Available online: February 13, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 2, Pages 1335–1340 (2019)
DOI: 10.31801/cfsuasmas.526518
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

VERTEX MAGIC TOTAL LABELING OF MIDDLE AND TOTAL GRAPH OF CYCLE

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Abstract. A vertex magic total labeling is a bijection from the union of the vertex set and edge set to the consecutive integers $1, 2, 3, ..., v + e$ with the property that for every u in the vertex set, the sum of the label of u and the label of the edges incident with u is equal to k, for some constant k. In this paper, we establish the vertex magic labeling of some classes of graphs and provide some open problems related to it.

1. Introduction and Preliminaries

Let G be a finite, undirected graph with no loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph(Michalak,1981) of G denoted by $M(G)$, is defined as follows. The vertex set of $M(G)$ is $V(G) \bigcup E(G)$. Two vertices v, w in the vertex set of $M(G)$ are adjacent in $M(G)$ if one of the following holds.

1. v, w are in $V(G)$ and v is adjacent to w in G.
2. v is in $V(G)$ and w is in $E(G)$ and v, w are incident in$G$.

The total graph (Michalak,1981; Harary 1969) of G, denoted by $T(G)$, is defined as follows. The vertex set of $T(G)$ is $V(G) \bigcup E(G)$ and any two vertices v, w in this set are adjacent in $T(G)$ if one of the following holds.

1. v, w are in $V(G)$ and v is adjacent to w in G.
2. v, w are in $E(G)$ and v,w are adjacent in G.
3. v is in $V(G)$, w is in $E(G)$ and v, w are incident in G.

Graph labeling is one of the most interesting areas of graph theory with wide range of applications. Graph labeling was first introduced in the year of 1960. A labeling of graph G is a mapping that carries a set of graph elements usually

Received by the editors: February 05, 2018; Accepted: June 25, 2018.

2010 *Mathematics Subject Classification.* 05C78 .

Key words and phrases. Graph labeling, middle graph, total graph,vertex magic graph.

Submitted via International Conference on Current Scenario in Pure and Applied Mathematics [ICCSPAM 2018].

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integers.A different kind of labelings have been studied and an excellent survey of graph labeling can be found in Gallian(2013).

A magic graph is a graph whose edges are labeled by positive integers, so that the sum over the edges incident with any vertex is the same, independent of the choice of vertex.

A graph is vertex magic if its vertices can be labeled so that the sum on any edge is the same. It is total magic if its edges and vertices can be labeled so that the vertex label plus the sum of labels on edges incident with that vertex is a constant.

Sedlacek(1963) introduced the concept of magic labeling. suppose that G is a graph with e edges, we say that G is magic if the edges of G can be labeled by the numbers $1, 2, 3, ..., e$ so that the sum of labels of all the edges incident with any vertex is same. Macdougall et.al.(2002) introduced the notion of vertex magic total labeling. If G is a finite simple undirected graph with v vertices and e edges, then a vertex magic total labeling is a bijection f from $V(G) \bigcup E(G)$ to the integers $1, 2, 3, ..., v+e$ with the property that for every u in $V(G)$

$$f(u)+\sum_{v\in N(u)} f(u,v) = k, \text{ for some constant } k.$$

In the following section, we find the vertex magic number for middle and total graph of cycle graph. In order to prove our results we shall use the following lemma and theorems by Daisy Cunningham et al.[2]

Lemma 1.1. *[2] If G is a vertex magic graph with v vertices and e edges, then*

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k.$$

Theorem 1.2. *Let G be a graph with v vertices and e edges. If G is a vertex magic graph, then the magic number k, is bounded such that*

$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leqslant k \leqslant e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}$$

Corollary 1.1. *[2] Let G be a cycle graph with v vertices. If G is a vertex magic graph, then the magic number k, is bounded such that*

$$\frac{5}{2}v + \frac{3}{2} \leqslant k, \text{where } v \text{ is odd.}$$

and

$$\frac{5}{2}v + 2 \leqslant k, \text{ where } v \text{ is even.}$$

Corollary 1.2. *[2] Let G be a graph with v vertices. If G is a vertex magic graph, then the magic number k, is bounded such that*

$$k \leqslant \frac{7}{2}v + \frac{3}{2}, \text{ where } v \text{ is odd}$$

and

$$k \leqslant \frac{7}{2}v + 1, \text{ where } v \text{ is even}$$

2. Main Results

Lemma 2.1. *For $n \geqslant 3$, If $M[C_n]$ is a vertex magic graph, then the magic number k is*

$$\frac{5}{4}(v+e+1) + \frac{E_{sum}}{v}.$$

Proof. Let C_n be a cycle graph with n vertices and n edges and let $M[C_n]$ be a middle graph of cycle with $2n$ (=v) vertices and $3n$ (=e) edges.

If G is a vertex magic graph, then

$$V_{sum} + 2E_{sum} = kv$$

(i.e) $T_{sum} + E_{sum} = kv$.

The labels of a graph are 1,2,3,...,v,v+1,..,v+e. Then total sum is

$$T_{sum} = \frac{(v+e)(v+e+1)}{2}$$
$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k$$

by observation, the number of vertices of $M[C_n]$ is twice of vertices in C_n and the number of edges in $M[C_n]$ is thrice of edges in C_n. □

$$\frac{(2n+3n)(2n+3n+1)}{4n} + \frac{E_{sum}}{2n} = k$$

(i.e) $\frac{5}{4}(v+e+1) + \frac{E_{sum}}{v} = k$.

Theorem 2.2. *For $n \geqslant 3$, If $M[C_n]$ is a vertex magic graph, then the magic number k is bounded and such that*

$$\frac{3(e+1)+5(v+e+1)}{4} \leqslant k \leqslant e + \frac{3(e+1)+5(v+e+1)}{4}.$$

Proof. From lemma (2.1), the magic constant k is

$$k = \frac{5}{4}(v+e+1) + \frac{E_{sum}}{v}$$

(i.e) $E_{sum} = kv - \frac{5v}{4}(v+e+1)$

The minimum edge sum E_{sum} occurred when numbers $1, 2, 3, ..., v+n$ are assigned to edges,

$$\begin{aligned} 1+2+.....+(v+n) &\leqslant E_{sum} \\ \frac{(v+n)(v+n+1)}{2} &\leqslant E_{sum} \end{aligned}$$

In middle graph of C_n, the number of vertices is $2n$,

$$\frac{(3n)(3n+1)}{2} \leqslant E_{sum} \tag{2.1}$$

The maximum edge sum E_{sum} occurred when numbers $v+1, v+2, ..., v+e$ are assigned to the edges.

$$E_{sum} \leqslant \sum_{i=1}^{e} v+i$$

$$E_{sum} \leqslant 6n^2 + \frac{3n(3n+1)}{2} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\frac{3n(3n+1)}{2} \leqslant E_{sum} \leqslant \frac{3n(3n+1)}{2}$$

(i.e) $\frac{3(e+1)+5(v+e+1)}{4} \leqslant k \leqslant e + \frac{3(e+1)+5(v+e+1)}{4}$ □

Corollary 2.1. *For $n \geqslant 3$, If $M[C_n]$ is a vertex magic graph, then the magic number k is bounded and below such that*

$$\frac{17}{4}v + \frac{5}{2} \leqslant k, \text{ where } v \text{ is even and } n \text{ is odd}$$

$$\frac{17}{4}v + 2 \leqslant k, \quad \text{where } v \text{ is even and } n \text{ is even}$$

Corollary 2.2. *For $n \geqslant 3$, If $M[C_n]$ is a vertex magic graph, then the magic number k is bounded and above such that*

$$k \leqslant \frac{23}{4}v + \frac{3}{2}, \text{ where } v \text{ is even and } n \text{ is odd}$$

$$k \leqslant \frac{23}{4}v + 2, \text{ where } v \text{ is even and } n \text{ is even}$$

Lemma 2.3. *For $n \geqslant 3$, If $T[C_n]$ is a vertex magic graph with v vertices and e edges, then the magic number k is*

$$\frac{3}{2}(v+e+1) + \frac{E_{sum}}{v}.$$

Proof. Let C_n be a cycle graph with n vertices and n edges and $T[C_n]$ be a total graph of cycle with $2n$(=v) vertices and $4n$(=e) edges. If G is a vertex magic graph, then

$$\begin{aligned} V_{sum} + 2E_{sum} &= kv \\ T_{sum} + E_{sum} &= kv \end{aligned}$$

The labels of a graph are 1,2,3,...,v,v+1,..,v+e. Then total sum is

$$T_{sum} = \frac{(v+e)(v+e+1)}{2}$$

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k$$

By observation, the number of vertices of $T[C_n]$ is twice of vertices in C_n and the number of edges in $T[C_n]$ is four times of edges in C_n. Therefore,

$$\frac{3}{2}(v+e+1) + \frac{E_{sum}}{v} = k.$$

□

Theorem 2.4. *For $n \geqslant 3$, If $T[C_n]$ is a vertex magic graph, then the magic number k is bounded and such that*

$$(e+1) + \frac{3}{2}(v+e+1) \leqslant k \leqslant e + (e+1) + \frac{3}{2}(v+e+1).$$

Proof. From lemma (2.3), the magic constant k is

$$k = \frac{3}{2}(v+e+1) + \frac{E_{sum}}{v}$$

(i.e) $E_{sum} = kv - \frac{3}{2}v(v+e+1)$. The minimum edge sum E_{sum} occurred when numbers $1, 2, 3, ..., (v+2n)$ are assigned to edges,

$$1 + 2 + ... + (v+2n) \leqslant E_{sum}$$
$$\frac{(v+2n)(v+2n+1)}{2} \leqslant E_{sum}$$

In total graph of C_n, the number of vertices is $2n$,

$$2n(4n+1) \leqslant E_{sum} \tag{2.3}$$

The maximum edge sum E_{sum} occurred when numbers $v+1, v+2, ..., v+e$ are assigned to the edges.

$$E_{sum} \leqslant \sum_{i=1}^{e} v + i$$
$$E_{sum} \leqslant 8n^2 + 3n(4n+1) \tag{2.4}$$

From (2.3) and (2.3), we have

$$2n(4n+1) \leqslant E_{sum} \leqslant 8n^2 + 2n(4n+1)$$

(i.e) $(e+1) + \frac{3}{2}(v+e+1) \leqslant k \leqslant e + (e+1) + \frac{3}{2}(v+e+1)$. □

Corollary 2.3. *For $n \geqslant 3$, If $T[C_n]$ is a vertex magic graph, then the magic number k is bounded and below such that $\frac{13}{2}v + 3 \leqslant k$, where v is even, n is odd and even.*

Corollary 2.4. *For $n \geqslant 3$, If $T[C_n]$ is a vertex magic graph, then the magic number k is bounded and above such that $k \leqslant \frac{17}{2}v + 2$, where v is even, n is odd and even.*

Conjecture 1. *For $n \geqslant 3$, If $M[C_n]$ and $T[C_n]$ is a vertex magic of odd cycle graph, then the vertex sum is of the form $[\frac{v(v+1)}{2} + n] + (i * v) = V_{sum}$, where $i = 0, 1, 2, ..., e-1$.*

Conjecture 2. *For $n \geqslant 3$, If $M[C_n]$ and $T[C_n]$ is a vertex magic of even cycle graph, then the vertex sum is of the form* $[\frac{v(v+1)}{2}] + (i * v) = V_{sum}$, *where* $i = 0, 1, 2, \ldots.., e$.

2.1. Open Problems.

(1) For $n \geqslant 3$ and n is even, then $M[C_n]$ is V-Super vertex magic graph or not. Justify?
(2) For $n \geqslant 3$ and n is even, then $M[C_n]$ is E-Super vertex magic graph or not. Justify?

3. Conclusion

This paper addresses labeling graphs in a such way that they are vertex magic. Exploring bounds for magic numbers is an interesting problem for all graphs. We are able to obtain more accurate bounds if we limit the graphs to be middle and total graph of cycle.Some interesting open problems that arise from this are E-Super and V-Super vertex magic of middle graph of even cycle.

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