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On M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds

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Abstract: Locally and globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds are studied. We show that a globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold is globally ϕ-symmetric. Some observations for a 3-dimensional locally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold are given. We also give an example of a 3-dimensional locally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold.

Keywords: (ε)-Kenmotsu manifold, 3-dimensional (ε)-Kenmotsu manifolds, ϕ-symmetric, M-projective curvature tensor, Einstein manifold, constant curvature.

1 Introduction

In 1969, Takahashi[15] introduced almost contact manifolds equipped with associated indefinite metrics. He studied Sasakian manifolds equipped with an associated indefinite metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also called as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds, respectively [1,7]. The concept of (ε)-Sasakian manifolds was introduced by Bejancu and Duggal [1] and further investigation was taken up by Xufeng and Xiaoli [17] and Rakesh Kumar et al [9]. The index of a metric plays significant roles in differential geometry on it generates variety of vector fields such as space-like, time-like, and light-like fields. In 1972, Kenmotsu [8] introduced a new class of almost contact manifolds which are now a days called as Kenmotsu manifolds. As our natural trend to study various types of contact manifolds with indefinite metric, De and Sarkar[6] introduced the concept of (ε)-Kenmotsu manifolds with indefinite metric and studied some interesting properties.

In 1971, Pokhariyal and Mishra [14] defined a tensor field M on a Riemannian manifold as

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{1}$$

Such a tensor field M is known as M-projective curvature tensor. In [10,11], Ojha studied some properties of M-projective curvature tensor in Sasakian manifolds and Kahler manifolds. He has also shown that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and H-projective curvature tensor on the other. From (1), we obtain

$$(\nabla_W M)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{1}{2(n-1)}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)]. \tag{2}$$

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The notion of local symmetry of Riemannian manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [16] introduced the notion of locally ϕ-symmetry on Sasakian manifolds. According to Takahashi, a Riemannian manifold is said to be locally ϕ-symmetric if it satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{3}$$

where X,Y,Z and W are horizontal vector fields which means that it is horizontal with respect to the connection form η of the local fibering; namely, a horizontal vector is nothing but a vector which is orthogonal to ξ. In (3), if X, Y, Z and W are not horizontal then we call the manifold is globally ϕ-symmetric. In the context of contact Geometry the notion of ϕ-symmetry was introduced and studied by Boeckx, Buecken and Vanhecke [3] with several examples. In [2], Blair, Koufogiorgos and Sharma studied locally ϕ-symmetric contact metric manifolds. The concept of ϕ-symmetry to Kenmotsu manifolds were studied in [4]. Later in [5], De, Ozgur and Mondal studied both locally and globally ϕ-quasiconformally symmetric Sasakian manifolds.

In this paper, we define locally M-projectively ϕ-symmetric and globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds. An (ε)-Kenmotsu manifold M is called $locally\ M-projectively\ \phi-symmetric$ if the condition

$$\phi^2((\nabla_W M)(X,Y)Z) = 0 \tag{4}$$

holds on M, where X, Y, Z and W are horizontal vectors. If X, Y, Z and W are arbitrary vectors then the manifold is called $globally\ M-projectively\ \phi-symmetric$.

The rest of the paper unfold as follows: Section 2 contains basic definitions of (ε)-Kenmotsu manifolds; Section 3 is devoted to the study of globally ϕ-symmetric (ε)-Kenmotsu manifolds; In this section we see that a globally ϕ-symmetric (ε)-Kenmotsu manifold is an indefinite space form. Section 4, we investigate the necessary and sufficient condition for a 3-dimensional (ε)-Kenmotsu manifold to be locally ϕ-symmetric; Section 5, provides some results on globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds; In section 6 we investigate that if an (ε)-Kenmotsu manifold is globally M-projectively ϕ-symmetric, then the manifold is an Einstein manifold. In addition, it is show that a globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold is globally ϕ-symmetric and hence is an indefinite space form; and finally section 7 provides two examples of a 3-dimensionallocally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds. Our results extend those obtain in [5] and [13] for the case of Sasakian manifolds and (ε)-para 3-Sasakian manifolds.

2 (ε)-Kenmotsu manifolds

Let M be an n-dimensional differential manifold endowed with an almost contact structure (ϕ,η,ξ), where ϕ is a tensor field of type $(1,1)$, η is a 1-form and ξ is a vector field on M satisfying

$$\phi^2 X = -X + \eta(X)\xi; \quad \eta(\xi) = 1 \quad \forall X \in \chi(M). \tag{5}$$

It follows that

$$\eta(\phi X) = 0; \quad \phi(\xi) = 0; \quad rank\phi = n-1, \tag{6}$$

then M is called an almost contact manifold. If there exists a semi-Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y) \quad \forall X,Y \in \chi(M), \tag{7}$$

with $\varepsilon = \pm 1$ then (ϕ, η, ξ, g) is called an (ε)-almost contact metric structure and M is known an (ε)-almost contact manifold. For an (ε)-almost contact manifold we also have

$$\eta(X) = \varepsilon g(X, \xi), \quad \forall X \in \chi(M) \tag{8}$$

$$\varepsilon = g(\xi, \xi). \tag{9}$$

Hence ξ is never a light like vector field on M. Here ε is 1 or -1 according as ξ is space like or time like vector field on M, and according to the casual character of ξ, we have two classes of (ε)-Kenmotsu manifolds. When $\varepsilon = -1$ and the index of g is an odd number ($\upsilon = 2s+1$), then M is time-like Kenmotsu manifold and M is a space-like Kenmotsu manifold when $\varepsilon = -1$ and $\upsilon = 2s$. For $\varepsilon = 1$ and $\upsilon = 0$, we obtain usual Kenmotsu manifold and for $\varepsilon = 1$ and $\upsilon = 1$, M is a Lorentz-Kenmotsu manifold.

If $d\eta(X,Y) = g(X, \phi Y)$ for every $X, Y \in \chi(M)$, then M is said to have (ε)-contact metric structure (ϕ, ξ, η, g). An (ε)-almost contact metric structure (ϕ, η, ξ, g) is (ε)-Kenmotsu if and only if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \varepsilon \eta(Y)\phi X, \quad \forall X, Y \in \chi(X) \tag{10}$$

where ∇ denotes the Levi-Civita connection with respect to g. Also one has

$$\nabla_X \xi = \varepsilon(X - \eta(X)\xi) \quad \forall X \in \chi(X). \tag{11}$$

Then for an (ε)-Kenmotsu manifold, we have following relations[6]

$$(\nabla_X \eta)(Y) = g(X,Y) - \varepsilon \eta(X)\eta(Y), \tag{12}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y), \tag{13}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{14}$$

If an (ε)-Kenmotsu manifold is a space of constant curvature then it is an indefinite space form.

3 Globally ϕ-symmetric (ε)-Kenmotsu manifolds

Let us suppose that an (ε)-Kenmotsu manifold is globally ϕ-symmetric. Then by virtue of (5) and (3) we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = 0. \tag{15}$$

This implies

$$-(\nabla_W R)(X,Y)Z + \varepsilon g((\nabla_W R)(X,Y)Z, \xi)\xi = 0. \tag{16}$$

Next, by using the property of curvature tensor we have

$$g((\nabla_W R)(X,Y)Z, \xi) = g(\nabla_W R(X,Y)Z, \xi) + g(R(X,Y)\xi, \nabla_W Z) + g(R(\nabla_W X, Y)\xi, Z) + g(R(X, \nabla_W Y)\xi, Z). \tag{17}$$

Since ∇ is a metric connection, it follows that

$$g(\nabla_W R(X,Y)Z, \xi) = g(R(X,Y)\nabla_W \xi, Z) - \nabla_W g(R(X,Y)\xi, Z) \tag{18}$$

and

$$\nabla_W g(R(X,Y)\xi,Z) = g(\nabla_W R(X,Y)\xi,Z) + g(R(X,Y)\xi,\nabla_W Z). \tag{19}$$

From (18) and (19), we have

$$g(\nabla_W R(X,Y)Z,\xi) = -g(\nabla_W R(X,Y)\xi,Z) - g(R(X,Y)\xi,\nabla_W Z) + g(R(X,Y)\nabla_W \xi,Z). \tag{20}$$

Using (20) in (17), we get

$$g((\nabla_W R)(X,Y)Z,\xi) = -g((\nabla_W R)(X,Y)\xi,Z). \tag{21}$$

Using (21), we obtain from (16) that

$$(\nabla_W R)(X,Y)Z = -\varepsilon g((\nabla_W R)(X,Y)\xi,Z)\xi. \tag{22}$$

Using (11) and (13), we have

$$(\nabla_W R)(X,Y)\xi = g(X,W)Y - g(Y,W)X - \varepsilon R(X,Y)W. \tag{23}$$

By taking account of (23) in (22), one can get

$$(\nabla_W R)(X,Y)Z = \{\varepsilon(g(Y,W)g(X,Z) - g(X,W)g(Y,Z)) + g(R(X,Y)W,Z)\}\xi. \tag{24}$$

Again, if (24) holds, then (21) and (23) implies that the manifold is globally ϕ-symmetric. Thus we can state the following.

Theorem 1. *An (ε)-Kenmotsu manifold is globally ϕ-symmetric if and only if the relation (24) holds for any vector fields X, Y, Z, W tangent to M.*

Next, putting $Z = \xi$ in (22) and using (21) we have

$$(\nabla_W R)(X,Y)\xi = 0, \tag{25}$$

for any vector fields X, Y, W on M. From (25) and (23) it follows that

$$R(X,Y)W = -\varepsilon\{g(Y,W)X - g(X,W)Y\}.$$

Thus the manifold is of constant curvature. This leads us to the following.

Theorem 2. *A globally ϕ-symmetric (ε)-Kenmotsu manifold is an indefinite space form.*

4 3-dimensional locally ϕ-symmetric (ε)-Kenmotsu manifolds

It is known that in a 3-dimensional Riemannian manifold

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X] \tag{26}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold. If we putting $Z = \xi$ in (26) and use (13) we get

$$\left(\frac{r}{2} + \varepsilon\right)(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY \tag{27}$$

By putting $Y=\xi$ in (27) and using (14) for $n=3$, we obtain

$$QX=\left(\frac{r}{2}+\varepsilon\right)X-\left(\frac{r}{2}+3\varepsilon\right)\eta(X)\xi, \tag{28}$$

that is,

$$S(X,Y)=\left(\frac{r}{2}+\varepsilon\right)g(X,Y)-\left(\frac{r}{2}+3\varepsilon\right)\varepsilon\eta(X)\eta(Y). \tag{29}$$

Thus from (28) and (29) in (26), we obtain

$$R(X,Y)Z=\left(\frac{r}{2}+2\varepsilon\right)[g(Y,Z)X-g(X,Z)Y]+\left(\frac{r}{2}+3\varepsilon\right)[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+\varepsilon\eta(X)\eta(Z)Y-\varepsilon\eta(Y)\eta(Z)X]. \tag{30}$$

By taking the covariant differentiation of (30) we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z=&\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y]+\frac{dr(W)}{2}[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\\&+\varepsilon\eta(X)\eta(Z)Y-\varepsilon\eta(Y)\eta(Z)X]+\left(\frac{r}{2}+3\varepsilon\right)[g(X,Z)(\nabla_W\eta)(Y)\xi+g(X,Z)\eta(Y)\nabla_W\xi\\&-g(Y,Z)(\nabla_W\eta)(X)\xi-g(Y,Z)\eta(X)\nabla_W\xi+\varepsilon(\nabla_W\eta)(X)\eta(Z)Y+\varepsilon(\nabla_W\eta)(Z)\eta(X)Y\\&-\varepsilon(\nabla_W\eta)(Y)\eta(Z)X-\varepsilon(\nabla_W\eta)(Z)\eta(Y)X].\end{aligned} \tag{31}$$

Now assume X,Y and Z are horizontal vector fields. So equation (31) becomes

$$(\nabla_W R)(X,Y)Z=\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y]+\left(\frac{r}{2}+3\varepsilon\right)[g(X,Z)(\nabla_W\eta)(Y)\xi-g(Y,Z)(\nabla_W\eta)(X)\xi]. \tag{32}$$

Applying ϕ^2 on both sides of above equation, we get

$$\phi^2((\nabla_W R)(X,Y)Z)=\frac{dr(W)}{2}[g(Y,Z)\phi^2X-g(X,Z)\phi^2Y]. \tag{33}$$

Since X, Y and Z are horizontal vectoir fields, using (5) equation (33) gives us

$$\phi^2((\nabla_W R)(X,Y)Z)=\frac{dr(W)}{2}[-g(Y,Z)X+g(X,Z)Y]. \tag{34}$$

Assume that $\phi^2((\nabla_W R)(X,Y)Z)=0$ for all horizontal vector fields. Then the equation (34) implies $dr(W)=0$. Hence we conclude the following theorem.

Theorem 3. *A* 3*-dimensional* (ε)*-Kenmotsu manifold is locally* ϕ*-symmetric if and only if the scalar curvature ris constant for all horizontal vector fields.*

In particular, by taking $Z=\xi$ in (31) we have

$$\begin{aligned}(\nabla_W R)(X,Y)\xi=&\left(\frac{r}{2}+3\varepsilon\right)[\varepsilon\eta(X)(\nabla_W\eta)(Y)\xi+\varepsilon\eta(X)\eta(Y)\nabla_W\xi-\varepsilon\eta(Y)(\nabla_W\eta)(X)\xi-\eta(Y)\eta(X)\nabla_W\xi\\&+\varepsilon(\nabla_W\eta)(X)Y+\varepsilon(\nabla_W\eta)(\xi)\eta(X)Y-\varepsilon(\nabla_W\eta)(Y)X-\varepsilon(\nabla_W\eta)(\xi)\eta(Y)X].\end{aligned} \tag{35}$$

If we assume X,Y,Z are horizontal vector fields, and using (12) in (35) we obtain

$$(\nabla_W R)(X,Y)\xi=\left(\frac{r}{2}+3\varepsilon\right)\varepsilon[g(X,W)Y-g(Y,W)X]. \tag{36}$$

Applying ϕ^2 to the both sides of (36) we get

$$\phi^2((\nabla_W R)(X,Y)\xi) = \left(\frac{r}{2}+3\varepsilon\right)\varepsilon[g(X,W)\phi^2 Y - g(Y,W)\phi^2 X]. \tag{37}$$

If we take X, Y are orthogonal to ξ in (36) and (37) we have

$$\phi^2(\nabla_W R)(X,Y)\xi = (\nabla_W R)(X,Y)\xi.$$

Now we can state the following:

Theorem 4. *Let M be a 3-dimensional (ε)-Kenmotsu manifold such that*

$$\phi^2(\nabla_W R)(X,Y)\xi = 0,$$

for all horizontal vector fields X,Y,W. Then M is an indefinite space form.

5 Globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds

An (ε)-Kenmotsu manifold M is said to be globally M-projectively ϕ-symmetric if the M-Projective curvature tensor M satisfies

$$\phi^2((\nabla_W M)(X,Y)Z) = 0, \tag{38}$$

for all vector fields $X,Y,Z,W \in \chi(M)$.

Let us suppose that M is globally M-projectively ϕ-symmetric. Then by virtue of (38) and (5), we have

$$-(\nabla_W M)(X,Y)Z + \eta((\nabla_W M)(X,Y)Z)\xi = 0. \tag{39}$$

From (2) it follows that

$$\begin{aligned}
&-g((\nabla_W R)(X,Y)Z,U) + \frac{1}{2(n-1)}[g(X,U)(\nabla_W S)(Y,Z) - g(Y,U)(\nabla_W S)(X,Z) + g(Y,Z)g((\nabla_W Q)X,U)\\
&-g(X,Z)g((\nabla_W Q)Y,U)] + \varepsilon\eta((\nabla_W R)(X,Y)Z)\eta(U) - \frac{\varepsilon}{2(n-1)}[(\nabla_W S)(Y,Z)\eta(X)\eta(U)\\
&-(\nabla_W S)(X,Z)\eta(U)\eta(Y) + g(Y,Z)\eta((\nabla_W Q)X)\eta(U) - g(X,Z)\eta((\nabla_W Q)Y)\eta(U)] = 0.
\end{aligned}$$

Putting $X = U = e_i$, where $\{e_i\}$, $i = 1,2,...,n$, is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, we get

$$\begin{aligned}
-\frac{n}{2(n-1)}(\nabla_W S)(Y,Z) + \varepsilon\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) + \frac{1}{2(n-1)}[g((\nabla_W Q)e_i,e_i) - \varepsilon\eta((\nabla_W Q)e_i)\eta(e_i)]g(Y,Z)\\
-\frac{1}{2(n-1)}[g((\nabla_W Q)Y,Z) - (\nabla_W S)(\xi,Z)\eta(Y) - \varepsilon\eta((\nabla_W Q)Y)\eta(Z)] = 0.
\end{aligned}$$

Putting $Z = \xi$, we obtain

$$-\frac{n}{2(n-1)}(\nabla_W S)(Y,\xi) + \varepsilon\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) + \frac{\varepsilon}{2(n-1)}[dr(W) - \varepsilon\eta((\nabla_W Q)e_i)\eta(e_i) + (\nabla_W S)(\xi,\xi)]\eta(Y) = 0. \tag{40}$$

Now

$$\eta((\nabla_W Q)e_i)\eta(e_i) = g((\nabla_W Q)e_i,\xi)g(e_i,\xi) = g((\nabla_W Q)\xi,\xi) \qquad (41)$$
$$= -\varepsilon g(Q(W-\eta(W)\xi),\xi) = -\varepsilon S(W,\xi) + \varepsilon\eta(W)S(\xi,\xi) = 0.$$

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi) \qquad (42)$$

and

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W\xi,\xi).$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ and using (13) we find

$$g(R(e_i,\nabla_W Y)\xi,\xi) = \varepsilon\{\eta(e_i)\eta(\nabla_W Y) - \eta(e_i)\eta(\nabla_W Y)\} = 0. \qquad (43)$$

As $g(R(e_i,Y)\xi,\xi) + g(R(\xi,\xi)Y,e_i) = 0$, we have

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0.$$

Using this we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \qquad (44)$$

By the use of (41)-(44), from (40) we obtain

$$(\nabla_W S)(Y,\xi) = \frac{1}{n}dr(W)\eta(Y). \qquad (45)$$

Put $Y=\xi$ in (45), we get $dr(W) = 0$. This implies r is constant. So from (45), we have

$$(\nabla_W S)(Y,\xi) = 0.$$

Using (11), this implies

$$S(Y,W) = -\varepsilon(n-1)g(Y,W).$$

Hence we can state the following theorem:

Theorem 5. *A globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold is an Einstein manifold.*

Next, suppose that $S(X,Y) = \lambda g(X,Y)$, i.e. $QX = \lambda X$. Then form (1) we have

$$M(X,Y)Z = R(X,Y)Z - \frac{\lambda}{(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

which gives us

$$(\nabla_W M)(X,Y)Z = (\nabla_W R)(X,Y)Z.$$

Applying ϕ^2 on both sides of the above equation we have

$$\phi^2(\nabla_W M)(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z.$$

Hence we can state the following theorem:

Theorem 6. *A globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold is globally ϕ-symmetric.*

Since a globally ϕ-symmetric (ε)-Kenmotsu manifold is always a globally M-projectively ϕ-symmetric manifold, from Theorem 6, we conclude that on an (ε)-Kenmotsu manifold, globally ϕ-symmetry and globally M-projective ϕ-symmetry are equivalent. Thus, we can state:

Corollary 1. *A globally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold is an indefinite space form.*

6 3-dimensional locally M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds

In a 3-dimensional (ε)-Kenmotsu manifold the curvature tensor R, the Ricci tensor S and the Ricci operator Q are as in (30), (29) and (28), respectively. Now putting (28), (29) and (30) into (1) we have

$$\begin{aligned} M(X,Y)Z =& \frac{1}{2}\left(\frac{r}{2}+3\varepsilon\right)[g(Y,Z)X-g(X,Z)Y]-\frac{3}{8}(r+6\varepsilon)[g(Y,Z)\eta(X)\xi \\ &-g(X,Z)\eta(Y)\xi+\varepsilon\eta(Y)\eta(Z)X-\varepsilon\eta(X)\eta(Z)Y]. \end{aligned} \tag{46}$$

Taking covariant differentiation of (46) we have

$$\begin{aligned} (\nabla_W M)(X,Y)Z =& \frac{dr(W)}{4}[g(Y,Z)X-g(X,Z)Y]-\frac{3dr(W)}{8}[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &+\varepsilon\eta(Y)\eta(Z)X-\varepsilon\eta(X)\eta(Z)Y]-\frac{3}{8}(r+6\varepsilon)[g(Y,Z)(\nabla_W\eta)(X)\xi+g(Y,Z)\eta(X)\nabla_W\xi \\ &-g(X,Z)(\nabla_W\eta)(Y)\xi-g(X,Z)\eta(Y)\nabla_W\xi+\varepsilon(\nabla_W\eta)(Y)\eta(Z)X+\varepsilon(\nabla_W\eta)(Z)\eta(Y)X \\ &-\varepsilon(\nabla_W\eta)(X)\eta(Z)Y-\varepsilon(\nabla_W\eta)(Z)\eta(X)Y]. \end{aligned} \tag{47}$$

Now assume X,Y and Z are horizontal vector fields. So equation (47) becomes

$$(\nabla_W M)(X,Y)Z = \frac{dr(W)}{4}[g(Y,Z)X-g(X,Z)Y]-\frac{3}{8}(r+6\varepsilon)[g(Y,Z)(\nabla_W\eta)(X)\xi+g(X,Z)(\nabla_W\eta)(Y)\xi]. \tag{48}$$

Applying ϕ^2 on both sides of above equation, we get

$$\phi^2((\nabla_W M)(X,Y)Z) = \frac{dr(W)}{2}[g(Y,Z)\phi^2X-g(X,Z)\phi^2Y]. \tag{49}$$

Since X, Y and Z are horizontal vector fields, using (5) equation (49) gives us

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{dr(W)}{2}[-g(Y,Z)X+g(X,Z)Y]. \tag{50}$$

Assume that $\phi^2((\nabla_W M)(X,Y)Z)=0$ with horizontal vector fields. Then the equation (50) implies $dr(W)=0$. Hence we conclude the following theorem:

Theorem 7. *A* 3*-dimensional (ε)-Kenmotsu manifold is locally M-projectively ϕ-symmetric if and only if the scalar curvature ris constant for all horizontal vector fields.*

Using Theorem 3 and Theorem 7, we state the following theorem.

Theorem 8. *A* 3*-dimensional (ε)-Kenmotsu manifold is locally M-projectively ϕ-symmetric if and only if it is locally ϕ-symmetric for all horizontal vector fields.*

In particular, by taking $Z=\xi$ in (47) we have

$$\begin{aligned}(\nabla_W M)(X,Y)\xi =& -\varepsilon\frac{dr(W)}{8}[\eta(Y)X-\eta(X)Y]-\frac{3}{8}(r+6\varepsilon)[\varepsilon\eta(Y)(\nabla_W\eta)(X)\xi+\varepsilon\eta(Y)\eta(X)\nabla_W\xi \\ &-\varepsilon\eta(X)(\nabla_W\eta)(Y)\xi-\varepsilon\eta(X)\eta(Y)\nabla_W\xi+\varepsilon(\nabla_W\eta)(Y)X+\varepsilon(\nabla_W\eta)(\xi)\eta(Y)X \\ &-\varepsilon(\nabla_W\eta)(X)Y-\varepsilon(\nabla_W\eta)(\xi)\eta(X)Y].\end{aligned} \tag{51}$$

If we assume X and Y are horizontal vector fields, And using (12) in (51) we obtain

$$(\nabla_W M)(X,Y)\xi = -\frac{3\varepsilon}{8}(r+6\varepsilon)[g(Y,W))X-g(X,W)Y]. \tag{52}$$

Applying ϕ^2 on the both sides of (52) we get

$$\phi^2((\nabla_W M)(X,Y)\xi) = -\frac{3\varepsilon}{8}(r+6\varepsilon)[g(Y,W)\phi^2X-g(X,W)\phi^2Y]. \tag{53}$$

If we take X,Y,W orthogonal to ξ in (52) and (53) we have

$$\phi^2(\nabla_W M)(X,Y)\xi = (\nabla_W M)(X,Y)\xi.$$

Now we can state the following:

Theorem 9. *Let M be a 3-dimensional (ε)-Kenmotsu manifold such that*

$$\phi^2(\nabla_W M)(X,Y)\xi = 0$$

for all horizontal vector fields X,Y,W. Then M is an indefinite space form.

7 Examples of M-projectively ϕ-symmetric (ε)-Kenmotsu manifolds

Example 1. We consider the 3-dimensional manifold $M^3=\{(x,y,z)\in\mathbf{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbf{R}^3$. The vector fields are

$$e_1=e^{-z}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right),\quad e_2=e^{-z}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right),\quad e_3=\frac{\partial}{\partial z}.$$

It is obvious that $\{e_1,e_2,e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=\varepsilon,\quad g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0,$$

where $\varepsilon=\pm1$. That is the form of the metric becomes $g=\varepsilon\left\{\frac{1}{2e^{-z}}(dx\otimes dx+dy\otimes dy)+dz\otimes dz\right\}$.

Let η be the 1-form defined by $\eta(Z)=\varepsilon g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=e_2$, $\phi(e_2)=-e_1$, $\phi(e_3)=0$. Then using the linearity of ϕ and g we have

$$\phi^2X=-Z+\eta(X)e_3,\quad \eta(e_3)=1,\quad g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y),$$

for any vector fields on M^3.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$$[e_1,e_2]=0, \quad [e_2,e_3]=\varepsilon e_2, \quad [e_1,e_3]=\varepsilon e_1. \tag{54}$$

Using Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])-g(Z,[X,Y]).$$

Koszul's formula yields

$$\nabla_{e_1}e_3=\varepsilon e_1,\ \nabla_{e_1}e_2=0,\ \nabla_{e_1}e_1=-\varepsilon e_3,\ \nabla_{e_2}e_3=\varepsilon e_2,\ \nabla_{e_2}e_2=\varepsilon e_3,\ \nabla_{e_2}e_1=0, \nabla_{e_3}e_3=0,\ \nabla_{e_3}e_2=0,\ \nabla_{e_3}e_1=0.$$

Thus it can be easily seen that (M^3,ϕ,ξ,η,g) is an (ε)-Kenmotsu manifold. Hence one can easily obtain by simple calculation that the curvature tensor and the Ricci tensor componenets are as follows

$$R(e_1,e_2)e_2=-e_1,\ R(e_1,e_3)e_3=-e_1,\ R(e_2,e_1)e_1=-e_2, R(e_2,e_3)e_3=-e_2,\ R(e_3,e_1)e_1=-e_3,\ R(e_3,e_2)e_2=-e_3.$$

and

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=-2\varepsilon. \tag{55}$$

Thus the scalar curvature r is constant. Hence from Theorem 3 and Theorem 8, M^3 is a locally M-projectively ϕ-symmetric (ε)-Kenmotsu manifold with horizontal vector fields.

Example 2. We consider the 3-dimensional manifold $M^3=\{(x,y,z)\in\mathbf{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbf{R}^3$. The vector fields are

$$e_1=z\frac{\partial}{\partial x}, \quad e_2=z\frac{\partial}{\partial y}, \quad e_3=-z\frac{\partial}{\partial z}.$$

It is obvious that $\{e_1,e_2,e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=\varepsilon, \quad g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0,$$

where $\varepsilon=\pm1$. That is the form of the metric becomes $g=\varepsilon\left\{\frac{1}{2e^{-z}}(dx\otimes dx+dy\otimes dy)+dz\otimes dz\right\}$.

Let η be the 1-form defined by $\eta(Z)=\varepsilon g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=e_2$, $\phi(e_2)=-e_1$, $\phi(e_3)=0$. Then as in the previous example, it can be easily seen that (M^3,ϕ,ξ,η,g) is an (ε)-Kenmotsu manifold which is locally M-projectively ϕ-symmetric with horizontal vector fields.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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