d}t \right\} \psi(x) \, \mathrm{d}x \\ &= \int_0^\infty \psi(x) e^{i\lambda x} \, \mathrm{d}x + \int_0^\infty \int_x^\infty \psi(x) K(x,t) e^{i\lambda t} \, \mathrm{d}t\mathrm{d}x \end{aligned}$$

Changing the order of integration, we get

$$e^{\pm}(x, \lambda) = \int_0^\infty \{(I+K)\psi(t)\} e^{i\lambda t} \, \mathrm{d}t \tag{4.13}$$

in which the operator I is the unit operator, and K is the operator defined by

$$K\psi(t) = \int_0^\infty K(x,t)\psi(x) \, \mathrm{d}t.$$

From (2.6) we understand K is a compact operator in $L_2(\mathbb{R}_+)$. Thus $(I+K)$ is a continuous and one-to-one on $L_2(\mathbb{R}_+)$. Using the Parseval's equality for the Fourier transforms and (4.13) we get

$$\int_{-\infty}^{\infty} \left| e^{\pm}(\psi, \lambda) \right|^2 \, \mathrm{d}\lambda \leq c \int_0^\infty |\psi(x)|^2 \, \mathrm{d}x \tag{4.14}$$

where $c > 0$ is a constant.

The proof of the theorem is completed by (2.12),(2.13) and (4.14). $\square$

By the preceding theorem, for every function $\psi \in L_2(\mathbb{R}_+)$ the limit

$$U(\psi, \lambda) = \lim_{N \to \infty} \int_0^N \psi(x) U(x, \lambda) \, \mathrm{d}x$$

exists in the sense of convergence in the mean square, relative to the measure $\lambda^2 \mathrm{d}\lambda$ on the real axis; that is,

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left| U(\psi,\lambda) - \int_0^N \psi(x) U(x,\lambda)\, \mathrm{d}x \right|^2 \lambda^2 \,\mathrm{d}\lambda = 0 \tag{4.15}$$

Since $C_0^\infty(\mathbb{R}_+)$ is dense in $L_2(\mathbb{R}_+)$, the estimate (4.11) may be extended onto $L_2(\mathbb{R}_+)$ for any $\psi \in L_2(\mathbb{R}_+)$ as

$$\int_{-\infty}^{\infty} |\lambda U(\psi,\lambda)|^2 \,\mathrm{d}\lambda \le c \int_0^\infty |\psi(x)|^2 \,\mathrm{d}x \tag{4.16}$$

where $U(\psi,\lambda)$ must be understood in the sense of (4.15). We shall need a generalization of this estimate.

Theorem 3. *If $\psi \in H_m$, then $U(\psi,\lambda)$ has a derivative of order $(m-1)$ which is absolutely continuous of every finite subinterval of the real axis and satisfies*

$$\int_{-\infty}^{\infty} \left| \left(\frac{d}{d\lambda}\right)^n [U(\psi,\lambda)] \,\mathrm{d}\lambda \right|^2 \le c_n \int_0^\infty (1+x)^{2n} |\psi(x)|^2 \,\mathrm{d}x \tag{4.17}$$

where $c_n > 0$ are constants, $n = 1, \ldots, m$.

The proof is similar to that of Theorem 2.

To transform (4.6) and (4.7) into the spectral expansion of L, we have to reform the integrals over Γ_+ and Γ_- onto the real axis.

Since the spectral singularities of L are the zeros of $E^{\pm}$, the integrals over the real axis are divergent in the norm of $L_2(\mathbb{R}_+)$. Now we will investigate the convergence of these integrals in a norm which is weaker than the norm of $L_2(\mathbb{R}_+)$. For this purpose we will use the technique of regularization of divergent integrals. So we define the following summability factor:

$$F^+_{p\beta}(\lambda) = \begin{cases} \frac{(\lambda-\lambda_p)^\beta}{\beta!} & , \quad |\lambda-\lambda_p| < \delta \quad , \quad p = 1,\ldots,n \\ 0 & , \quad |\lambda-\lambda_p| \geqslant \delta \quad , \quad p = 1,\ldots,n \end{cases} \tag{4.18}$$

$$F^-_{p\beta}(\lambda) = \begin{cases} \frac{(\lambda-\lambda_p)^\beta}{\beta!} & , \quad |\lambda-\lambda_p| < \delta \quad , \quad p = n+1,...,k \\ 0 & , \quad |\lambda-\lambda_p| \geqslant \delta \quad , \quad p = n+1,...,k \end{cases} \tag{4.19}$$

with $\delta > \delta_0$. We can choose $\delta > 0$ so small that the $\delta-$neighborhoods of λ_p, $p = 1, ..., n, n+1, ..., k$ have no common points and do not contain the point $\lambda = 0$. Define the functions

$$F^{+}\left\{g_1(\lambda)\right\} = g_1(\lambda) - \sum_{p=1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{d}{d\lambda}\right)^{\beta} g_1(\lambda)\right\}_{\lambda=\lambda_p} F^{+}_{p\beta}(\lambda) \tag{4.20}$$

$$F^{-}\left\{g_2(\lambda)\right\} = g_2(\lambda) - \sum_{p=n+1}^{k}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{d}{d\lambda}\right)^{\beta} g_2(\lambda)\right\}_{\lambda=\lambda_p} F^{-}_{p\beta}(\lambda) \tag{4.21}$$

where g_1 and g_2 is chosen so that the right hand side of the above formulae is meaningful. It is evident from (4.18)-(4.19) that $\lambda_1, ..., \lambda_n$ are the roots of $F^{+}\left\{g_1(\lambda)\right\} = 0$ and λ_{n+1},...,$\lambda_k$are the roots of $F^{-}\left\{g_2(\lambda)\right\} = 0$ at least of orders $m_1, ..., m_n$and $m_{n+1}, ..., m_k$, respectively.

In the following, we will use the operators

$$P^{+}\psi(x) = \frac{1}{2\pi i}\int_{\Gamma_+} \frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} U\left(x,\lambda\right) U\left(\psi,\lambda\right) d\lambda \tag{4.22}$$

$$P^{-}\psi(x) = \frac{1}{2\pi i}\int_{\Gamma_-} \frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} U\left(x,\lambda\right) U\left(\psi,\lambda\right) d\lambda \tag{4.23}$$

and

$$\begin{aligned} I^{+}\psi(x) = \frac{1}{2\pi i}&\sum_{p=1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right]\right\}_{\lambda=\lambda_p} \\ &\times \int_{\Gamma_+}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} F^{+}_{p\beta}(\lambda) d\lambda \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} F^{+}\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\} d\lambda \end{aligned}$$

$$\begin{aligned} I^{-}\psi(x) = \frac{1}{2\pi i}&\sum_{p=n+1}^{k}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right]\right\}_{\lambda=\lambda_p} \\ &\times \int_{\Gamma_-}\frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} F^{-}_{p\beta}(\lambda) d\lambda \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} F^{-}\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\} d\lambda \end{aligned}$$

Since under the condition (1.4) $e^{+}(x,\lambda)$ and $e^{-}(x,\lambda)$ have an analytic continuation to the half-planes $Imk > -\frac{\epsilon}{2}$ and $Imk < \frac{\varepsilon}{2}$, respectively, we get

$$P^{\pm}\psi = I^{\pm}\psi$$

for $\psi \in C_0^{\infty}\left(\mathbb{R}_+\right)$.

Theorem 4. *For each $\psi \in H_{(m_0+1)}$, there exist a constant $c > 0$ such that*

$$\left\| I^{\pm}\psi \right\|_{-(m_0+1)} \leq c_1 \left\| \psi \right\|_{(m_0+1)} \tag{4.24}$$

where $m_0 = \max\{m_1, ..., m_n, m_{n+1}, ..., m_k\}$.

Proof. Define

$$\Lambda_p^+ = (\lambda_p - \delta, \lambda_p + \delta), \quad p = 1, ..., n \tag{4.25}$$

Then $0 \notin \Lambda_p^+$, $p = 1, ..., n$. Using the integral form of remainder in the Taylor formula, we get

$$\begin{aligned} &F^+\{U(x,\lambda)U(\psi,\lambda)\} \\ &= \begin{cases} U(x,\lambda)U(\psi,\lambda) & , \lambda \in \Lambda_0^+ \\ \frac{1}{(m_p-1)!}\int_{\lambda_p}^{\lambda}(\lambda-\xi)^{m_p-1}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}\left[U(x,\xi)U(\psi,\xi)\right]\right\}\mathrm{d}\xi, & \lambda \in \Lambda_p^+ \end{cases} \end{aligned} \tag{4.26}$$

where $\Lambda_0^+ = R \diagdown \left\{ \bigcup\limits_{p=1}^{n} \Lambda_p^+ \right\}$. If we use the notation

$$I_p^+\psi(x) = \frac{1}{2\pi i}\int_{\Lambda_p^+} \frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)} \left\{U(x,\lambda)U(\psi,\lambda)\right\} \mathrm{d}\lambda, \quad p = 1, ..., k$$

$$\begin{aligned} \tilde{I}^+\psi(x) = \frac{1}{2\pi i}&\sum_{p=1}^{k}\sum_{\beta=0}^{m_p-1}\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U(x,\lambda)U(\psi,\lambda)\right]_{\lambda=\lambda_p} \\ &\times \int_{\Gamma} \frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)} F_{p\beta}^+(\lambda)\,\mathrm{d}\lambda \end{aligned}$$

we obtain

$$I^+ = I_0^+ + ... + I_k^+ + \tilde{I}^+ \tag{4.27}$$

from (4.25) and (4.26). We now show that each of the operators $I_0^+, ..., I_p^+$ and $\tilde{I}^+$ is continuous from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. We start from with $\tilde{I}^+$. From (4.18) we find the absolute convergence of

$$\int_{\Gamma_+} \frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)} F_{p\beta}^+(\lambda)\,\mathrm{d}\lambda$$

Using (3.6) and the isomorphism $H_{-m_0} \sim H'_{m_0}$ we see that $\tilde{I}^+$ is continuous from H_m into H_{-m_0} or from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. Hence there exists a constant $\tilde{c} > 0$ such that

$$\left\|\tilde{I}^+\psi(x)\right\|_{-(m_0+1)} \leqslant \tilde{c}\left\|\psi\right\|_{(m_0+1)} \tag{4.28}$$

for any $\psi \in H_{(m_0+1)}$.

Next we want to show the continuity of I_p^+, $p=1,...,n$ from $H_{(m_0+1)}$ into $H_{-(m_0+1)}$. From (4.26) we see that

$$I_p^+\psi(x) = \frac{1}{2\pi i(m_p-1)!}\int_{\Lambda_p^+}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}\int_{\lambda_p}^{\lambda}(\lambda-\xi)^{m_p-1} \times\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}[U(x,\lambda)U(\psi,\lambda)]\right\}d\xi d\lambda \tag{4.29}$$

Interchanging the order of integration, we get

$$\begin{aligned} I_p^+\psi(x) = \frac{1}{2\pi i(m_p-1)!}&\left\{\int\limits_{\lambda_p}^{\lambda_p+\delta}\int\limits_{\xi}^{\lambda_p+\delta}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}[U(x,\lambda)U(\psi,\lambda)]\right\}\right.\\ &\times(\lambda-\xi)^{m_p-1}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}d\lambda d\xi\\ &-\int\limits_{\lambda_p-\delta}^{\lambda_p}\int\limits_{\lambda_p-\delta}^{\xi}\left\{\left(\frac{\partial}{\partial\xi}\right)^{m_p}[U(x,\lambda)U(\psi,\lambda)]\right\}\\ &\times(\lambda-\xi)^{m_p-1}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}d\lambda d\xi. \end{aligned}$$

Since λ_p is a zero of $E^+(\lambda)$ order m_p, there exists a continuous function $E_p^+(\lambda)$ such that $E_p^+(\lambda_p)\neq 0$ and $E^+(\lambda)=(\lambda-\lambda_p)^{m_p}E_p^+(\lambda_p)$. On the other hand,

$$\left|\int\limits_{\xi}^{\lambda_p+\delta}(\lambda-\xi)^{m_p-1}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}\,\mathrm{d}\lambda\right|\leq h_p^{(1)}(\xi)\left[\ln\delta-\ln(\xi-\lambda_p)\right] \tag{4.30}$$

if $\xi>\lambda_p$, and

$$\left|\int_{\lambda_p-\delta}^{\xi}(\lambda-\xi)^{m_p-1}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}\,\mathrm{d}\lambda\right|\leq h_p^{(2)}(\xi)\left[\ln(\lambda_p-\xi)-\ln\delta\right] \tag{4.31}$$

if $\xi<\lambda_p$, where

$$h_p^{(1)}(\xi)=\max_{\lambda\in[\xi,\lambda_p+\delta]}\left|\frac{\lambda e_x^+(0,\lambda)}{E_p^+(\lambda)}\right|,\quad h_p^{(2)}(\xi)=\max_{\lambda\in[\lambda_p-\delta,\xi]}\left|\frac{\lambda e_x^+(0,\lambda)}{E_p^+(\lambda)}\right|.$$

(4.30) and (4.31) show that I_p^+, $p=1,...,n$ are integral operators with kernels having logarithmic singularities.

(4.29) can be written as

$$I_p^+\psi(x)=\int\limits_{\Lambda_p}\sum_{s=0}^{m_p}b_{sp}^+(x,\xi)\left\{\left(\frac{d}{d\xi}\right)^{s}U(\psi,\xi)\right\}d\xi.$$

Define

$$B_{sp}=\int_0^\infty\int_{\Lambda_p}\left|\frac{b^+_{sp}(x,\xi)}{(1+x)^{m_0+1}}\right|^2 d\xi dx.$$

We see that $B_{sp}<\infty$, by (3.6), (4.30) and (4.31). Since

$$\begin{aligned}\left\|I_p^+\psi\right\|^2_{-(m_0+1)} &= \int_0^\infty\left|\frac{I_p^+\psi(x)}{(1+x)^{m_0+1}}\right|^2 dx\\ &\le\sum_{s=0}^{m_p}\int_0^\infty\int_{\Lambda_p^+}\left|\frac{b^+_{sp}(x,\xi)}{(1+x)^{m_0+1}}\right|^2 d\xi dx\int_{\Lambda_p^+}\left|\left(\frac{d}{d\xi}\right)^s U(\psi,\xi)\right|^2 d\xi\\ &=\sum_{k=0}^{m_p}B_{sp}\int_{\Lambda_p^+}\left|\left(\frac{d}{d\xi}\right)^s U(\psi,\xi)\right|^2 d\xi\end{aligned}$$

Utilizing (4.16) and (4.17) we obtain

$$\left\|I_p^+\psi\right\|_{-(m_0+1)}\le c_p\left\|\psi\right\|_{m_0}\le c_p\left\|\psi\right\|_{(m_0+1)},\ p=1,...,n \tag{4.32}$$

where c_p are constants.

We consider the operator I_0^+ which is defined by

$$I_0^+\psi=\frac{1}{2\pi i}\int_{-\infty}^\infty \varkappa_0^+(\lambda)\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}\left[U(x,\lambda)U(\psi,\lambda)\right]\mathrm{d}\lambda \tag{4.33}$$

where $\varkappa_0^+$ is the characteristic function of the interval Λ_0^+. From (4.33), similar to the proof of Theorem 4.2, we get

$$\int_0^\infty\left|I_p^+\psi(x)\right|^2 dx\le c_0\int_0^\infty\left|\psi(x)\right|^2 dx,$$

where $c_0>0$ is a constant. Since

$$H_{(m_0+1)}\subsetneqq L_2\left(\mathbb{R}_+\right)\subsetneqq H_{-(m_0+1)},$$

we find

$$\left\|I_0\psi\right\|_{-(m_0+1)}\le\ c_0\left\|\psi\right\|_{(m_0+1)} \tag{4.34}$$

From (4.27), (4.28),(4.32) and (4.34) we have

$$\left\|I^+\psi\right\|_{-(m_0+1)}\le c\left\|\psi\right\|_{(m_0+1)}$$

□

In a similar way it follows that

$$\left\|I^{-}\psi\right\|_{-(m_0+1)} \le c\left\|\psi\right\|_{(m_0+1)}$$

Then for every $\psi \in H_{(m_0+1)}$,

$$\begin{aligned} I^{+}\psi(x) = \frac{1}{2\pi i}\sum_{p=1}^{n}\sum_{\beta=0}^{m_p-1}&\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U(x,\lambda)U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_p} \\ &\times\int\limits_{\Gamma}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)}F_{p\beta}^{+}(\lambda)\,\mathrm{d}\lambda \\ &+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)}F^{+}\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\}\,\mathrm{d}\lambda \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} I^{-}\psi(x) = \frac{1}{2\pi i}\sum_{p=n+1}^{k}\sum_{\beta=0}^{m_p-1}&\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U(x,\lambda)U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_p} \\ &\times\int\limits_{\Gamma_-}\frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)}F_{p\beta}^{-}(\lambda)\,\mathrm{d}\lambda \\ &+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)}F^{-}\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\}\,\mathrm{d}\lambda \end{aligned} \tag{4.36}$$

Let $a_p(\lambda)$ denote any function which is defined and differentiable in a neighbourhood of λ_p, and which satisfies the condition

$$\left\{\left(\frac{d}{d\lambda}\right)^{m_p-1-\beta}a_p(\lambda)\right\}_{\lambda=\lambda p} = \begin{cases} \frac{1}{2\pi i}\binom{m_p-1}{\beta}\int\limits_{\Gamma_+}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)}F_{p\beta}^{+}(\lambda)\,\mathrm{d}\lambda, & p=1,...,n \\ -\frac{1}{2\pi i}\binom{m_p-1}{\beta}\int\limits_{\Gamma_-}\frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)}F_{p\beta}^{-}(\lambda)\,\mathrm{d}\lambda, & p=n+1,...,k \end{cases} \tag{4.37}$$

Then (4.35) and (4.36) can be written as

$$\begin{aligned} I^{+}\psi(x) = \sum_{p=1}^{n}&\left\{\left(\frac{\partial}{\partial\lambda}\right)^{m_p-1}\left[a_p(\lambda)U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right]\right\}_{\lambda=\lambda_p} \\ &+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)}F^{+}\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\}\,\mathrm{d}\lambda \end{aligned} \tag{4.38}$$

$$I^{-}\psi(x)=\sum_{p=n+1}^{k}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{m_p-1}\left[a_p(\lambda)U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right]\right\}_{\lambda=\lambda_p}\tag{4.39}$$
$$+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^-(0,\lambda)}{E^-(\lambda)}F^-\left\{U\left(x,\lambda\right)U\left(\psi,\lambda\right)\right\}\,\mathrm{d}\lambda$$

we shall also use the following integral operator (see (4.7)):

$$Q^{+}\psi(x)=\frac{1}{2\pi i}\int\limits_{\Gamma_+}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}F^+\left[U(x,\lambda)U(\psi,\lambda)\right]\,\mathrm{d}\lambda\tag{4.40}$$

$$Q^{-}\psi(x)=\frac{1}{2\pi i}\int\limits_{\Gamma_-}\frac{\lambda e_x^-(0,\lambda)}{E^-(\lambda)}F^-\left[U(x,\lambda)U(\psi,\lambda)\right]\,\mathrm{d}\lambda\tag{4.41}$$

$$J^{+}\psi(x)=\frac{1}{2\pi i}\sum_{p=1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U(x,\lambda)U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_p}$$
$$\times\int\limits_{\Gamma}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}F_{p\beta}^+(\lambda)d\lambda$$
$$+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^+(0,\lambda)}{E^+(\lambda)}F^+\left[U(x,\lambda)U(\psi,\lambda)\right]\,\mathrm{d}\lambda$$

$$J^{-}\psi(x)=\frac{1}{2\pi i}\sum_{p=1}^{n}\sum_{\beta=0}^{m_p-1}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{\beta}\left[U(x,\lambda)U(\psi,\lambda)\right]\right\}_{\lambda=\lambda_p}$$
$$\times\int\limits_{\Gamma}\frac{\lambda e_x^-(0,\lambda)}{E^-(\lambda)}F_{p\beta}^-(\lambda)d\lambda$$
$$+\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\lambda e_x^-(0,\lambda)}{E^-(\lambda)}F^-\left[U(x,\lambda)U(\psi,\lambda)\right]\,\mathrm{d}\lambda$$

It is evident that

$$Q^{\pm}\psi=J^{\pm}\psi,$$

for $\psi\in C_0^{\infty}(R_+)$.

Theorem 5. *For every each $\psi\in H_{(m_0+1)}$, there exist a constant $c>0$ such that*

$$\left\|J^{\pm}\psi\right\|_{-(m_0+1)}\leq c\left\|\psi\right\|_{(m_0+1)}.$$

It is evident that, for every $\psi \in H_{(m_0+1)}$

$$J^{+}\psi(x) = \sum_{p=1}^{n} \left\{ \left(\frac{\partial}{\partial \lambda} \right)^{m_p - 1} \left[b_p(\lambda) \left[U(x,\lambda) U(\psi,\lambda) \right] \right] \right\}_{\lambda=\lambda_p} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} F^{+} \left\{ U\left(x,\lambda\right) U\left(\psi,\lambda\right) \right\} d\lambda \tag{4.42}$$

where

$$\left\{ \left(\frac{d}{d\lambda} \right)^{m_p - 1 - \beta} b_p(\lambda) \right\}_{\lambda=\lambda_p} = \begin{cases} \frac{1}{2\pi i} \binom{m_p - 1}{\beta} \int\limits_{\Gamma_+} \frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} F_{p\beta}^{+}(\lambda)\, \mathrm{d}\lambda & p = 1, .., n \\ -\frac{1}{2\pi i} \binom{m_p - 1}{\beta} \int\limits_{\Gamma_-} \frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} F_{p\beta}^{-}(\lambda)\, \mathrm{d}\lambda & p = n+1, .., k \end{cases} \tag{4.43}$$

Theorem 6. *Under the condition (1.4) the following two-fold spectral expansion in terms of the principal functions of L holds,*

$$\begin{aligned} \psi(x) \quad = \quad & \sum_{i=1}^{p} \left\{ \left(\frac{\partial}{\partial \lambda} \right)^{m_i^{+} - 1} \left[a_i(\lambda) \left[U(x,\lambda) U(\psi,\lambda) \right] \right] \right\}_{\lambda=\lambda_i} \\ & + \sum_{i=p+1}^{n} \left\{ \left(\frac{\partial}{\partial \lambda} \right)^{m_p - 1} \left[a_p(\lambda) \left[U(x,\lambda) U(\psi,\lambda) \right] \right] \right\}_{\lambda=\lambda_p} \\ & + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^{+}(0,\lambda)}{E^{+}(\lambda)} F^{+} \left\{ U(x,\lambda) U(\psi,\lambda) \right\} \mathrm{d}\lambda \\ & - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} F^{-} \left\{ U(x,\lambda) U(\psi,\lambda) \right\} \mathrm{d}\lambda \end{aligned} \tag{4.44}$$

$$
\begin{aligned}
0 \;=\; & \sum_{i=1}^{p} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_i^+ -1} \left[b_i(\lambda)\left[U(x,\lambda)U(\psi,\lambda)\right]\right] \right\}_{\lambda=\lambda_i} \\
& + \sum_{i=p+1}^{n} \left\{ \left(\frac{\partial}{\partial\lambda}\right)^{m_p -1} \left[b_p(\lambda)\left[U(x,\lambda)U(\psi,\lambda)\right]\right] \right\}_{\lambda=\lambda_p} \qquad (4.45) \\
& + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e^{+}(0,\lambda)}{E^{+}(\lambda)} F^{+}\left\{U(x,\lambda)U(\psi,\lambda)\right\} \mathrm{d}\lambda \\
& - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\lambda e_x^{-}(0,\lambda)}{E^{-}(\lambda)} F^{-}\left\{U(x,\lambda)U(\psi,\lambda)\right\} \mathrm{d}\lambda
\end{aligned}
$$

for every function $\psi \in H_{(m_0+1)}$. *The integrals in* (4.44) *and* (4.45) *converge in the norm of* $H_{-(m_0+1)}$ *where* a_i, b_i, F, a_p *and* b_p *defined by* (4.8), (4.9), (4.20), (4.21), (4.37), *and* (4.43) *respectively.*

Proof. We obtain (4.44) and (4.45) for $\psi \in C_0^\infty(R_+) \subset H_{(m_0+1)}$, by use of (4.6), (4.7), (4.22), (4.23) and (4.38)-(**??**). The convergence of the integrals appearing in (4.44) and (4.45) in the norm of $H_{-(m_0+1)}$, has been given in Theorem 4 and Theorem 5. As $C_0^\infty(R_+)$ is dense in $H_{(m_0+1)}$, the proof is finished. □

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