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Iterative Algorithm for extended mixed equilibrium problem

Rais Ahmad[1], *Haider Abbas Rizvi*[1] *and Adem Kilicman*[2]

[1]Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[2]Department of Mathematics and Institute for Mathematical Research, University of Putra Malaysia, 43400, Selangor, Malaysia

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Abstract: In this paper, we introduce and study an extended mixed equilibrium problem by using auxiliary principle technique. A generalized predictor-corrector iterative algorithm is defined for solving extended mixed equilibrium problem. The convergence of the method mentioned requires some condition $(^*)$, g-relatively relaxed Lipschitz continuity and relatively g-relaxed monotonicity of the mappings.

Keywords: Equilibrium, algorithm, relatively-relaxed, auxiliary principle, mapping.

1 Introduction

An important generalization of the variational inequality problem is an equilibrium problem. It has been shown that the equilibrium problem provides a natural, novel and unified framework to study a wide class of problems arising in nonlinear analysis, optimization, economics, finance and game theory. The equilibrium problem includes many mathematical problems as particular cases such as mathematical programming problems, complementarity problem, variational inequality problems, fixed point problems, minimax inequality problems, Nash equilibrium problems in non-cooperative games, etc., see [1,2,4,6,11].

It is well known that there are many numerical methods including projection method, resolvent operator technique, Wiener-Hopf equation, extragradient and descent methods for solving different variational inequality (inclusion) problems. Since it is impossible to find the projection in case of equilibrium problems, thus the above mentioned methods can not be applied. To over come this difficulty Noor [8] and Ding [9] has used the auxiliary principle technique to suggest some predictor-corrector iterative algorithm for solving some equilibrium problems. In this paper, we introduce and study an extended mixed equilibrium problem which involves single as well as multi-valued mappings by using auxiliary principles technique. By using condition $(^*)$, relatively g-relaxed Lipschitz continuity and relatively g-relaxed monotonocity of the mappings, we define a generalized predictor-corrector iterative algorithm to approximate the solutions of extended mixed equilibrium problem. Finally, convergence analysis is discussed.

2 Preliminaries

Let H be a real Hilbert space equipped with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $CB(H)$ be the family of all nonempty bounded closed subsets of H. Let $A,B : H \to CB(H)$ be the multi-valued mappings and $P,G,g : H \to H$ be the single

* Corresponding author e-mail: raisain_123@rediffmail.com

valued mappings. Let $F : H\times H \to H$ be the mappings and K be the closed convex subset of H. we introduce the following extended mixed equilibrium problem:

Find $x,y \in H, g(x) \in K, u \in A(x), v \in B(x)$ such that

$$F(P(u)-G(v),g(y)-g(x)) \geq 0,\ \forall\, g(y) \in\ K. \tag{1}$$

If P is an identity mapping, $G,B \equiv 0$ and $g(y)-g(x) = g(y)$, then problem (1) reduced to the problem of finding $x \in H, u \in A(x)$ such that

$$F(u,g(y)) \geq 0,\ \forall\, g(y)\ \in\ K. \tag{2}$$

Problem (2) is introduced and studied by Noor [8]. If A is single-valued mapping, then problem (2) is equivalent to finding $x \in H$

$$F(A(x),g(y)) \geq 0,\ \forall\, g(y)\ \in\ K. \tag{3}$$

Problem (3) is called general equilibrium problem.

If $A = I = g$, the identity mappings, then the problem (3) encounters with the original equilibrium problem introduced and studied by Blum and Oettli [2] and Noor and Oettli [7].

If $F(x,y) = \langle x,y\rangle,\ \forall\, x,y\ \in\ H$, then from problem (1), we can obtain many variational inequality problems studied in recent past. It is clear that for appropriate choices of mappings involved in the formulation of extended mixed equilibrium problem (1), one can obtain many existing equilibrium problems, variational inequality problems and complementarity problems.

Definition 1. *Let $A,B :\to CB(H)$ be the multi-valued mappings, $P,G,g : H \to H$ be the single-valued mappings and K a closed convex subset of H. Then*

(i) *P is said to be relatively g-relaxed Lipschitz continuous if there is a constant $k > 0$ such that*

$$\langle P(u_1)-P(u_2),g(x)-g(y)\rangle \leq -k\|g(z)-g(y)\|^2, \quad \forall\, u_1 \in A(z), u_2 \in A(x), g(x), g(y), g(z) \in K.$$

(ii) *G is said to be relatively g-relaxed monotone if there is a constant $c > 0$ such that*

$$\langle G(v_1)-G(v_2),g(y)-g(x)\rangle \geq -c\|g(z)-g(x)\|^2, \quad \forall\, v_1 \in B(z),\ v_2 \in B(x),\ g(x),g(y),g(z)\ \in K.$$

We use the following condition for the proof of our results.

Condition$(*)$:

$$\begin{aligned} &F(P(u_1)-G(v_1),g(x)-g(y))+F(P(u_2)-G(v_2),g(y)-g(x)) \\ &\leq \langle P(u_1)-P(u_2),g(x)-g(y)\rangle - \langle G(v_1)-G(v_2),g(y)-g(x)\rangle + \alpha\|g(z)-g(y)\|^2, \end{aligned}$$

for some $\alpha > 0$, $u_1 \in A(z),\ u_2 \in A(x),\ v_1 \in B(z),\ v_2 \in B(x),\ g(x),g(y),g(z) \in K$.

Lemma 1. [10] *Let X and Y be the topological spaces and $T : X \to 2^X$ be an upper semi-continuous, multi-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a sequence in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there exist a $y_0 \in T(x_o)$ and a subsequence $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

3 Iterative algorithm and existence result

We begin this section by defining a generalized predictor-corrector iterative algorithm for solving the problem (1).

For given $x,y \in H, g(x) \in K$, $u \in A(x), v \in B(x)$, consider the following corresponding auxiliary extended mixed variational inequality problem:

Find $\hat{x} \in H, g(\hat{x}) \in K$ such that

$$\langle g(\hat{x}) - g(x), g(y) - g(\hat{x})\rangle + \rho F(P(u) - G(v), g(y) - g(\hat{x})) \geq 0, \ \forall g(y) \ \in K, \tag{4}$$

where $\rho > 0$ is a constant.

If $\hat{x} = x$, then the problem (4) is nothing but problem (1). This enables us to define the following generalized predictor-corrector algorithm for solving extended mixed equilibrium problem (1).

3.1 Iterative Algorithm

For a given $x_0, y_0 \in H$, $g(x_0) \in K$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $0 < \varepsilon < 1$, compute the approximate solution (x_n, u_n, v_n) of problem (1) by the following schemes:

$$\langle g(y_n) - g(x_n), g(y) - g(y_n)\rangle + \mu F(P(u_n) - G(v_n), g(y) - g(y_n)) \geq 0, \quad \forall\, g(y) \ \in \ K. \tag{5}$$

$$\langle g(z_n) - g(y_n), g(y) - g(z_n)\rangle + \beta F(P(a_n) - G(b_n), g(y) - g(z_n)) \geq 0, \quad \forall\, g(y) \ \in \ K. \tag{6}$$

$$\langle g(x_{n+1}) - g(z_n), g(y) - g(x_{n+1})\rangle + \rho F(P(e_n) - G(f_n), g(y) - g(x_{n+1})) \geq 0, \quad \forall g(y) \in K, \tag{7}$$

where

$$\begin{aligned}
&u_n \in A(x_n), \ \|u_{n+1} - u_n\| \leq D(A(x_{n+1}), A(x_n)) + \varepsilon\|x_{n+1} - x_n\|,\\
&v_n \in B(x_n), \ \|v_{n+1} - v_n\| \leq D(B(x_{n+1}), B(x_n)) + \varepsilon\|x_{n+1} - x_n\|,\\
&a_n \in A(y_n), \ \|a_{n+1} - a_n\| \leq D(A(y_{n+1}), A(y_n)) + \varepsilon\|y_{n+1} - y_n\|,\\
&b_n \in B(y_n), \ \|b_{n+1} - b_n\| \leq D(B(y_{n+1}), B(y_n)) + \varepsilon\|y_{n+1} - y_n\|,\\
&e_n \in A(z_n), \ \|e_{n+1} - e_n\| \leq D(A(z_{n+1}), A(z_n)) + \varepsilon\|z_{n+1} - z_n\|,\\
&f_n \in B(z_n), \ \|f_{n+1} - f_n\| \leq D(B(z_{n+1}), B(z_n)) + \varepsilon\|z_{n+1} - z_n\|,
\end{aligned} \tag{8}$$

where $\mu, \beta, \rho > 0$ are constants and D is the Hausdorff metric on $CB(H)$.

The following lemma is useful for our convergence analysis.

Lemma 2. *Let (x,u,v) be an exact solution of extended mixed equilibrium problem (1) and $\{x_n\}, \{u_n\}$ and $\{v_n\}$ be the approximate sequences generated by Algorithm (**??**). Suppose that P is relatively g-relaxed Lipschitz continuous with constant $k > 0$, G is relatively g-relaxed monotone with constant $c > 0$, and condition $(^*)$ is satisfied. Then the following* *inequalities hold:*

$$\|g(x_{n+1})-g(x_n)\|^2 \leq \|g(x_n)-g(x)\|^2-[1-2\rho(\alpha+c-k)]\|g(x_{n+1})-g(z_n)\|^2, \tag{9}$$

$$\|g(z_n)-g(x)\|^2 \leq \|g(z_{n-1})-g(x)\|^2-[1-2\beta(\alpha+c-k)]\|g(z_n)-g(y_n)\|^2, \tag{10}$$

$$\|g(y_n)-g(x_n)\|^2 \leq \|g(y_{n-1})-g(x)\|^2-[1-2\mu(\alpha+c-k)]\|g(y_n)-g(x_n)\|^2. \tag{11}$$

Proof. Let (x,u,v) be the solution of extended mixed equilibrium problem (1).i.e., $x \in H$, $u \in A(x), v \in B(x), g(x) \in K$ such that

$$\mu F(P(u)-G(v),g(y)-g(x)) \geq 0,\ \forall\, g(y)\ \in K, \tag{12}$$

$$\beta F(P(u)-G(v),g(y)-g(x)) \geq 0,\ \forall\, g(y)\ \in K, \tag{13}$$

$$\rho F(P(u)-G(v),g(y)-g(x)) \geq 0,\ \forall\, g(y)\ \in K. \tag{14}$$

Putting $y = x_{n+1}$ in (3.11) and $y = x$ in (7) we have

$$\rho F(P(u)-G(v),g(x_{n+1})-g(x)) \geq 0, \forall\, g(x)\ \in\ K, \tag{15}$$

$$\langle g(x_{n+1})-g(z_n),g(x)-g(x_{n+1})\rangle+\rho F(P(e_n)-G(f_n),g(x)-g(x_{n+1})) \geq 0. \tag{16}$$

Adding (15) and (16) we have

$$\langle g(x_{n+1})-g(z_n),g(x)-g(x_{n+1})\rangle \geq -\rho\left[F(P(e_n)-G(f_n),g(x)-g(x_{n+1}))+F(P(u)-G(v),g(x_{n+1})-g(x))\right]. \tag{17}$$

Using condition $(^*)$ we have

$$\begin{aligned} &F(P(e_n)-G(f_n),g(x)-g(x_{n+1}))+F(P(u)-G(v),g(x_{n+1})-g(x)) \\ &\leq \langle P(e_n)-P(u),g(x)-g(x_{n+1})\rangle-\langle G(f_n)-G(v),g(x_{n+1})-g(x)\rangle+\alpha\|g(z_n)-g(x_{n+1})\|^2. \end{aligned} \tag{18}$$

Since P is relatively g-relaxed Lipschitz continuous and G is relatively g-relaxed monotone, (18) becomes

$$\begin{aligned} &F(P(e_n)-G(f_n),g(x)-g(x_{n+1}))+F(P(u)-G(v),g(x_{n+1})-g(x)) \\ &\leq -k\|g(z_n)-g(x_{n+1})\|^2+c\|g(z_n)-g(x_{n+1})\|^2+\alpha\|g(z_n)-g(x_{n+1})\|^2=(\alpha+c-k)\|g(z_n)-g(x_{n+1})\|^2. \end{aligned} \tag{19}$$

Using (19), (17) becomes

$$\langle g(x_{n+1})-g(z_n),g(x)-g(x_{n+1})\rangle \geq -\rho(\alpha+c-k)\|g(z_n)-g(x_{n+1})\|^2. \tag{20}$$

Since

$$\begin{aligned} \|g(x)-g(z_n)\|^2 &= \|g(x)-g(x_{n+1})+g(x_{n+1})-g(z_n)\|^2 \\ &= \|g(x)-g(x_{n+1})\|^2+\|g(x_{n+1})-g(z_n)\|^2+2\langle g(x_{n+1})-g(z_n),g(x)-g(x_{n+1})\rangle. \end{aligned}$$

It follows from (20) that

$$\langle g(x_{n+1})-g(z_n),g(x)-g(x_{n+1})\rangle = \frac{1}{2}\left[\|g(x)-g(z_n)\|^2-\|g(x_{n+1})-g(x)\|^2-\|g(x_{n+1})-g(z_n)\|^2\right]$$
$$\geq -2\rho(\alpha+c-k)\|g(x_{n+1})-g(z_n)\|^2.$$

Therefore, we have

$$\|g(x_{n+1})-g(x)\|^2 \leq \|g(z_n)-g(x)\|^2-[1-2\rho(\alpha+c-k)]\|g(x_{n+1})-g(z_n)\|^2$$
$$\leq \|g(z_n)-g(x)\|^2, \tag{21}$$

for $\rho < \frac{1}{2(\alpha+c-k)}$ and $(\alpha+c) > k$. Putting $y = z_n$ in (3.10) and $y = x$ in (6), we have

$$\beta F(P(u)-G(v),g(z_n)-g(x)) \geq 0, \tag{22}$$

$$\langle g(z_n)-g(y_n),g(x)-g(z_n)\rangle + \beta F(P(a_n)-G(b_n),g(x)-g(z_n)) \geq 0. \tag{23}$$

Adding (22) and 23 we have

$$\langle g(z_n)-g(y_n),g(x)-g(z_n)\rangle \geq -\beta\left[F(P(a_n)-G(b_n),g(x)-g(z_n))+F(P(u)-G(v),g(z_n)-g(x))\right]. \tag{24}$$

Using the condition $(^*)$ we have

$$F(P(a_n)-G(b_n),g(x)-g(z_n))+F(P(u)-G(v),g(z_n)-g(x))$$
$$\leq \langle P(a_n)-P(u),g(x)-g(z_n)\rangle - \langle G(b_n)-G(v),g(z_n)-g(x)\rangle + \alpha\|g(z_n)-g(y_n)\|^2. \tag{25}$$

Since P is relatively g-relaxed Lipschitz continuous and G is relatively g-relaxed monotone, (25) becomes

$$F(P(a_n)-G(b_n),g(x)-g(z_n))+F(P(u)-G(v),g(z_n)-g(x))$$
$$\leq -k\|g(z_n)-g(y_n)\|^2+c\|g(z_n)-g(y_n)\|^2+\alpha\|g(z_n)-g(y_n)\|^2$$
$$=(\alpha+c-k)\|g(z_n)-g(y_n)\|^2. \tag{26}$$

Using (26) ,(24) becomes

$$\langle g(z_n)-g(y_n),g(x)-g(z_n)\rangle \geq -2\beta(\alpha+c-k)\|g(z_n)-g(y_n)\|^2. \tag{27}$$

Since

$$\|g(x)-g(y_n)\|^2 = \|g(x)-g(z_n)+g(z_n)-g(y_n)\|^2$$
$$= \|g(x)-g(z_n)\|^2+\|g(z_n)-g(y_n)\|^2+2\langle g(x)-g(z_n),g(z_n)-g(y_n)\rangle.$$

It follows from (27) that

$$\langle g(z_n)-g(y_n),g(x)-g(z_n)\rangle = \frac{1}{2}\left[\|g(x)-g(y_n)\|^2-\|g(x)-g(z_n)\|^2-\|g(z_n)-g(y_n)\|^2\right]$$
$$\geq -2\beta(\alpha+c-k)\|g(z_n)-g(y_n)\|^2.$$

Therefore, we have

$$\|g(z_n)-g(x)\|^2=\|g(y_n)-g(x)\|^2-(1-2\beta(\alpha+c-k))\|g(z_n)-g(y_n)\|^2\leq\|g(y_n)-g(x)\|^2, \tag{28}$$

for $\beta<\frac{1}{2(\alpha+c+k)}$ and $(\alpha+c)>k$. Putting $y=y_n$ in 22 and $y=x$ in (5), we have

$$\mu F(P(u)-G(u),g(y_n)-g(x))\geq 0. \tag{29}$$

$$\langle g(y_n)-g(x_n),g(x)-g(y_n)\rangle+\mu F(P(u)-G(u),g(x)-g(y_n))\geq 0. \tag{30}$$

Adding (29) and (30), we have

$$\langle g(y_n)-g(x_n),g(x)-g(y_n)\rangle\geq-\mu\left[F(P(u)-G(u),g(x)-g(y_n))+F(P(u)-G(u),g(y_n)-g(x))\right]. \tag{31}$$

Using condition $(^*)$, we have

$$F(P(u_n)-G(v_n),g(x)-g(y_n))+F(P(u)-G(v),g(y_n)-g(x))$$
$$\leq\langle P(u_n)-P(u),g(x)-g(y_n)\rangle-\langle G(v_n)-G(v),g(y_n)-g(x)\rangle+\alpha\|g(y_n)-g(x_n)\|^2. \tag{32}$$

Since P is relatively g-relaxed Lipschitz continuous and G is relatively g-relaxed monotone, (32) becomes

$$F(P(u_n)-G(v_n),g(x)-g(y_n))+F(P(u)-G(v),g(y_n)-g(x))$$
$$\leq -k\|g(y_n)-g(x_n)\|^2+c\|g(y_n)-g(x_n)\|^2+\alpha\|g(y_n)-g(x_n)\|^2$$
$$=(\alpha+c-k)\|g(y_n)-g(x_n)\|^2. \tag{33}$$

Using (33), (31) becomes

$$\langle g(y_n)-g(x_n),g(x)-g(y_n)\rangle\geq-2\mu(\alpha+c-k)\|g(y_n)-g(x_n)\|^2. \tag{34}$$

Since

$$\|g(x_n)-g(x)\|^2=\|g(x)-g(y_n)+g(y_n)-g(x_n)\|^2$$
$$=\|g(x)-g(y_n)\|^2+\|g(y_n)-g(x_n)\|^2+2\langle g(x)-g(y_n),g(y_n)-g(x_n)\rangle. \tag{35}$$

It follows from (34) that

$$\langle g(y_n)-g(x_n),g(x)-g(y_n)\rangle = \frac{1}{2}[\|g(x)-g(y_n)\|^2-\|g(x)-g(y_n)\|^2-\|g(y_n)-g(x_n)\|^2]$$
$$\geq 2\mu(\alpha+c-k)\|g(y_n)-g(x_n)\|^2.$$

Therefore, we have

$$\|g(y_n)-g(x)\|^2 \leq \|g(x_n)-g(x)\|^2-[1-2\mu(\alpha+c-k)]\|g(y_n)-g(x)\|^2 \leq \|g(x_n)-g(x)\|^2, \tag{36}$$

for $\mu < \frac{1}{2(\alpha+c-k)}, (\alpha+c)>k$. From (21), (28) and (36), the inequalities (9), (10) and (11) hold.

Theorem 1. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $A,B: H \to CB(H)$ be upper semi continuous, compact valued and D-continuous multi-valued mappings with t as D-continuous constant of A and γ as D-continuous constant of B. Let $g: H \to H$ be a single valued continuous mapping such that g is one-to-one. Let $F: H\times H \to H$ be a continuous mapping such that condition $(^*)$ is satisfied. Let $P: H \to H$ is relatively g-relaxed Lipschitz continuous with constant $k>0$ and $G: H\to H$ is relatively g-relaxed monotone with constant $c>0$. Then for any given $x_o \in H, g(x_o)\in K, u_o \in A(x_o), v_o \in B(x_o)$, the iterative sequences $\{x_n\},\{u_n\}$and$\{v_n\}$ generated by the Algorithm 3.1 with*

$$\alpha>0, 0<\rho,\mu,\beta<\frac{1}{2(\alpha+c-k)}, \ (\alpha+c)>k,$$

converge strongly to a solution (x_o,u_o,v_o) of extended mixed equilibrium problem (1).

Proof. Let (x,u,v) be a solution of extended mixed equilibrium problem (1). Since $\alpha,c,k>0$ and

$$\alpha>0, 0<\rho,\mu,\beta<\frac{1}{2(\alpha+c-k)}, \ (\alpha+c)>k,$$

it follows from the inequalities (9) to (11) that the sequences $\{\|g(x_n)-g(x)\|\}$, $\{\|g(z_n)-g(x)\|\}$, $\{\|g(y_n)-g(x)\|\}$ are non-increasing and consequently $\{x_n\}$, $\{z_n\}$ and $\{v_n\}$ are bounded sequences. Also we have

$$\sum_{n=0}^{\infty}[1-2\rho(\alpha+c-k)]\|g(x_{n+1})-g(z_n)\|^2 \leq \|g(x_o)-g(x)\|^2,$$

$$\sum_{n=o}^{\infty}[1-2\mu(\alpha+c-k)]\|g(y_n)-g(x_n))\|^2 \leq \|g(y_o)-g(x)\|^2,$$

$$\sum_{n=0}^{\infty}[1-2\beta(\alpha+c-k)]\|g(z_n)-g(y_n)\|^2 \leq \|g(z_o)-g(x)\|^2.$$

The above inequalities imply that

$$\lim_{n\to\infty}\|g(x_{n+1})-g(z_n)\|^2=0,$$

$$\lim_{n\to\infty}\|g(y_n)-g(x_n)\|^2=0,$$

$$\lim_{n\to\infty}\|g(z_n)-g(y_n)\|^2=0.$$

Thus, we have

$$\|g(x_{n+1})-g(x_n)\| \leq \|g(x_{n+1})-g(z_n)\|+\|g(z_n)-g(y_n)\|+\|g(y_n)-g(x_n)\|$$
$$\to 0,\ as\ n\to\infty.$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences and g is one-to-one, we have subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ such that $\{x_{n_j}\}\to x_o$ and $\{y_{n_j}\}\to x_o$ and $g(x_{n_j})\to g(x_o), g(y_{n_j})\to g(x_o)$ where$g(x_o)\in K$. As A,B are upper semi continuous and compact valued, by Lemma 2.1, there exist subsequences $\{u_{n_{ij}}\}$ of $\{u_{n_i}\}$ and $\{v_{n_{ij}}\}$ of $\{v_{n_i}\}$ such that $u_{n_{ij}}\to u_o$ and $v_{n_{ij}}\to v_o$, $u_o\in T(x_o)$ and $v_o\in B(x_o)$. By (5), we have

$$\langle g(y_{n_{ij}})-g(x_{n_j}), g(y)-g(y_{n_{ij}})\rangle+\mu F(P(u_{n_{ij}})-G(v_{n_{ij}}), g(y)-g(y_{n_{ij}}))\geq 0, \quad \forall\, g(y)\ \in\ K. \tag{37}$$

By the continuity of F,P,G and g, letting $j\to\infty$ in (37), we have

$$F(P(u_o)-G(v_o), g(y)-g(x_o))\geq 0,\ \forall\, g(y)\ \in\ K,$$

i.e., (x_o,u_o,v_o) is a solution of extended mixed equilibrium problem (1). It follows from Algorithm 3.1, that

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq D(A(x_{n+1}),A(x_o))+\varepsilon\|x_{n+1}-x_n\|\\ &\leq t\|x_{n+1}-x_n\|+\varepsilon\|x_{n+1}-x_n\|\\ &=(t+\varepsilon)\|x_{n+1}-x_n\|\to 0,\ as\ n\to\infty.\end{aligned}$$

It remains to show that $u\in A(x)$. In fact $u_n\in A(x)$ and

$$\begin{aligned}d(u,A(x)) &\leq \max\{d(u_n,A(x)), \sup_{w_1\in A(x)} d(A(x_n),w_1)\}\\ &\leq \max\{\sup_{w_2\in A(x)} d(A(w_2),x_n), \sup_{w_1\in A(x)} d(A(x_n),w_1)\}\\ &= D(A(x_n),A(x)).\end{aligned}$$

Then we have

$$\begin{aligned}d(u,A(x)) &\leq \|u-u_n\|+d(u_n,A(x))\\ &\leq \|u-u_n\|+D(A(x_n),A(x))\\ &\leq \|u-u_n\|+\|x_n-x\|\to 0,\ as\ x\to\infty,\end{aligned}$$

which implies that $d(u,A(x))=0$. Since $A(x)\in CB(H)$, it follows that $u\in A(x)$. Similarly, we can show that $v\in B(x)$. This completes the proof.

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