JOURNAL OF SCIENCE
SAKARYA UNIVERSITY

Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University | http://www.saujs.sakarya.edu.tr/

Title: A Characterization of Some Class Nonlinear Eigenvalue Problem in VELS

Authors: Lütfi Akın
Recieved: 2018-11-03 00:00:00

Accepted: 2019-01-23 00:00:00

Article Type: Research Article
Volume: 23
Issue: 4
Month: August
Year: 2019
Pages: 583-587

How to cite
Lütfi Akın; (2019), A Characterization of Some Class Nonlinear Eigenvalue Problem in VELS. Sakarya University Journal of Science, 23(4), 583-587, DOI: 10.16984/saufenbilder.478197
Access link
http://www.saujs.sakarya.edu.tr/issue/43328/478197

Sakarya University Journal of Science 23(4), 583-587, 2019

A Characterization of Some Class Nonlinear Eigenvalue Problem in VELS

Lütfi AKIN[*1]

Abstract

In last the quarter century, many researchers have been interested by the theory of the variable exponent function space and its applications. We well-know that a normal mode analysis of a vibrating mechanical or electrical system gives rise to an eigenvalue problem. We will investigate a characterization of some class nonlinear eigenvalue problem in variable exponent Lebesgue spaces.

Keywords: Variable exponent, operator theory, Lebesgue spaces

1. INTRODUCTION

In this paper, we derive a new boundedness and compactness result for the Hardy operator in variable exponent Lebesgue spaces (VELS), $L^{p(.)}(0,l)$. A maximally weak condition is assumed on the exponent function. The last time, such a study was carry out in [1,2,3,4,5,6,7,8,9,13,14]. For a study the Dirichlet problem of some class nonlinear eigenvalue problem with nonstandard growth condition the obtained results is applied. Such equations arise in the studies of the so called Winslow effect physical phenomena [11] in the smart materials. In this connection, we mention recent studies for the multidimensional cases with application of Ambrosetti-Rabinoviche's Mountain pass theorem approaches (see, e.g. in [1,10, 12]) .

Theorem 1.1. Let $q, p : (0,l) \to (1,\infty)$ be measurable functions with $q(x) \geq p(x)$ on $(0,l)$. Assume p be monotony increasing and the function $x^{-\frac{1}{p'(x)}+\delta}$ is almost decreasing on $(0,l)$.

Then operator H boundedly acts the space $L^{p}(0,l)$ into $L^{q(.),\ -\frac{1}{p'}-\frac{1}{q(.)}}(0,l)$. Moreover, the norm of mapping depends on p^-, p^+, δ, β.

In the given assertions, $L^{p,\alpha}(0,l)$ denotes the space of measurable functions with finite norm $\|yx^{\alpha}\|_{L^{p(.)}(0,l)}$, while $W^{1}_{p(.),\alpha}(0,l)$ stands the space of absolutely continuous functions y with $y(0)=0$ and finite norm

$$\|y\|_{W^1_{p(.)}} = \|y'\|_{L^{p(.)}}.$$

We say, the function $\alpha: (0,l) \to (0,\infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$ such that for any $0 < t_1 < t_2 < l$ it holds

$$\alpha(t_1) \leq C\alpha(t_2) \quad (\alpha(t_1) \geq C\alpha(t_2))$$

We need the following assertion.

Lemma 1.2. Let $p(x)$ be increasing for $x \in (0,l)$. Let $t \in A_n(x) = (2^{-n-1}x,\ 2^{-n}x]$. Then it holds

$$t^{-\frac{1}{p'(t)}} \leq Ct^{-\frac{1}{(p^-_{x,n})'}}, \tag{1}$$

where $p_{x,n} = \inf_{t \in A_n(x)} p(t)$.

Proof. Let $y \in A_n(x)$ be a point with $t^{-\frac{1}{p'(y)}} \leq 2t^{-\frac{1}{(p^-_{x,n})'}}$. Let $y < t$ and both lie in $A_n(x)$. Then using almost decreasing of $x^{-\frac{1}{p'}+\varepsilon}$ it follows that

$$t^{-\frac{1}{p'(t)}+\varepsilon} \leq cy^{-\frac{1}{p'(y)}+\varepsilon}$$

Using $t, y \in A_n(x)$, $(p^-_{x,n})' > 1$ it follows

$$t^{-\frac{1}{p'(t)}} \leq 2^{+\varepsilon}Cy^{-\frac{1}{p'(y)}} \leq 2^{+2+\varepsilon}C\, t^{-\frac{1}{(p^-_{x,n})'}},$$

[*] Corresponding Author: lutfiakin@artuklu.edu.tr

[1] Mardin Artuklu University, Department of Bussiness Administration, Mardin, Turkey. ORCİD: 0000-0002-5653-9393

Now let $y > t$, then using increasing of p, $\frac{1}{p'}$ also will be increasing. Since $\frac{1}{p'(t)} < \frac{1}{p'(y)}$, it follows that

$$\left(\frac{1}{t}\right)^{\frac{1}{p'(t)}} \leq C\left(\frac{1}{t}\right)^{\frac{1}{p'(y)}} \leq 2Ct^{-\frac{1}{(p_{\bar{x},n})'}},$$

where $c = l^{\frac{1}{p^{-\prime}}} + l^{\frac{1}{p^{+\prime}}}$.

The Lemma 1.2 has been proved.

Proof of Theorem 1.1. Let $f:(0,l) \to (0,\infty)$ be a positive measurable function. It holds the identity

$$Hf(x) = \sum_{n=1}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t)dt \qquad (2)$$

Assume $\|f\|_p = 1$. Using the triangle property of $p(.) -$ norms

$$\|x^\alpha Hf\|_{q(.)} \leq \sum_{n=1}^{\infty} \left\| x^\alpha \int_{A_n(x)} f(t)dt \right\|_{q(.)} \qquad (3)$$

With

$\alpha(x) = -\frac{1}{p'(x)} - \frac{1}{q(x)}$ (recall $A_n(x) = (2^{-n-1}x, 2^{-n}x]$) .

Derive estimation for every summand in (3). In this purpose get estimation for the proper modular

$$I_{q(.)}\left(x^{\alpha(.)} \int_{A_n(x)} f(t)dt\right) = \int_0^l \left(x^{\alpha(.)} \int_{A_n(x)} f(t)dt\right)^{q(x)} dx.$$

Applying the assumption on p (decreasing of $x^{-\frac{1}{p'}+\varepsilon}$), and using the expression for

$q(x) = \frac{1}{-\alpha - \frac{1}{p'(x)}}$ we have

$$I_q\left(x^{-\frac{1}{p'}-\frac{1}{q}} \int_{A_n(x)} f(t)dt\right) = \int_0^l \left(x^{-\frac{1}{p'}+\varepsilon} \int_{A_n(x)} f(t)dt\right)^{q(x)} \frac{dx}{x^{1+\varepsilon q(x)}}$$

$$\leq C^{q^+} 2^{-n\varepsilon q^-} \int_0^l \frac{dx}{x} \left(\int_{A_n(x)} f(t) t^{-\frac{1}{p'(t)}} dt\right)^{q(x)} \qquad (4)$$

Notice, we have used that $x^{-\frac{1}{p'(x)}+\varepsilon} \leq ct^{-\frac{1}{p'(t)}+\varepsilon}$ for any $0 < x < l$ and that $2^{-n-1}x < t \leq 2^{-n}x$ by using the almost decreasing of $x^{-\frac{1}{p'(x)}+\varepsilon}$.

Therefore, from (3) using Holder's inequality, it follows

$$I_q\left(x^{\alpha(.)} \int_{A_n(x)} f(t)dt\right)$$

$$\leq C^{q^+} 2^{-n\varepsilon q^-} \int_0^l \frac{dx}{x} \left(\int_{A_n(x)} (f(t))^{p_{\bar{x},n}} dt\right)^{\frac{q(x)}{p_{\bar{x},n}}}$$

$$\cdot \left(\int_{A_n(x)} t^{-\frac{(p_{\bar{x},n})'}{p'(t)}} dt\right)^{\frac{q(x)}{(p_{\bar{x},n})}} \qquad (5)$$

Applying this Lemma 1 and estimate (1) it follows from (6) that

$$I_q\left(x^{\alpha(.)} \int_{A_{n(x)}} f(t)dt\right) \leq$$

$$\int_0^l \frac{dx}{x} \left(\int_{A_n(x)} (f(t))^{p_{\bar{x},n}} dt\right)^{\frac{q(x)}{p_{\bar{x},n}}}.$$

$$(c \ln 2)^{\frac{q^+}{p^-}} 2^{-n\varepsilon q^-} C^{q^+}$$

Since

$$\int_{A_n(x)} (f(t))^{p_{\bar{x},n}} dt \leq \int_{A_n(x)} (f(t))^{p(t)} dt + \int_{A_n(x)} dt$$

$$\leq 1 + 2^{-n}x \leq 1 + 2^{-n}l \leq l + 1.$$

it follows

$$I_q\left(x^\alpha \int_{A_n(x)} f(t)dt\right)$$

$$\leq (c \ln 2)^{q^+} 2^{-n\varepsilon q^-}.$$

$$\cdot \int_0^l \frac{dx}{x} \left(\frac{1}{l+1} \int_{A_n(x)} (f(t))^{p_{\bar{x},n}} dt\right)^{\frac{q(x)}{p_{\bar{x},n}}} (l+1)^{q^+}$$

$$\leq \left(c \ln 2\,(l+1)\right)^{q^+}.$$

$$\cdot \int_0^l \left(\frac{1}{l+1} \int_{A_n(x)} \left[(f(t)^{p(t)} + 1)\right] dt\right)^{\frac{p(x)}{p_{\bar{x},n}}} \frac{dx}{x}$$

$$\leq 2^{-n\varepsilon q^-} C^{q^+} (c \ln 2)^{q^+} (l+1)^{q^+-1}.$$

$$\cdot\int_0^l \frac{dx}{x}\left(\int_{A_n(x)} \left[\left(f(t)^{p(t)}+1\right)\right]dt\right).$$

Hence,

$I_q\left(x^{\alpha(.)}\int_{A_{n(x)}} f(t)dt\right) \le$
$c_3 2^{-n\varepsilon q^-} C^{q^+}\int_0^l \left(\int_{A_n(x)}\left[\left(f(t)\right)^{p(t)}+1\right)\right]dt\right)\frac{dx}{x}$

$$\le c_3\int_0^{2^{-n}l}\left[\left(f(t)^{p(t)}+1\right)\right]\int_{2^n t}^{2^{n+1}t}\frac{dx}{x}$$

$$= C^{q^+}c_3 2^{-n\varepsilon q^-}\ln 2\int_0^{2^{-n}l}\left[\left(f(t)^{p(t)}+1\right)\right]dt$$

$$\le c^{q^+}c_3 2^{-n\varepsilon q^-}\ln 2\,(1+2^{-n}l) = c_4 2^{-n\varepsilon q^-}.$$

Therefore, it has been proved that

$$I_q\left(x^{-\frac{1}{p'}-\frac{1}{q}}\int_{A_{n(x)}} f(t)dt\right)\le c_4 2^{-n\varepsilon q^-},$$

which implies

$$\left\|x^{-\frac{1}{p'}-\frac{1}{q}}\int_{A_{n(x)}} f(t)dt\right\|_{q(.);(0,l)} \le c_4^{\frac{1}{q^+}}2^{-n\varepsilon\frac{q^-}{q^+}} \qquad (6)$$

Inserting (6) in (3), we get

$$\left\|x^{-\frac{1}{p'}-\frac{1}{q}}Hf\right\|_{q(.);(0,l)} \le c_4^{\frac{1}{q^+}}\sum_{n=1}^{\infty}2^{-n\varepsilon\frac{q^-}{q^+}} = c_5$$

The Theorem A has been proved.

Theorem 1.3. Let

$q,p:(0,l)\to(1,\infty)$ be measurable functions such that $\infty > q^+ \ge q(x) \ge q^- > p^+ \ge p(x) \ge p^- > 1$

Assume that the function p increases on $(0,l)$ and $x^{-\frac{1}{p'}+\varepsilon}$ is almost decreasing in $(0,l)$. Then operator H acts compactly the space $L^{p,\beta}(0,l)$ into $L^{q,-\frac{1}{p'}-\frac{1-\delta}{q}}(0,l)$ for any $\delta\in(0,1)$.

Proof. In order to proof Theorem 1.3, we may to apply the approaches from [3,4,5]. In this way, insert the operators

$$P_1 f(x) = X_{(0,a)}(x)x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_0^x f(t)dt;$$

$$P_2 f(x) = X_{(a,l)}(x)x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_0^a f(t)dt;$$

$$P_3 f(x) = X_{(a,l)}(x)x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_a^x f(t)dt;$$

As it was stated in [3], P_3 is a limit of finite rank operators, while P_2 is a finite rank operator. From the condition $\lim_{t\to 0} B(t) = 0$ it follows that

$$\|Hf-P_2f-P_3f\|_{L^{q(.)}(0,l)} \le \|P_1 f\|_{L^{q(.)}(0,l)} \le ca^{\frac{\delta}{p^+}}\|f\|_p$$

or

$\|H-P_2-P_3\|_{L^p\to L^{q,-\frac{1}{p'}-\frac{1-\delta}{q}}} \le \|P_1\|_{L^{p,\beta}\to L^{q,-\frac{1}{p'}-\frac{1-\delta}{q}}}$ as

$$\le ca^{\frac{\delta}{p}}\to 0,\ \ a\to 0. \qquad (7)$$

To show the last estimation we shall use the argues of Theorem 1.1. Repeating all constructions there, we get the following estimates

$$I_q\left(x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_{A_{n(x)}} f(t)dt\right)$$
$$=\int_0^l\frac{dx}{x^{1-\delta+\varepsilon}}\left(x^{-\frac{1}{p'}+\varepsilon}\int_{A_{n(x)}} f(t)dt\right)^{q(x)}$$
$$\le C^{q^+}2^{-n\varepsilon q^-}\int_0^l\frac{dx}{x^{1-\delta}}\left(\int_{A_{n(x)}} t^{-\frac{1}{p'(t)}}f(t)dt\right)^{q(x)}.$$

Notice, we have used $x^{-\frac{1}{p'}+\varepsilon}\le ct^{-\frac{1}{p'(t)}+\varepsilon}$ for any $t\in A_n(x)$.

Therefore, and using Holder's inequality

$$I_q\left(x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_{A_{n(x)}} f(t)dt\right)$$
$$\le C^{q^+}2^{-n\varepsilon q^-}\int_0^l\frac{dx}{x^{1-\delta}}\left(\int_{A_{n(x)}}(f(t))^{p_{\bar{x},n}}dt\right)^{\frac{q(x)}{p_{\bar{x},n}}}.$$

$$\cdot\left(\int_{A_{n(x)}} t^{-\frac{(p_{x,n}^{-})'}{p'(t)}}dt\right)^{\frac{q(x)}{(p_{x,n}^{-})'}}$$

Applying Lemma 1.2 and argues above, we attain the estimates

$$I_q\left(x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_{A_{n(x)}} f(t)dt\right)$$

$$\leq C^{q^+}2^{-n\varepsilon q^-}(c\ln 2)^{q^+}(l+1)^{q^+-1}.$$

$$\cdot\int_0^l \frac{dx}{x^{1-\delta}}\left(\int_{A_n(x)}\left[(f(t))^{p(t)}+1\right]dt\right)$$

$$\leq C_3 2^{-n\varepsilon q^-}\int_0^{-2^{-n}}\left[(f(t))^{p(t)}+1\right]\left(\int_{2^n t}^{2^{n+1}t}\frac{dx}{x^{1-\delta}}\right)dt$$

$$\leq 2^{-n\varepsilon q^-}(1+2^{-n}l)C_3C^{q^+}l^{\delta}$$

Therefore, it has been shown that

$$I_q\left(x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_{A_{n(x)}} f(t)dt\right)\leq cl^{\delta}2^{-n\varepsilon^{-}}$$

if $\|f\|_p \leq 1$. This implies

$$\left\|x^{-\frac{1}{p'}-\frac{1-\delta}{q}}\int_{A_{n(x)}} f(t)dt\right\|_{q(.);(0,l)} \leq C^{\frac{1}{q^+}}l^{\frac{\delta}{q^+}}2^{\frac{-n\varepsilon q^-}{q^+}}$$

Inserting this estimates over $n = 1,2,\ldots$ in the expression

$$\left\|x^{-\frac{1}{p'}-\frac{1-\delta}{q}}Hf\right\|_{q(.);(0,l)} \leq c^{\frac{1}{q^+}}l^{\frac{\delta}{q^+}}\sum_{n=1}^{\infty}2^{\frac{-n\varepsilon q^-}{q^+}}=c_5 l^{\frac{\delta}{q^+}}$$

The last estimate is a needed estimation which completes the proof of Theorem 1.3.

Conclusion. In this study, we obtained a new boundedness and compactness for the Hardy operator in variable exponential Lebesgue spaces (VELS), $L^{p(.)}(0,l)$.

Akcnowledgement. The author would like to thank the referee for careful reading of the paper and valuable suggestions.

References

1. A. Ambrosetti and P. Rabinowitz,(1973)," Dual variational methods in critical point theory and applications", *J. Funct. Anal.* **14** , 349–381

2. D.Cruz-Uribe, SFO and F.I. Mamedov ,(2012), "On a general weighted Hardy type inequality in the variable exponent Lebesgue spaces", *Rev. Mat. Compl.*, **25**(2), 335-367

3. D.E.Edmunds , P.Gurka and L.Pick ,(1994), "Compactness of Hardy type integral operators in weighted Banach function spaces", *Studia Math.* , **109** (1), 73-90

4. D.E.Edmunds, V.Kokilashvili and A.Meskhy,(2005), "On the boundedness and compactness of weighted Hardy operator in space $L^{p(x)}$", *Georgian Math. J.,* **12**(1), 27–44

5. F.I.Mamedov, Y. Zeren and L. Akın,(2017), "Compactification of weighted Hardy operator in variable exponent Lebesgue spaces", *Asian Journal of Mathematics and Computer Research.17*(1), 38-47.

6. L. Akın, (2018),"On two weight criterions for the Hardy-Littlewood maximal operator in BFS", *Asian Journal of Science and Technology,* Vol. 09, Issue: 5, pp.8085-8089.

7. F.I.Mamedov and Y. Zeren ,(2014), A necessary and sufficient condition for Hardy's operator in the variable Lebesgue space, *Abst. Appl. Anal.*, 5/6, 7 pages.

8. F.I.Mamedov and Y. Zeren ,(2012), "On equivalent conditions for the general weighted Hardy type inequality in space $L^{p(.)}(0,l)$ ", *Zeitsch. fur Anal. und ihre Anwend.*, **34**(1), 55-74

9. L. Akın, (2018), " A Characterization of Approximation of Hardy Operators in VLS", *Celal Bayar University Journal of Science,* Volume 14, Issue 3, p 333-336

10. F.I.Mamedov and A.Harman ,(2010), "On a Hardy type general weighted inequality in spaces $L^{p(.)}(0,l)$", *Integr. Equ. Oper. Th.*, **66**(1), 565-592

11. M. Willem,(1996)," Minimax Theorems", Birkhauser, Boston.

12. V.D.Radulescu,(2015), "Nonlinear elliptic equations with variable exponent: Old and new", *Nonlinear Analysis,* **121,** 336–369

13. E. Piskin, (2018),"Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon

equation with variable exponents", *Honam Mathematical Journal,* 40(4), pp. 771-783.

14. E. Piskin, (2017),"Sobolev Uzayları", Seçkin Yayıncılık.