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On the Generalized Baskakov Durrmeyer Operators

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Abstract

The intent of this article is to construct Baskakov Durrmeyer type operators. Their structure depends on a function τ. We exude the uniform convergence of the operators using the weighted modulus of continuity. Moreover we obtain pointwise convergence of $\tilde{C}_m^\tau$ by obtaining Voronovskaya type theorem.

Keywords: Durrmeyer operators, uniform convergence, asymptotic formula.

1. INTRODUCTION

In approximation theory, the positive approximation processes worked out by Korovkin and rise in many problems. The most useful samples of such operators are Baskakov operators. In 1957, Baskakov [5] introduced the following positive linear operators on unbounded the interval [0,∞) for suitable functions defined on the interval [0,∞).

$$l_m(g;x) = \sum_{l=0}^{\infty} \vartheta_{m,l}(x) g\left(\frac{l}{m}\right), \quad x \in [0,\infty), m \in \mathbb{N},$$

where $\vartheta_{m,l}(x) = \binom{m+l-1}{l}\left(\frac{x^l}{(1+x)^{m+l}}\right)$.

Cardenas Morales et al. in 2011 [6] studied Bernstein type operators described for g∈C[0,1] by $C_m(g \circ \tau^{-1}) \circ \tau$. $C_m$being the classical Bernstein operators and τ being any function that is continuously differentiable ∞ times on [0,1], such that τ(0)=0, τ(1)=1 and τ′(x)>0 for x∈[0,1]. In addition, the Durrmeyer type generalization of processed operators was found in [1]. Moreover Aral [4] studied simulant alterations of the Szasz -Mirakyan operators. They offered quantitative type theorems to explore the degree of weighted convergence with the help of a weighted modulus of continuity constructed using the function τ of the operators. Moreover in [2] a durrmeyer type generalization of Szasz operators was introduced. Many writers have studied in this way, see [3-10], and the references therein. Very recently Patel et al. [11] studied generalization of Baskakov operators.

Set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and let R^+ be the positive real semi-axis [0,∞). Suppose that τ is any function satisfying the conditions:

(p_1) τ is a continuously differentiable function on R^+,

(p_2) τ(0)=0, $\inf_{x\in[0,\infty)} \tau'(x) \geq 1$.

The generalized Baskakov operators are defined by

$$C_m^\tau(g;x) = \sum_{l=0}^{\infty}(g \circ \tau^{-1})\left(\left(\frac{l}{m}\right)\right) P_{m,\tau,l}(x), \qquad (1)$$

where $P_{m,\tau,l}(x) := \binom{m+l-1}{l}\left(\frac{\tau(x)^l}{(1+\tau(x))^{m+l}}\right)$. $C_m$are the classical Baskakov operators and can be obtained from C_m^τ as a particular case τ(x)=x.

The general integral modification of (1) to approximate Lebesgue integrable functions on R^+ can be defined as

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$$\tilde{C}_m^{\tau}(g;x)=(m-1)\sum_{l=0}^{\infty}P_{m,\tau,l}(x)$$
$$\times\int_0^{\infty}(g\circ\tau^{-1})(t)p_{m,l}(t)dt, \quad (2)$$

where m∈ℕ, $p_{m,l}(t)=\binom{m+l-1}{l}\left(\frac{x^l}{(1+x)^{m+l}}\right)$ and τ is any function with the assumptions (p_1) and (p_2).

The structure of the article is as follows. In section 2, we give some lemmas about new operators. Section 3 includings the proof of uniform convergence of the operators and also a statement concerning the degree of this uniform cenvergence. Finally, in chapter 4 we find an asymptotic formula for $\tilde{C}_m^{\tau}$ using Taylor's theorem.

2. BASIC RESULTS

In this section we offer the moments.

Lemma 1. We have

$$\tilde{C}_m^{\tau}(1;x)=1,\qquad \tilde{C}_m^{\tau}(\tau;x)=\frac{1+m\tau(x)}{m-2}, \quad (3)$$

$$\tilde{C}_m^{\tau}(\tau^2;x)=\frac{2+4m\tau(x)+m(m+1)\tau^2(x)}{(m-2)(m-3)}, \quad (4)$$

$$\tilde{C}_m^{\tau}(\tau^3;x)=\frac{6+\tau(x)(17m-3)+\tau^2(x)(9m^2+8m-3)}{(m-2)(m-3)(m-4)}$$
$$+\frac{\tau_3(x)(m^3+3m^2+2m)}{(m-2)(m-3)(m-4)}. \quad (5)$$

Lemma 2. If we define the central moment of degree k,

$$\eta_{m,k}^{\tau}(x)=\tilde{C}_m^{\tau}(\left(\tau(t)-\tau(x)\right)^k;x)$$

then we have

$$[m-(k+2)]\eta_{m,k+1}^{\tau}(x)=$$
$$\left(\tau(x)+\tau^2(x)\right)[(D\eta_{m,k}^{\tau}(x)+2k\tau(x)\eta_{m,k-1}^{\tau}(x)]$$
$$+(k+1)\left(1+2\tau(x)\right)\eta_{m,k}^{\tau}(x).$$

$$\eta_{m,1}^{\tau}(x)=\frac{2\tau(x)+1}{m-2}. \quad (6)$$

$$\eta_{m,2}^{\tau}(x)=\frac{2\left[\tau^2(x)(m+3)+\tau(x)(m+3)+1\right]}{(m-2)(m-3)}. \quad (7)$$

$$\eta_{m,3}^{\tau}(x)=\frac{\tau^3(x)(24m+24)+\tau^2(x)(35m+33)}{(m-2)(m-3)(m-4)(m-5)}$$
$$+\frac{\tau(x)(11m+21)+6)}{(m-2)(m-3)(m-4)(m-5)}$$

$$\eta_{m,4}^{\tau}(x)=\frac{\tau^4(x)(12m^2+252m+120)}{(m-2)(m-3)(m-4)(m-5)}$$
$$+\frac{\tau^3(x)(24m^2+494m+210)}{(m-2)(m-3)(m-4)(m-5)}$$
$$+\frac{\tau^2(x)(12m^2+309m+195)+\tau(x)(67m+105)+24}{(m-2)(m-3)(m-4)(m-5)}.$$

Throughout the article we will utilize the following function classes. $C_B(R^+)$is the space of all real valued continuous and bounded functions g on R^+. Let ψ(x)=1+τ²(x).

$$B_{\psi}(R^+)=\{g:R^+\to R, |g(x)|\le M_g\psi(x), x\ge 0\},$$

$$C_{\psi}(R^+)=\left\{g\in C_{\psi}(R^+), g\ is\ continuous\ on\ R^+\right\},$$

$$C_{\psi}^{*}(R^+)=\{g\in C_{\psi}(R^+), lim_{x\to\infty}=\frac{g(x)}{\psi(x)}const.\},$$

$$U_{\psi}(R^+)=\{g\in C_{\psi}(R^+),$$
$$\frac{g(x)}{\psi(x)}\ is\ uniformly\ continuous\ on\ R^+\},$$

where M_g is a constant depending only on g. $C_B(R^+)$ is the linear normed space with the norm $\|g\|=sup_{x\in R^+}|g(x)|$ and the other spaces are normed linear spaces with the norm

$$\|g\|_{\psi}=\frac{sup_{x\in R^+}|g(x)|}{\psi(x)}.$$

3. UNIFORM CONVERGENCE OF $\tilde{C}_m^{\tau}$

The properties of linear positive operators acting from $C_{\psi}R^+$ to $B_{\psi}R^+$. Also Korovkin type theorems have been introduced in [7-8].

Lemma 3. [7] The positive linear operators L_m, m≥1, act from $C_{\psi}R^+$ to $B_{\psi}R^+$ if and only if the inequality

$$|L_m(\psi;x)|\le l_m\psi(x),$$

holds, where l_m is a positive constant depending on m.

Teorem 1. [7] Let the sequence of linear positive operators (L_m), m≥1, acting from $C_{\psi}R^+$ to $B_{\psi}R^+$ satisfy the three conditions

$$lim_{m\to\infty}\|L_m\tau^{\nu}-\tau^{\nu}\|_{\psi}=0, \nu=0,1,2.$$

Then for any function $g \in C_{\psi}^{*}(R^{+})$,

$$lim_{m\to\infty}\|L_m g - g\|_{\psi} = 0.$$

Teorem 2. For each function $g \in C_{\psi}^{*}(R^{+})$, $lim_{m\to\infty}\|C_m^{\tau} g - g\|_{\psi} = 0.$

Proof. Let's show this first

$\tilde{C}_m^{\tau}: C_{\psi}(R^{+}) \to B_{\psi}(R^{+})$. Using (3) and (4) we get

$$\left|\tilde{C}_m^{\tau}(\psi;x)\right| = 1 + \frac{2+4m\tau(x)+\tau^2(x)m(m+1)}{(m-2)(m-3)}.$$

We get

$$|\tilde{C}_m^{\tau}(\psi;x)| \le (1+\tau^2(x))\frac{(m^2+5m+8)}{(m^2-5m+6)}$$

however, since

$$\left\|\tilde{C}_m^{\tau}1 - 1\right\|_{\psi} = 0,$$

$$\left\|\tilde{C}_m^{\tau}\tau - \tau\right\|_{\psi} = sup_{x\in R^{+}}\frac{2\tau(x)+1}{(m-2)(1+\tau^2(x))}.$$

$$\left\|\tilde{C}_m^{\tau}\tau - \tau\right\|_{\psi} \le \frac{3}{m-2}.$$

$$\left\|\tilde{C}_m^{\tau}\tau^2 - \tau^2\right\|_{\psi} = sup_{x\in R^{+}}\frac{m(m+1)\tau^2(x)+4m\tau(x)+2-(m-2)(m-3)\tau^2(x)}{(1+\tau^2(x))(m-2)(m-3)}$$

$$\le \frac{10m-4}{m^2-5m+6}.$$

We deduce

$$lim_{m\to\infty}\left\|\tilde{C}_m^{\tau}g - g\right\|_{\psi} = 0$$

by Theorem 1.

For our aim we recollect the following theorem proved in [9].

Teorem 3. [9] Let $L_m: C_{\psi}(\mathbb{R}^{+}) \to B_{\psi}(\mathbb{R}^{+})$ be a sequence of positive linear operators with

$$\|L_m(\tau^0) - \tau^0\|_{\psi^0} = k_m,$$

$$\|L_m(\tau) - \tau\|_{\psi^{\frac{1}{2}}} = l_m,$$

$$\|L_m(\tau^2) - \tau^2\|_{\psi} = n_m,$$

$$\|L_m(\tau^3) - \tau^3\|_{\psi^{\frac{3}{2}}} = p_m,$$

where k_m, l_m, n_m *and* p_m tend to zero as m→∞. Then

$$\|L_m(g) - g\|_{\psi^{\frac{3}{2}}} \le (7 + 4k_m + 2n_m)\omega_{\tau}(g;\delta_m) + \|g\|_{\psi}k_m$$

for all $g \in C_{\psi}(\mathbb{R}^{+})$, where

$$\delta_m = 2\sqrt{(k_m+2l_m+n_m)(1+k_m)} + k_m + 3l_m + 3n_m + p_m.$$

Teorem 4. For all $g \in C_{\psi}(\mathbb{R}^{+})$, we get

$$\left\|\tilde{C}_m^{\tau}(g) - g\right\|_{\psi^{\frac{3}{2}}} \le \left(7 + \left(\frac{20m}{(m-4)^2}\right)\right)\omega_{\tau}\left(g; 2\sqrt{\left(\frac{16m-2}{(m-4)^2}\right)} + \left(\frac{99m^2-192m+144}{(m-4)^3}\right)\right).$$

Proof. On account of apply Theorem 3, we must calculate the sequences k_m, l_m, n_m *and* p_m. Using (3) and (4) we find

$$\left\|\tilde{C}_m^{\tau}(\tau^0) - \tau^0\right\|_{\psi^0} = k_m = 0$$

and

$$l_m = \left\|\tilde{C}_m^{\tau}(\tau) - \tau\right\|_{\psi^{\frac{1}{2}}} = sup_{x\in\mathbb{R}^{+}}\frac{1+2\tau(x)}{\sqrt{(1+\tau^2(x))}(m-2)} \le \frac{3}{m-4}.$$

Also we get

$$n_m = \left\|\tilde{C}_m^{\tau}(\tau^2) - \tau^2\right\|_{\psi} = sup_{x\in\mathbb{R}^{+}}\frac{2+4m\tau(x)+6(m-1)\tau^2(x)}{(1+\tau^2(x))(m-2)(m-3)} \le \frac{10m}{(m-4)^2}.$$

Finally using (5), we have

$$p_m = \left\|\tilde{C}_m^{\tau}(\tau^3) - \tau^3\right\|_{\psi^{\frac{3}{2}}}$$

$$= sup_{x\in\mathbb{R}^{+}}\frac{12\tau^3(x)(m^2-2m+2)}{(1+\tau^2(x))^{\left(\frac{3}{2}\right)}(m-2)(m-3)(m-4)} + \frac{\tau^2(x)(9m^2+8m-3)+\tau(x)(17m-3)+6}{(1+\tau^2(x))^{\left(\frac{3}{2}\right)}(m-2)(m-3)(m-4)} \le \frac{60m^2}{(m-4)^3}.$$

Using (10), we find result.

4. A VORONOVSKAYA TYPE THEOREM

In this chapter, we find some asymtotic estimates of $\tilde{C}_m^{\tau}$ by obtaining Voronovskaya type theorem. Let's remember the following lemma given in [9].

Lemma 4. For every $g \in C_{\psi}(R^{+})$, for δ>0 and for all u, x≥0,

$$|g(u) - g(x)| \leq \big(\psi(u) + \psi(x)\big)\left(2 + \left(\frac{|\tau(u) - \tau(x)|}{\delta}\right)\right)\omega_{\tau}(g, \delta) \qquad (11)$$

holds.

Teorem 5. Let $g \in C_{\psi}(R^{+})$, x∈I and suppose that the first and second derivatives of $g\circ\tau^{-1}$ exist at τ(x). If the second derivative of $g\circ\tau^{-1}$ is bounded on R^{+}, then we have

$$lim_{m\to\infty} m\left[\tilde{C}_m^{\tau}(g;x) - g(x)\right] =$$

$$(1 + 2\tau(x))(g \circ \tau^{-1})'(\tau(x)) + (1 + \tau(x) + \tau^2(x))(g \circ \tau^{-1})''(\tau(x)).$$

Proof. By the Taylor expansion of $g\circ\tau^{-1}$ at the point $\tau(x)\in R^{+}$, there exists ξ lying between x and t such that

$$g(t) = (g \circ \tau^{-1})(\tau(t)) = (g \circ \tau^{-1})(\tau(x)) + (g \circ \tau^{-1})'(\tau(x))(\tau(t) - \tau(x))$$
$$+(((g \circ \tau^{-1})''(\tau(x))(\tau(t) - \tau(x))^2)/2) + \gamma_x(t)(\tau(t) - \tau(x))^2,$$

where

$$\gamma_x(t) := \left(\frac{(g \circ \tau^{-1})''\big(\tau(\xi)\big) - (g \circ \tau^{-1})''\big(\tau(x)\big)}{2}\right). \qquad (12)$$

We get

$$m[\tilde{C}_m^{\tau}(g;x) - g(x)] = (g \circ \tau^{-1})'\big(\tau(x)\big) m\tilde{C}_m^{\tau}(\tau(t) - \tau(x))$$
$$+(((g \circ \tau^{-1})''\big(\tau(x)\big) m\tilde{C}_m^{\tau}(\tau(t) - \tau(x))^2)/2) + m\tilde{C}_m^{\tau}(\gamma_x(t)(\tau(t) - \tau(x))^2; x).$$

Using (6) and (7), we have

$$lim_{m\to\infty} m\tilde{C}_m^{\tau}(\tau(t) - \tau(x); x) = 1 + 2\tau(x).$$

$$lim_{m\to\infty} m\tilde{C}_m^{\tau}\Big(\big(\tau(t) - \tau(x)\big)^2; x\Big) = 2\big(1 + \tau(x) + \tau^2(x)\big).$$

Let calculate the last term.

$|m\tilde{C}_m^{\tau}(|\gamma_x(t)|(\tau(t) - \tau(x))^2; x)|$.
Since $lim_{t\to x}\gamma_x(t) = 0$ for every ε>0, let δ>0 such that $|\gamma_x(t)| < \varepsilon$ for every t≥0. Cauchy-Schwarz inequality applied we have

$$lim_{m\to\infty} m\tilde{C}_m^{\tau}(|\gamma_x(t)|(\tau(t) - \tau(x))^2; x) \leq \varepsilon lim_{m\to\infty} m\tilde{C}_m^{\tau}((\tau(t) - \tau(x))^2; x)$$
$$+\left(\frac{M}{\delta^2}\right) lim_{m\to\infty} m\tilde{C}_m^{\tau}((\tau(t) - \tau(x))^4; x).$$

Since

$$lim_{(m\to\infty)} m\tilde{C}_m^{\tau}((\tau(t) - \tau(x))^4; x) = 0,$$

we get

$$lim_{m\to\infty} m\tilde{C}_m^{\tau}(|\gamma_x(t)|(\tau(t) - \tau(x))^2; x) = 0.$$

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