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## Conjugate Mates For Non-Null Frenet Curves

Alev Kelleci\*

### ABSTRACT

For each non-null Frenet curve  $\gamma$  in Minkowski 3-space, there exists a unique unit speed non-null curve  $\bar{\gamma}$  tangent to the principal binormal vector field of  $\gamma$ . We briefly call this curve  $\bar{\gamma}$  the conjugate mate of  $\gamma$ . The aim of this paper is to prove some relationships between a non-null Frenet curve and its non-null conjugate mate.

**Keywords:** slant helix, conjugate mate, general helix, adjoint curve, Salkowski curve, anti-Salkowski curve.

### 1. INTRODUCTION

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are very interesting and important problem not only in Euclidean space but also in Lorentzian space. One of the most popular among these curves is Frenet curve. If a curve  $\gamma$  is a Frenet curve, then its curvature  $\kappa > 0$  and torsion  $\tau=0$ . These curves are studied in [3, 4, 5, 6, 7] by many geometers. Some important kinds of these curves are rectifying curves, slant helices characterized in [10,11]. Since there exist three kinds of curves (time-like, space-like, and null or light-like curves) depending on their causal characters in a Lorentzian space, working in Lorentzian space is more complicated than working in Euclidean space. Also, it is well-known that the studies of space-like curves and time-like curves have many analogies and similarities because they have the natural geometric invariant

parameter by the arc-length parameter which normalizes the tangent vector, [1,2].

One another popular curve is a Salkowski (resp. anti-Salkowski) curve in Minkowski space  $E_1^3$ , generally known as family of non-null curves with constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [15, 16, 17].

Recently, the theory of the associated curve of a given curve has been one of interesting topics. Many geometers have investigated this problem from different viewpoints: For example, the Bertrand partner and the involute-evolute curves in Minkowski 3-space, are two important types of associated curves, characterized by the curvature and the torsion, (see in [8, 9,12,13]). In [5], the authors studied the general helices and the slant helices in Minkowski 3-spaces are by using some special associated curves of a given curve. Subsequently, they studied the Euclidean version in [4]. They called the special associated curve the principal-directional (-donor) curve and the

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binormal-directional (-donor) curve. These notions gave us a certain method constructing the general helices and the slant helices (see [4, 5]). In [3], Deshmukh, Chen and Alghanemi studied some new type associated curve called as the natural mate and the conjugate mate of a Frenet curve in Euclidean 3-space, closely related with the principal (binormal)-directional curve defined in [4, 5] and also the adjoint curve in [14]. In [4, 5, 14], authors characterized these curves and also gave new results for them. In this paper, we will recall the concept of conjugate mate for non-null Frenet curves in Minkowski 3-space by moving from the notion of conjugate mate defined in [3]. We have also shown that the conjugate mate of non-null Frenet curves is unique and give some relation between conjugate mates in Minkowski 3-space.

## 2. PRELIMINARIES

In this section, we would like to give a brief summary of basic definitions, facts and equations in the theory of curves in Minkowski 3-space (see for details, [1,2]).

Let  $E_1^3$  denote the Minkowski 3-space with canonical Lorentzian metric tensor given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where  $(x_1, x_2, x_3)$  are rectangular coordinates of the points of  $E_1^3$ .

The causality of a vector in Minkowski space is defined as follows: A non-zero vector  $v$  in  $E_1^3$  is said to be space-like, time-like and light-like (null) regarding to  $\langle v, v \rangle > 0, \langle v, v \rangle < 0$  and  $\langle v, v \rangle = 0$ , respectively. We consider the zero vector as a space-like vector. Note that  $v$  is said to be causal if it is not space-like. Two non-zero vectors  $u$  and  $v$  in  $E_1^3$  are said to be orthogonal if  $\langle u, v \rangle = 0$ . A set of  $\{e_1, e_2, e_3\}$  of vectors in  $E_1^3$  is called as an orthonormal frame if it satisfies that

$$\begin{aligned} \langle e_1, e_1 \rangle &= -1, \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \\ \langle e_i, e_j \rangle &= 0, i \neq j. \end{aligned}$$

For two non-zero vectors  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  in  $E_1^3$ , we define the Lorentzian product of  $u$  and  $v$  as in the following:

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

One can check that the vector product is skew-symmetric, i.e.,  $u \times v = -v \times u$ .

A curve  $\gamma = \gamma(t)$  in  $E_1^3$  is said to be space-like, time-like or light-like (null) if its tangent vector field  $\gamma'(t)$  is space-like, time-like or light-like (null), respectively, for all  $t$ .

Let  $\gamma$  be a non-null curve in  $E_1^3$  parametrized by arc-length, i.e.,  $|\langle \gamma', \gamma' \rangle| = 1$ , and we suppose that  $|\langle \gamma'', \gamma'' \rangle| \neq 0$ . Then this curve induces a Frenet frame  $\left\{ T = \gamma', N = \frac{\gamma''}{\sqrt{|\langle \gamma'', \gamma'' \rangle|}}, B = T \times N \right\}$  satisfying the following Frenet equations:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa \epsilon_1 & 0 \\ -\kappa \epsilon_0 & 0 & -\tau \epsilon_0 \epsilon_1 \\ 0 & -\tau \epsilon_1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (2.1)$$

where  $\langle T, T \rangle = \epsilon_0, \langle N, N \rangle = \epsilon_1, \langle B, B \rangle = -\epsilon_0 \epsilon_1, \langle T', N \rangle = \kappa$  and  $\langle N', B \rangle = \tau$ . The vector fields  $T, N, B$  and the functions  $\kappa, \tau$  are called the tangent, principal normal, binormal and curvature and torsion of  $\gamma$ , respectively. Accordingly, the Frenet frame of  $\gamma$  satisfies

$$T \times N = B, N \times B = -\epsilon_1 T, B \times T = -\epsilon_0 N.$$

In (2.1), if  $\epsilon_0 = 1$  or  $\epsilon_0 = -1$ , then  $\gamma$  is space-like or time-like, respectively. A space-like curve  $\gamma$  satisfying (2.1) is said to be *type1* or *type2* if  $\epsilon_1 = 1$  or  $\epsilon_1 = -1$ .

When the Frenet frame moves along a curve in  $E_1^3$ , there exist an axis of instantaneous frame's rotation. The direction of such axis is given by Darboux vector. If  $\gamma$  is a unit speed non-null curve with Frenet-Serret apparatus  $\{\kappa, \tau, T, N, B\}$ , the Darboux vector of  $\gamma$  is that

$$D = -\epsilon_0 \epsilon_1 \tau T - \epsilon_0 \epsilon_1 \kappa B. \quad (2.2)$$

A space-like curve in  $E_1^3$  with a space-like or time-like principal normal is a slant helix if and only if the function

$$\sigma_{s1,2} = \frac{\kappa^2}{|\kappa^2 - \tau^2|^{3/2}} \left( \frac{\tau}{\kappa} \right)' \quad (2.3)$$

or

$$\sigma_{s3} = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)' \quad (2.4)$$

is constant, [11].

A time-like curve in  $E_1^3$  is a slant helix if and only if the function

$$\sigma_{t1,2} = \frac{\kappa^2}{|\kappa^2 - \tau^2|^{3/2}} \left( \frac{\tau}{\kappa} \right)' \quad (2.5)$$

is constant everywhere  $\kappa^2 - \tau^2$  does not vanish, [11].

### 3. CHARACTERIZATIONS OF CONJUGATE MATES OF NON-NULL FRENET CURVES

We provide the following easy results for later use.

**Proposition 1.** Assume that  $\gamma$  is a space-like Frenet curve of *type1* in  $E_1^3$  with Frenet-Serret apparatus  $\{\kappa, \tau, T, N, B\}$  with torsion  $\tau > 0$  (resp.,  $\tau < 0$ ). Then there exist a time-like Frenet curve  $\bar{\gamma}$  in  $E_1^3$  with Frenet-Serret apparatus  $\{\tau, -\kappa, B, -N, T\}$  (resp.,  $\{-\tau, \kappa, B, N, -T\}$ ).

**Proof.** Since the proof of the curve will be similar to the condition of the torsion, we will only prove the positive condition. Let the torsion of  $\gamma$  be positive, then by Equation (2.1) such that  $\epsilon_0 = 1$  and  $\epsilon_1 = 1$  we have

$$B' = \tau(-N), (-N)' = \tau B + \kappa T, T' = (-\kappa)(-N).$$

Hence, there is a unique unit speed curve  $\bar{\gamma}$  in  $E_1^3$  with curvature  $\tau$ , torsion  $-\kappa$  and Frenet frame  $\{B, -N, T\}$  according to the existence theorem. Furthermore, since torsion  $-\kappa \neq 0$  and tangent vector field  $B$  is a time-like, so it is a time-like Frenet curve. Thus the proof is completed.

Now we will give the following results that the proof can be made in exactly the same way as the previous proposition.

**Proposition 2.** Assume that  $\gamma$  is a space-like Frenet curve of *type2* in  $E_1^3$  with Frenet-Serret apparatus  $\{\kappa, \tau, T, N, B\}$  with torsion  $\tau > 0$  (resp.,  $\tau < 0$ ). Then there exist a space-like Frenet curve of *type2*  $\bar{\gamma}$  in  $E_1^3$  with Frenet-Serret apparatus  $\{\tau, -\kappa, B, -N, T\}$  (resp.,  $\{-\tau, \kappa, B, N, -T\}$ ).

**Proposition 3.** Assume that  $\gamma$  is a time-like Frenet curve in  $E_1^3$  with Frenet-Serret apparatus  $\{\kappa, \tau, T, N, B\}$  with torsion  $\tau > 0$  (resp.,  $\tau < 0$ ). Then there exist a space-like Frenet curve of *type1*  $\bar{\gamma}$  in  $E_1^3$  with Frenet-Serret apparatus  $\{\tau, -\kappa, B, -N, T\}$  (resp.,  $\{-\tau, \kappa, B, N, -T\}$ ).

For the non-null Frenet curve  $\gamma$  in  $E_1^3$ , the non-null curve  $\bar{\gamma}$  in the above propositions is given by

$$\bar{\gamma} = \int B ds$$

which was called the binormal direction curve of  $\gamma$  in [5] and also called adjoint curve of  $\gamma$  in [14]. In this paper, we call  $\bar{\gamma}$  the conjugate mate of  $\gamma$  in  $E_1^3$ .

**Corollary 1.** A non-null Frenet curve  $\gamma$  in  $E_1^3$  such that  $\tau \neq \kappa$  is a general helix if and only if its non-null conjugate mate  $\bar{\gamma}$  is a general helix.

**Proof.** Let  $\gamma$  be a non-null general helix with non-zero curvatures, so the ratio of curvatures is constant, i.e.,  $\frac{\tau}{\kappa} = c$ . On the other hand from Proposition 1, we have  $\bar{\kappa} = \tau$  and  $\bar{\tau} = -\kappa$ . Thus, we get the ratio of curvatures of  $\bar{\gamma}$  as

$$\frac{\bar{\tau}}{\bar{\kappa}} = \frac{-\kappa}{\tau} = -\frac{1}{c}$$

which is also constant.

**Corollary 2.** Let  $\gamma$  be a non-null Frenet curve in  $E_1^3$  such that  $\tau \neq \kappa$  and  $\bar{\gamma}$  be its non-null conjugate mate. Then, the conjugate mates  $\{\gamma, \bar{\gamma}\}$  is not only Bertrand pairs but also involute-evolute curves.

**Proof.** The proof of this corollary is proved exactly the same way as in the Euclidean case such as, for instance, the elementary proof given in [14].

**Theorem 1.** A space-like Frenet curve of *type1*  $\gamma$  in  $E_1^3$  with a positive torsion (resp., negative torsion) is a slant helix if and only if its time-like conjugate mate  $\bar{\gamma}$  is a slant helix.

**Proof.** Let  $\gamma$  be a space-like Frenet curve of *type1* with curvature  $\kappa$  and torsion  $\tau > 0$  and let  $\bar{\gamma}$  be its time-like conjugate mate with curvature  $\bar{\kappa}$  and torsion  $\bar{\tau}$ . From Proposition 1, we have  $\bar{\kappa} = \tau$  and  $\bar{\tau} = -\kappa$ . So we have

$$\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)' = \left(\frac{-\kappa}{\tau}\right)' = \frac{\kappa^2}{\tau^2} \left(\frac{\tau}{\kappa}\right)'$$

which yields

$$\frac{\bar{\kappa}^2 \left(\frac{\bar{\tau}}{\bar{\kappa}}\right)'}{|\bar{\kappa}^2 - \bar{\tau}^2|^{3/2}} = \frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{|\kappa^2 - \tau^2|^{3/2}}$$

Therefore, by considering the function given in (2.3), the last equation implies that  $\gamma$  is a slant helix if and only if its time-like conjugate mate  $\bar{\gamma}$  is a slant helix. The proof of this corollary can be done similar to the condition of the torsion being negative.

Note that the proof of the followings can be made in exactly the same way as the previous theorem.

**Theorem 2.** A space-like Frenet curve of *type2*  $\gamma$  in  $E_1^3$  with a positive torsion (resp., negative torsion) is a slant helix if and only if its space-like conjugate mate of *type2*  $\bar{\gamma}$  is a slant helix.

**Theorem 3.** A time-like Frenet curve of *type2*  $\gamma$  in  $E_1^3$  with a positive torsion (resp., negative torsion) is a slant helix if and only if its space-like conjugate mate of *type1*  $\bar{\gamma}$  is a slant helix.

**Theorem 4.** Let  $\gamma$  be a space-like Frenet curve of *type1* in  $E_1^3$  with a positive torsion (resp., negative torsion). The curve  $\gamma$  is a Salkowski curve if and only if its time-like conjugate mate  $\bar{\gamma}$  is a anti-Salkowski curve.

**Proof.** Let  $\gamma$  be a space-like Frenet curve of *type1* with curvature  $\kappa$  and torsion  $\tau > 0$  and let  $\bar{\gamma}$  be its time-like conjugate mate with curvature  $\bar{\kappa}$  and torsion  $\bar{\tau}$ . Assume that  $\gamma$  is a Salkowski curve. By the definition, we have  $\kappa = c$  and  $\tau$  is non-constant. Also, we have  $\bar{\kappa} = \tau$  and  $\bar{\tau} = -\kappa$  from Proposition 1, which yields  $\bar{\tau} = -c$  and  $\bar{\kappa}$  is non-constant. Therefore, we get the conjugate mate  $\bar{\gamma}$  is a anti-Salkowski curve. Conversely, it can be proved in a similar way.

The followings can be demonstrated in a completely similar way as in the previous theorem.

**Theorem 5.** Let  $\gamma$  be a space-like Frenet curve of *type2* in  $E_1^3$  with a positive torsion (resp., negative torsion). The curve  $\gamma$  is a Salkowski curve if and only if its space-like conjugate mate of *type2*  $\bar{\gamma}$  is a anti-Salkowski curve.

**Theorem 6.** Let  $\gamma$  be a time-like Frenet curve in  $E_1^3$  with a positive torsion (resp., negative torsion). The curve  $\gamma$  is a Salkowski curve if and only if its space-like conjugate mate of *type1*  $\bar{\gamma}$  is a anti-Salkowski curve.

**Example 1.** Let  $\gamma: \mathbb{R} \rightarrow E_1^3$  be the unit speed space-like Frenet curve of *type2* by

$$\gamma(s) = \left( \frac{s}{\sqrt{2}}, \frac{\sinh s}{\sqrt{2}}, \frac{\cosh s}{\sqrt{2}} \right)$$

which is plotted in Figure 1.

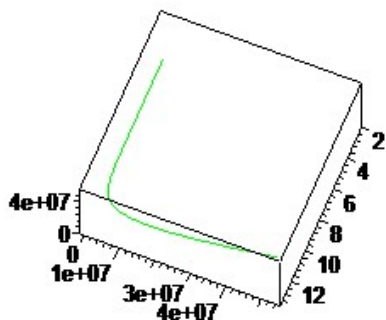


Figure. 1 The space-like Frenet curve of *type2*  $\gamma$  with negative torsion.

The Frenet-Serret apparatus of  $\gamma$  is given by

$$\left\{ \begin{array}{l} \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \left( \frac{1}{\sqrt{2}}, \frac{\cosh s}{\sqrt{2}}, \frac{\sinh s}{\sqrt{2}} \right), \\ (0, \sinh s, \cosh s), \left( \frac{1}{\sqrt{2}}, -\frac{\cosh s}{\sqrt{2}}, -\frac{\sinh s}{\sqrt{2}} \right) \end{array} \right\}$$

By Proposition 2, the conjugate mate  $\bar{\gamma}$  of  $\gamma$  has the following Frenet-Serret apparatus

$$\left\{ \begin{array}{l} \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \left( \frac{1}{\sqrt{2}}, -\frac{\cosh s}{\sqrt{2}}, -\frac{\sinh s}{\sqrt{2}} \right), \\ (0, \sinh s, \cosh s), \left( -\frac{1}{\sqrt{2}}, -\frac{\cosh s}{\sqrt{2}}, -\frac{\sinh s}{\sqrt{2}} \right) \end{array} \right\}$$

By integrating  $B(s)$ , we obtain the conjugate mate  $\bar{\gamma}$  of  $\gamma$  as

$$\bar{\gamma}(s) = \left( \frac{s}{\sqrt{2}}, -\frac{\sinh s}{\sqrt{2}}, -\frac{\cosh s}{\sqrt{2}} \right)$$

which is plotted in Figure 2.

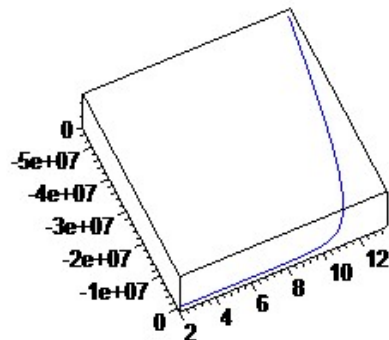


Figure. 2 The space-like Frenet curve of *type2*  $\bar{\gamma}$  being conjugate mate of  $\gamma$ .

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