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A note on the basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations

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Abstract

In this paper we generalize the basic Lichnerowicz cohomology on transversally locally conformally Kählerian foliations and we study its relation with basic Bott-Chern cohomology and 0–th basic Dolbeault cohomology with values in the associated foliated weight bundle.

Keywords: transversally locally conformally Kählerian foliation, Lichnerowicz cohomology, Bott-Chern cohomology.

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1. Introduction and preliminaries

1.1. Introduction. A locally conformally Kähler (LCK) manifold M is a complex manifold whose universal cover $\widetilde{M}$ has a Kähler metric g such that $\pi_1(M)$ acts on $(\widetilde{M}, g)$ holomorphically and conformally. The fundamental properties of LCK manifolds were studied by Vaisman, Kashiwada, Dragomir, Ornea and Verbitsky.

The Lichnerowicz cohomology, also known in literature as Morse-Novikov cohomology, is a cohomology defined for a smooth manifold M and a closed 1–form θ. It is defined by twisting the usual differential of the de Rham complex $\Omega^\bullet(M)$ of M; namely, the Lichnerowicz cohomology is the cohomology of a complex $(\Omega^\bullet(M), d_\theta)$, where d_θ is defined by $d_\theta\varphi = d\varphi - \theta \wedge \varphi$. This cohomology was originally defined by Lichnerowicz and Novikov in the context of Poisson geometry and Hamiltonian mechanics, respectively. Lichnerowicz cohomology is naturally defined for a LCK manifold with its canonical closed 1–form called the Lee form, [18, 24]. Using the complex structures, variants of Lichnerowicz cohomology are defined for the Dolbeault cohomology and the Bott-Chern cohomology. These can be considered as fundamental invariants of LCK manifolds as established mainly by Vaisman, Ornea and Verbitsky.

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In a paper by Barletta and Dragomir [2] is introduced a new class of foliations called transversally locally conformally Kähler foliations (transversally LCK foliations), which is a foliated version of LCK manifolds roughly in the following sense: This class of foliations has a LCK structure on the direction transverse to the leaves. For instance, the simple foliation defined by a C^∞ submersion $f : \mathcal{M} \to M$ of $\mathcal{M}$ onto a LCK manifold M is transversally LCK. The case where the dimension of the leaves is zero corresponds to the original LCK manifolds.

The aim of this note is to extend some theories related to Lichnerowicz cohomology and its variants to basic forms on transversally LCK foliations. In this sense, in the preliminary subsection following [4, 5], we make a short review on the de Rham and Dolbeault theory for basic forms on transversally (holomorphic) foliations. The second section is dedicated to study of the basic Lichnerowicz cohomology of transversally LCK foliations. The relation of this cohomology with basic Bott-Chern cohomology and 0–th basic Dolbeault cohomology with values in the foliated weight bundle of a transversally LCK foliation is also studied, obtaining a basic version of some known results in the case of LCK manifolds due to Ornea and Verbitski [18].

1.2. Preliminaries. Let us consider $\mathcal{M}$ an $(n+m)$–dimensional manifold which will be assumed to be connected and orientable. Differential forms (and in particular functions) will take their values in the field of complex numbers $\mathbb{C}$. If φ is a form, then $\overline{\varphi}$ denote its complex conjugate and we say that φ is *real* if $\varphi = \overline{\varphi}$.

1.1. Definition. A codimension n foliation $\mathcal{F}$ on $\mathcal{M}$ is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of $\mathcal{M}$;
(ii) For every $i \in I$, $\varphi_i : U_i \to M$ are submersions, where M is an n–dimensional manifold;
(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i \tag{1.1}$$

for every $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

We recall that a *plaque* of the foliation $\mathcal{F}$ is given by any fibre of φ_i and by (1.1) we have that, on the intersection $U_i \cap U_j$, the plaques defined respectively by φ_i and φ_j coincide. The manifold $\mathcal{M}$ is decomposed into a family of disjoint immersed connected submanifolds of dimension m called the *leafs* of $\mathcal{F}$.

The foliation $\mathcal{F}$ is said *transversally orientable* if on M can be given an orientation which is preserved by all $f_{i,j}$. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and by $\Gamma(\mathcal{F})$ the space of its global sections i.e. vector fields tangent to $\mathcal{F}$. Also, a differential form φ is called *basic* if it satisfies $i_X\varphi = \mathcal{L}_X\varphi = 0$ for every $X \in \Gamma(\mathcal{F})$, where i_X and $\mathcal{L}_X$ denote the interior product and Lie derivative with respect to X, respectively. A *basic function* is a function constant on the leaves; such functions form an algebra denoted by $\mathcal{F}_b(\mathcal{M})$. The quotient $Q\mathcal{F} = T\mathcal{M}/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. A vector field $Y \in \mathcal{X}(\mathcal{M})$ is said to be *foliated* if, for every $X \in \Gamma(\mathcal{F})$ we have $[X, Y] \in \Gamma(\mathcal{F})$ and we denote by $\mathcal{X}(\mathcal{M}, \mathcal{F})$ the algebra of foliated vector fields on $\mathcal{M}$. The space $\mathcal{X}(\mathcal{M}/\mathcal{F}) = \mathcal{X}(\mathcal{M}, \mathcal{F})/\Gamma(\mathcal{F})$ is called the algebra of *basic vector fields* on $\mathcal{M}$.

In this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(z^1,\dots,z^n,y^1,\dots,y^m)$ on an open subset U on which the foliation is defined by the equations $dz^i=0$, $i=1,\dots,n$. If $\mathcal{F}$ is transversally holomorphic (see the below discussion) $z^1,\dots,z^n$ will be complex coordinates.

We recall that a transverse structure to $\mathcal{F}$ is a geometric structure on M invariant by all the local diffeomorphisms $f_{i,j}$. Such a transverse structure can be considered as a geometric structure on the leaf space $\mathcal{M}/\mathcal{F}$ which generally is not a manifold.

If M is a complex manifold and all $f_{i,j}$ are biholomorphic maps then we say that $\mathcal{F}$ is *transversally holomorphic*. In this case, any transversal to $\mathcal{F}$ inherits a complex structure. If, moreover, M is endowed with a Hermitian structure which is preserved by all $f_{i,j}$ then we say that $\mathcal{F}$ is *Hermitian*. In this case $f_{i,j}$ are in particular biholomorphic maps and isometries. Also, the normal bundle $Q\mathcal{F}$ is equipped with a Hermitian metric g "invariant along the leaves" which can be writen in a transverse local system of coordinates $(z^1,\dots,z^n)$ in the form $g=g_{j\overline{k}}(z,\overline{z})dz^j\otimes d\overline{z}^k$. If the associated basic 2–form $\omega=\frac{i}{2}g_{j\overline{k}}(z,\overline{z})dz^j\wedge d\overline{z}^k$ is closed then $\mathcal{F}$ is called *transversally Kählerian.*

Throughout this paper we consider $\mathcal{F}$ to be transversally holomorphic with $2n$ codimension. Let $\Omega^r(\mathcal{M}/\mathcal{F})$ be the space of all basic forms of degree r. It is easy to see that the exterior derivative of a basic form is also a basic form. Indeed, if $\varphi\in\Omega^r(\mathcal{M}/\mathcal{F})$ then $i_X\varphi=\mathcal{L}_X\varphi=0$ for any $X\in\Gamma(\mathcal{F})$ and, then by Cartan's formulas $\mathcal{L}_X=i_Xd+di_X$ and $d^2=0$ it follows that $i_Xd\varphi=\mathcal{L}_Xd\varphi=0$ for any $X\in\Gamma(\mathcal{F})$. Let us denote by $d_b=d|_{\Omega^\bullet(\mathcal{M}/\mathcal{F})}$ the restriction of exterior derivative to basic forms. Then we have $d_b:\Omega^\bullet(\mathcal{M}/\mathcal{F})\longrightarrow\Omega^{\bullet+1}(\mathcal{M}/\mathcal{F})$ and the differential complex

$$(1.2)\qquad 0\longrightarrow\Omega^0(\mathcal{M}/\mathcal{F})\xrightarrow{d_b}\Omega^1(\mathcal{M}/\mathcal{F})\xrightarrow{d_b}\dots\xrightarrow{d_b}\Omega^{2n}(\mathcal{M}/\mathcal{F})\longrightarrow 0$$

which is called the *basic de Rham complex* of $\mathcal{F}$; its cohomology is the basic de Rham cohomology $H^\bullet(\mathcal{M}/\mathcal{F})$. Now, we consider $Q_{\mathbb{C}}\mathcal{F}=Q\mathcal{F}\otimes_{\mathbb{R}}\mathbb{C}$ the complexified normal bundle of $\mathcal{F}$. Let J be the automorphism of $Q_{\mathbb{C}}\mathcal{F}$ associated to the complex structure; J satisfies $J^2=-\mathrm{Id}$ and then has two eigenvalues i and $-i$ with associated eigensubbundles respectively denoted by $Q^{1,0}\mathcal{F}$ and $Q^{0,1}\mathcal{F}=\overline{Q^{1,0}\mathcal{F}}$. We have a splitting $Q_{\mathbb{C}}\mathcal{F}=Q^{1,0}\mathcal{F}\oplus Q^{0,1}\mathcal{F}$ which gives rise to decomposition

$$\bigwedge^r(Q^*_{\mathbb{C}}\mathcal{F})=\bigoplus_{p+q=r}\bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F}),$$

where $\bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F})=\bigwedge^p(Q^{1,0*}\mathcal{F})\otimes\bigwedge^q(Q^{0,1*}\mathcal{F})$. Basic sections of $\bigwedge^{p,q}(Q^*_{\mathbb{C}}\mathcal{F})$ are called *basic forms of type* (p,q) on $(\mathcal{M},\mathcal{F})$. They form a vector space denoted by $\Omega^{p,q}(\mathcal{M}/\mathcal{F})$. We have

$$(1.3)\qquad \Omega^r(\mathcal{M}/\mathcal{F})=\bigoplus_{p+q=r}\Omega^{p,q}(\mathcal{M}/\mathcal{F}).$$

As in the classical case of a complex manifold, see [15], the basic exterior derivative is decomposed into two operators

$$\partial_b:\Omega^{p,q}(\mathcal{M}/\mathcal{F})\to\Omega^{p+1,q}(\mathcal{M}/\mathcal{F})\,,\ \overline{\partial}_b:\Omega^{p,q}(\mathcal{M}/\mathcal{F})\to\Omega^{p,q+1}(\mathcal{M}/\mathcal{F}).$$

We have $\partial_b^2 = \overline{\partial}_b^2 = 0$ and $\partial_b\overline{\partial}_b + \overline{\partial}_b\partial_b = 0$. The differential complex

$$(1.4)\qquad 0 \longrightarrow \Omega^{p,0}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \Omega^{p,1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} \Omega^{p,n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

is called the *basic Dolbeault complex* of $\mathcal{F}$; its cohomology $H^{p,\bullet}(\mathcal{M}/\mathcal{F})$ is the basic Dolbeault cohomology of foliation $\mathcal{F}$.

2. Basic Lichnerowicz cohomology of transversally locally conformally Kählerian foliations

2.1. Basic Lichnerowicz cohomology. Let $(\mathcal{M},\mathcal{F})$ be a transversally holomorphic foliation and $\theta \in \Omega^1(\mathcal{M}/\mathcal{F})$ be a closed basic 1-form. Denote by $d_{b,\theta} : \Omega^r(\mathcal{M}/\mathcal{F}) \to \Omega^{r+1}(\mathcal{M}/\mathcal{F})$ the map $d_{b,\theta} = d_b - \theta\wedge$.

Since $d_b\theta = 0$, we easily obtain that $d^2_{b,\theta} = 0$. The differential complex

$$(2.1)\qquad 0 \longrightarrow \Omega^0(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \Omega^1(\mathcal{M}/\mathcal{F}) \xrightarrow{d_{b,\theta}} \dots \xrightarrow{d_{b,\theta}} \Omega^{2n}(\mathcal{M}/\mathcal{F}) \longrightarrow 0$$

is called the *basic Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$ and its cohomology groups $H^\bullet_\theta(\mathcal{M}/\mathcal{F})$ are called the *basic Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

This is a basic version of the classical Lichnerowicz cohomology, motivated by Lichnerowicz's work [13] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic manifolds, see [1, 12]. We also notice that Vaisman in [24] studied it under the name of "adapted cohomology" on locally conformal Kähler (LCK) manifolds. Some notions concerning to a such basic Lichnerowicz cohomology of real foliations may be found in [7].

We notice that, locally, the basic Lichnerowicz complex becames the basic de Rham complex after a change $\varphi \mapsto e^f\varphi$ with f a basic function which satisfies $d_bf = \theta$, namely $d_{b,\theta}$ is the unique differential in $\Omega^\bullet(\mathcal{M}/\mathcal{F})$ which makes the multiplication by the smooth basic function e^f an isomorphism of cochain basic complexes $e^f : (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta}) \to (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_b)$.

2.1. Proposition. *The basic Lichnerowicz cohomology depends only on the basic class of θ. In fact, we have the isomorphism $H^r_{\theta-d_bf}(\mathcal{M}/\mathcal{F}) \approx H^r_\theta(\mathcal{M}/\mathcal{F})$.*

Proof. Since $d_{b,\theta}(e^f\varphi) = e^f d_{b,\theta-d_bf}\varphi$ it results that the map $[\varphi] \mapsto [e^f\varphi]$ is an isomorphism between $H^r_{\theta-d_bf}(\mathcal{M}/\mathcal{F})$ and $H^r_\theta(\mathcal{M}/\mathcal{F})$. □

For the basic Lichnerowicz cohomology, similar basic complexes of Dolbeault and Bott-Chern type can be defined. Taking into account the decomposition $\theta = \theta^{1,0} + \theta^{0,1}$, consider the Hodge components of the basic Lichnerowicz differential $d_{b,\theta} = d_b - \theta\wedge$ as

$$(2.2)\qquad d_{b,\theta} = \partial_{b,\theta} + \overline{\partial}_{b,\theta}\,,\ \partial_{b,\theta} = \partial_b - \theta^{1,0}\wedge\,,\ \overline{\partial}_{b,\theta} = \overline{\partial}_b - \theta^{0,1}\wedge\,.$$

The diferential complex

$$(2.3)\qquad \dots \xrightarrow{\overline{\partial}_{b,\theta}} \Omega^{p,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,\theta}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{\overline{\partial}_{b,\theta}} \dots$$

is called the *basic Dolbeault-Lichnerowicz complex* of $(\mathcal{M},\mathcal{F})$; its cohomology groups denoted by $H^{p,\bullet}_\theta(\mathcal{M}/\mathcal{F})$ are called the *basic Dolbeault-Lichnerowicz cohomology groups* of $(\mathcal{M},\mathcal{F})$.

The differential complex

$$\Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F}) \xrightarrow{\partial_{b,\theta}\overline{\partial}_{b,\theta}} \Omega^{p,q}(\mathcal{M}/\mathcal{F}) \xrightarrow{\partial_{b,\theta}\oplus\overline{\partial}_{b,\theta}} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}) \tag{2.4}$$

is called the *basic Bott-Chern-Lichnerowicz complex* of $(\mathcal{M}, \mathcal{F})$ and its cohomology groups

$$H^{\bullet,\bullet}_{BCL}(\mathcal{M}/\mathcal{F}) = \frac{\mathrm{Ker}\{\Omega^{\bullet,\bullet} \xrightarrow{\partial_{b,\theta}} \Omega^{\bullet+1,\bullet}\} \cap \mathrm{Ker}\{\Omega^{\bullet,\bullet} \xrightarrow{\overline{\partial}_{b,\theta}} \Omega^{\bullet,\bullet+1}\}}{\mathrm{Im}\{\Omega^{\bullet-1,\bullet-1} \xrightarrow{\partial_{b,\theta}\overline{\partial}_{b,\theta}} \Omega^{\bullet,\bullet}\}}$$

are called the *basic Bott-Chern-Lichnerowicz cohomology groups* of $(\mathcal{M}, \mathcal{F})$.

In the end of this subsection we apply some considerations from [24] for basic forms and we obtain a relation between a twisted basic cohomology associated to θ and basic real cohomology of $(\mathcal{M}, \mathcal{F})$. For every θ as above, let us consider now the auxiliary basic operator $\widetilde{d}_b = d_b - \frac{r}{2}\theta\wedge$ where r is the degree of the basic form acted on. We notice that $\widetilde{d}_b$ is an antiderivation of basic differential forms and it is easy to see that $\widetilde{d}_b^2 = -\frac{1}{2}\theta \wedge d_b$. Then $\widetilde{d}_b$ defines a *twisted basic cohomology* of basic differential forms of $(\mathcal{M}, \mathcal{F})$, which is given by

$$H^{\bullet}_{\widetilde{d}_b}(\mathcal{M}/\mathcal{F}) = \frac{\mathrm{Ker}\,\widetilde{d}_b}{\mathrm{Im}\,\widetilde{d}_b \cap \mathrm{Ker}\,\widetilde{d}_b} \tag{2.5}$$

and is isomorphic to the cohomology of the basic complex $(\widetilde{\Omega}^{\bullet}(\mathcal{M}/\mathcal{F}), \widetilde{d}_b)$ consisting of the basic differential forms $\varphi \in \Omega^{\bullet}(\mathcal{M}/\mathcal{F})$ satisfying $\widetilde{d}_b^2\varphi = -\theta \wedge d_b\varphi = 0$.

The basic complex $\widetilde{\Omega}^{\bullet}(\mathcal{M}/\mathcal{F})$ admits a basic subcomplex $\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})$, namely, the ideal generated by θ. On this subcomplex, $\widetilde{d}_b = d_b$, which means that it is a basic subcomplex of the usual basic de Rham complex of $(\mathcal{M}, \mathcal{F})$. Hence, one has the homomorphisms

$$a : H^r(\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})) \to H^r_{\widetilde{d}_b}(\mathcal{M}/\mathcal{F})\,,\; b : H^r(\Omega^{\bullet}_{\theta}(\mathcal{M}/\mathcal{F})) \to H^r(\mathcal{M}/\mathcal{F}, \mathbb{R}). \tag{2.6}$$

Now, we can easily construct a homomorphism

$$c : H^r_{\widetilde{d}_b}(\mathcal{M}/\mathcal{F}) \to H^{r+1}(\mathcal{M}/\mathcal{F}, \mathbb{R}). \tag{2.7}$$

Indeed, if $[\varphi] \in H^r_{\widetilde{d}_b}(\mathcal{M}/\mathcal{F})$, where φ is $\widetilde{d}_b$-closed basic form, then we put $c([\varphi]) = [\theta \wedge \varphi]$, and this produces the homomorphism from (2.7). We notice that the existence of c gives some relation between $\widetilde{d}_b$ and the basic real cohomology of $(\mathcal{M}, \mathcal{F})$.

2.2. Remark. If we consider the decomposition $\widetilde{d}_b = \widetilde{\partial}_b + \widetilde{\overline{\partial}}_b$ we can construct analogous of homomorphisms a, b and c from (2.6) and (2.7), respectively, for corresponding basic Dolbeault cohomology.

2.2. Basic Lichnerowicz cohomology of transversally LCK foliations. In this subsection we consider the notion of transversally locally conformally Kählerian foliation that is a version of locally conformally Kähler manifold notion, see [22, 23, 24], for transversally Kählerian foliations, and we investigate some problems related to basic Lichnerowicz cohomology for such structures.

2.3. Definition. A locally conformally transversally Kählerian foliation, briefly transversally LCK foliation, is a transversally Hermitian foliation $(\mathcal{M}, \mathcal{F}, g)$ for which an open covering $\{U_i\}$ of $\mathcal{M}$ exists, and for each i a basic function $\sigma_i : U_i \to \mathbb{R}$ such that $\widetilde{g}_i = e^{-\sigma_i}(g|_{U_i})$ is a transverse Kähler metric on U_i.

It is easy to see that $\theta|_{U_i} = d_b\sigma_i$ defines a global d_b–closed 1–form, and $(\mathcal{M}, \mathcal{F}, \omega)$ has the characteristic property [2]:

$$d_b\omega = \theta \wedge \omega, \tag{2.8}$$

where ω is the basic Hermitian form on $(\mathcal{M}, \mathcal{F})$. If we take $U_i = \mathcal{M}$, then $(\mathcal{M}, \mathcal{F}, \omega)$ is called globally conformally Kählerian foliation. The basic form θ is called the basic Lee form of $(\mathcal{M}, \mathcal{F}, \omega)$. It is exact iff $(\mathcal{M}, \mathcal{F}, \omega)$ is globally conformal Kählerian foliation.

2.4. Example. A simple foliation defined by a C^∞ submersion $f : \mathcal{M} \to M$ of $\mathcal{M}$ onto a LCK manifold M is transversally LCK foliation. The case where the dimension of the leaves is zero corresponds to the original LCK manifolds.

2.5. Remark. If $(\mathcal{M}, \mathcal{F})$ is a complex analytic foliated manifold, [20], then similarly to Proposition 1.1 from [22], the foliation $\mathcal{F}$ is a transversally LCK foliation if and only if its transverse bundle $Q\mathcal{F}$ has a Kähler metric which is locally conformally with a foliated Hermitian metric, or equivalently $(\mathcal{M}, \mathcal{F})$ has a Hermitian metric which is locally conformally with a bundle-like metric.

Now, if $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation with basic Lee form θ, then due to (2.8) we have $d_{b,\theta}\omega = 0$. Therefore, ω represents a cohomology class $[\omega]_L$ in the basic Lichnerowicz complex $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta})$.

2.6. Definition. The basic cohomology class $[\omega]_L \in H^2_\theta(\mathcal{M}/\mathcal{F})$ is called the *basic Lichnerowicz class* of the transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega)$.

This invariant is a basic version of Morse-Novikov class of LCK manifolds, see [18].

Also, if we consider the decomposition $d_{b,\theta} = \partial_{b,\theta} + \overline{\partial}_{b,\theta}$ we have $\partial_{b,\theta}\omega = \overline{\partial}_{b,\theta}\omega = 0$ and so ω represents a cohomology class $[\omega]_{BCL}$ in the basic Bott-Chern-Lichnerowicz complex of $(\mathcal{M}, \mathcal{F}, \omega)$.

2.7. Definition. If $(\mathcal{M}, \mathcal{F}, \omega)$ is a transversally LCK foliation then the cohomology class $[\omega]_{BCL} \in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$ is called the *basic Bott-Chern-Lichnerowicz class* of $(\mathcal{M}, \mathcal{F}, \omega)$.

Thus, for any transversally LCK foliation we have three basic cohomological invariants:

- the basic Lee class $[\theta] \in H^1(\mathcal{M}/\mathcal{F})$;
- the basic Lichnerowicz class $[\omega]_L \in H^2_\theta(\mathcal{M}/\mathcal{F})$;
- the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL} \in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$.

Now, using an argument inspired from [11], we briefly present an another basic cohomology associated to transversally LCK foliations which is connected with the basic Lichnerowicz cohomology of transversally LCK foliations. Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ. We consider the basic closed 1-forms θ_0 and θ_1 defined by

(2.9) $\quad \theta_0 = m\theta$ and $\theta_1 = (m+1)\theta,\ m \in \mathbb{R}$.

Denote by $H^\bullet_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{\theta_1}(\mathcal{M}/\mathcal{F})$ the basic Lichnerowicz cohomologies of the basic complexes $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_0})$ and $(\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{\theta_1})$, repectively.

Now, let $\widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) = \Omega^k(\mathcal{M}/\mathcal{F}) \oplus \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ and $\widehat{d}_b : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \to \widehat{\Omega}^{k+1}(\mathcal{M}/\mathcal{F})$ be the basic differential operator defined by

(2.10) $\quad \widehat{d}_b(\varphi, \psi) = (d_{b,\theta_1}\varphi - \omega \wedge \psi, -d_{b,\theta_0}\psi)$.

Using (2.8), by direct calculus it follows $\widehat{d}_b^2 = 0$. Thus, we can consider the basic complex $(\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b)$ and $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$ the associated basic cohomology. We have the following result which relates $\widehat{H}^\bullet(\mathcal{M}/\mathcal{F})$ with basic Lichnerowicz cohomologies $H^\bullet_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^\bullet_{\theta_1}(\mathcal{M}/\mathcal{F})$.

2.8. Proposition. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ. Suppose that $i^k : \Omega^k(\mathcal{M}/\mathcal{F}) \to \widehat{\Omega}^k(\mathcal{M}/\mathcal{F})$ and $\pi_2^k : \widehat{\Omega}^k(\mathcal{M}/\mathcal{F}) \to \Omega^{k-1}(\mathcal{M}/\mathcal{F})$ are homomorphisms of $\mathcal{F}_b(\mathcal{M}, \mathbb{R})$-modules defined by*

$$i^k(\varphi) = (\varphi, 0) \text{ and } \pi_2^k(\varphi, \psi) = \psi,$$

for $\varphi \in \Omega^k(\mathcal{M}/\mathcal{F})$ and $\psi \in \Omega^{k-1}(\mathcal{M}/\mathcal{F})$. Then:

i) *The mappings i^k and π_2^k induce an exact sequence of basic complexes*

$$0 \longrightarrow (\Omega^\bullet(\mathcal{M}/\mathcal{F}), d_{b,\theta_1}) \xrightarrow{i^k} (\widehat{\Omega}^\bullet(\mathcal{M}/\mathcal{F}), \widehat{d}_b) \xrightarrow{\pi_2^k} (\Omega^{\bullet-1}(\mathcal{M}/\mathcal{F}), -d_{b,\theta_0}) \longrightarrow 0.$$

ii) *This exact sequence induces a long exact sequence*

$$\ldots\ H^k_{\theta_1}(\mathcal{M}/\mathcal{F}) \xrightarrow{(i^k)^*} \widehat{H}^k(\mathcal{M}/\mathcal{F}) \xrightarrow{(\pi_2^k)^*} H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F}) \xrightarrow{-\delta^{k-1}} H^{k+1}_{\theta_1}(\mathcal{M}/\mathcal{F})\ \ldots,$$

where the connecting homomorphism $-\delta^{k-1}$ is defined by

(2.11) $\quad (-\delta^{k-1})[\varphi] = [\varphi \wedge \omega],\ \forall\, [\varphi] \in H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F})$.

From the above proposition, we obtain

2.9. Corollary. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ and such that the basic Lichnerowicz cohomology groups $H^k_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^k_{\theta_1}(\mathcal{M}/\mathcal{F})$ have finite dimension, for all k. Then, the basic cohomology group $\widehat{H}^k(\mathcal{M}/\mathcal{F})$ has also finite dimension, for all k, and*

(2.12) $$\widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong \frac{H^k_{\theta_1}(\mathcal{M}/\mathcal{F})}{\operatorname{Im}\delta^{k-2}} \oplus \ker \delta^{k-1},$$

where $\delta^k : H^k_{\theta_0}(\mathcal{M}/\mathcal{F}) \to H^{k+2}_{\theta_1}(\mathcal{M}/\mathcal{F})$ is the homomorphism given by (2.11).

2.10. Corollary. *Let $(\mathcal{M}, \mathcal{F}, \omega)$ be a transversally LCK foliation with basic Lee form θ such that the dimensions of the basic cohomology groups $H^k_{\theta_0}(\mathcal{M}/\mathcal{F})$ and $H^k_{\theta_1}(\mathcal{M}/\mathcal{F})$ are finite, for all k. If the basic Lichnerowicz class $[\omega]_L$ vanish then, for all k, we have*

(2.13) $\quad \widehat{H}^k(\mathcal{M}/\mathcal{F}) \cong H^k_{\theta_1}(\mathcal{M}/\mathcal{F}) \oplus H^{k-1}_{\theta_0}(\mathcal{M}/\mathcal{F})$.

2.3. Basic Bott-Chern cohomology of transversally LCK foliations. In this subsection we present a link between basic Bott-Chern cohomology, basic Lichnerowicz cohomology and 0-th basic Dolbeault cohomology with values in the associated foliated weight bundle of a transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega)$. The notions are introduced by analogy with the corresponding notions in the case of LCK manifolds, [18].

Let us consider further $(\mathcal{M}, \mathcal{F}, \omega)$ to be a transversally Kählerian foliation. Then the basic Kähler form ω determines the basic Kähler class $[\omega] \in H^{1,1}(\mathcal{M}/\mathcal{F})$, and the difference of basic Kähler forms which have the same basic Kähler class is measured by a basic potential f:

$$\omega_1 - \omega = \partial_b \overline{\partial}_b f$$

see Proposition 3.5.1. from [4]. (This also follows from $\partial_b\overline{\partial}_b$-Lemma for transversally Kähler foliations, [17]). Thus the space of all basic Kähler structures on a transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ is locally modeled on $H^{1,1}(\mathcal{M}/\mathcal{F}, \mathbb{R}) \times (\mathcal{F}_b(\mathcal{M})/\mathrm{const})$. A similar local description exists for the set of all basic LCK-structures on a given transversally holomorphic foliation, if we fix the basic cohomology class $[\theta]$ of a basic Lee form. Using the basic Bott-Chern-Lichnerowicz class $[\omega]_{BCL} \in H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F})$ of a basic LCK form ω, similarly to [18], we can obtain that the difference of two basic LCK forms in the same basic Bott-Chern-Lichnerowicz class is expressed by a basic potential, just like in transversally Kähler case, and the set of all basic LCK structures on a given transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ is locally parametrized by

$$H^{1,1}_{BCL}(\mathcal{M}/\mathcal{F}) \times (\mathcal{F}_b(\mathcal{M})/\mathrm{Ker}\, d_{b,\theta} d^c_{b,\theta}), \tag{2.14}$$

where $d_{b,\theta} d^c_{b,\theta} = -2\sqrt{-1}\partial_{b,\theta}\overline{\partial}_{b,\theta}$.

In order to find a connection between basic Bott-Chern cohomology, basic Lichnerowicz cohomology and 0–th basic Dolbeault cohomology with values in the associated foliated weight bundle of a transversally LCK foliation $(\mathcal{M}, \mathcal{F}, \omega)$, we briefly recall some definitions concerning to foliated bundles and basic connections, see [5, 9, 14].

Let $G \hookrightarrow P \to M$ be a principal bundle with structural group $G \subset \mathrm{GL}(n, \mathbb{C})$. The group G acts on P on the right and on its Lie algebra $\mathcal{G}$ by the adjoint representation Ad i.e., for $g \in G$ and $X \in \mathcal{G}$, $\mathrm{Ad}_g(X) = gXg^{-1}$. We say that a principal G-bundle $P \to (\mathcal{M}, \mathcal{F})$ is a *foliated principal bundle* if it is equipped with a foliation $\mathcal{F}_P$ (*the lifted foliation*) such that the distribution $T\mathcal{F}_P$ is invariant under the right action of G, is transversal to the tangent space to the fiber, and projects to $T\mathcal{F}$. A connection ω on P is called *adapted* to $\mathcal{F}_P$ if the associated horizontal distribution contains $T\mathcal{F}_P$. An adapted connection γ is called a *basic connection* if it is basic as a $\mathcal{G}$-valued form on $(P, \mathcal{F}_P)$.

Let us consider now $E \to (\mathcal{M}, \mathcal{F})$ be a complex vector bundle defined by a cocycle $\{U_i, g_{ij}, G\}$ where $\{U_i\}$ is an open cover of $\mathcal{M}$ and $g_{ij} : U_i \cap U_j \to G \subset \mathrm{GL}(n, \mathbb{C})$ are the transition functions. To such a vector bundle we can always associate a principal G-bundle $P \to (\mathcal{M}, \mathcal{F})$ whose fibre is the group G and the transition functions are g_{ij} (viewed as translations on G). The complex vector bundle $E \to (\mathcal{M}, \mathcal{F})$ is *foliated* if E is associated to a foliated principal G-bundle $P \to (\mathcal{M}, \mathcal{F})$. Let $\Omega^\bullet(\mathcal{M}, E)$ denote the space of forms on $(\mathcal{M}, \mathcal{F})$ with values in E.

If a connection form γ on P is adapted, then we say that an associated covariant derivative operator ∇ on $\Omega^\bullet(\mathcal{M}, E)$ is *adapted* to the foliated bundle. We say that ∇ is a *basic* connection on E if in addition the associated curvature operator ∇^2 satisfies $i_X\nabla^2 = 0$ for every $X \in \Gamma(\mathcal{F})$. Note that ∇ is basic if the principal connection γ associated to ∇ is basic. Let $\Gamma(E)$ denote the smooth sections of E, and let ∇ denote a basic connection on E. We say that a section $s : \mathcal{M} \to E$ is a *basic section* if and only if $\nabla_X s = 0$ for all $X \in \Gamma(\mathcal{F})$. Let ∇_b denote the basic connection and $\Gamma_b(E)$ denote the space of basic sections of E.

Now, let us consider E to be a foliated complex line bundle over the transversally holomorphic foliation $(\mathcal{M}, \mathcal{F})$ with a flat basic connection ∇_b. We denote by $\Omega^{p,q}(\mathcal{M}/\mathcal{F}, E)$ the set of all basic (p, q)–forms on $(\mathcal{M}, \mathcal{F})$ with values in E. Consider the basic complex

$$\ldots \longrightarrow \Omega^{p-1,q-1}(\mathcal{M}/\mathcal{F}, E) \overset{\partial_{b,E}\overline{\partial}_{b,E}}{\longrightarrow} \Omega^{p,q}(\mathcal{M}/\mathcal{F}, E)$$

$$\overset{\partial_{b,E}\oplus\overline{\partial}_{b,E}}{\longrightarrow} \Omega^{p+1,q}(\mathcal{M}/\mathcal{F}, E) \oplus \Omega^{p,q+1}(\mathcal{M}/\mathcal{F}, E) \longrightarrow \ldots, \tag{2.15}$$

where $\partial_{b,E}$ and $\overline{\partial}_{b,E}$ denote the $(1, 0)$ and $(0, 1)$-parts of the basic connection operator $\nabla_b : \Omega^\bullet(\mathcal{M}/\mathcal{F}, E) \to \Omega^{\bullet+1}(\mathcal{M}/\mathcal{F}, E)$. The cohomology of (2.15) denoted by $H^{p,q}_{BC}(\mathcal{M}/\mathcal{F}, E)$ is called the *basic Bott-Chern cohomology of* $(\mathcal{M}, \mathcal{F})$ *with values in* E.

2.11. Definition. Let $(\mathcal{M}, \mathcal{F}, \omega, \theta)$ be a transversally LCK foliation, and E its foliated weight bundle, that is, a trivial complex foliated line bundle with the flat basic connection $d_b - \theta$. Consider ω as a closed basic $(1, 1)$-form on $(\mathcal{M}, \mathcal{F})$ with values in E. Its basic Bott-Chern class $[\omega]_{BC} \in H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E)$ is called the *basic Bott-Chern class of the transversally LCK foliation* $(\mathcal{M}, \mathcal{F}, \omega, \theta)$.

Now, similarly to [18], we give a characterization of $H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E)$ in terms of basic Lichnerowicz cohomology of $(\mathcal{M}, \mathcal{F}, \theta)$ and 0–th basic Dolbeault cohomology of the foliated weight bundle E.

The 0–th basic Dolbeault cohomology with values in a foliated bundle E can be realized as cohomology of the complex

$$\Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \overset{\overline{\partial}_{b,E}}{\longrightarrow} \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \overset{\overline{\partial}_{b,E}}{\longrightarrow} \Omega^{0,2}(\mathcal{M}/\mathcal{F}, E) \overset{\overline{\partial}_{b,E}}{\longrightarrow} \ldots \tag{2.16}$$

If E has a flat basic connection, then $\partial_{b,E} : \Omega^{0,1}(\mathcal{M}/\mathcal{F}, E) \to \Omega^{1,1}(\mathcal{M}/\mathcal{F}, E)$ induces a map

$$H^{0,1}(\mathcal{M}/\mathcal{F}, \mathcal{E}) \overset{\partial^*_{b,E}}{\longrightarrow} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F}, E) \tag{2.17}$$

from the 0-th basic Dolbeault cohomology with values in the underlying holomorphic foliated bundle (denoted as $\mathcal{E}$) to the basic Bott-Chern cohomology with values in E. The basic complex

$$\Gamma_b(E) = \Omega^{0,0}(\mathcal{M}/\mathcal{F}, E) \overset{\nabla^{1,0}_b}{\longrightarrow} \Omega^{1,0}(\mathcal{M}/\mathcal{F}, E) \overset{\nabla^{1,0}_b}{\longrightarrow} \Omega^{2,0}(\mathcal{M}/\mathcal{F}, E) \overset{\nabla^{1,0}_b}{\longrightarrow} \ldots \tag{2.18}$$

computes the 0-th basic Dolbeault cohomology with values in holomorphic foliated bundle $\mathcal{E}'$ with a holomorphic structure defined by the complex conjugate of the $\nabla^{1,0}_b$-part of the basic connection. When the foliated bundle E is real, we have $\mathcal{E} \approx \mathcal{E}'$. Then the cohomology of the basic complex (2.18) is naturally identified with $\overline{H^{0,\bullet}(\mathcal{M}/\mathcal{F},\mathcal{E})}$. The map $\overline{\partial}_{b,E} : \Omega^{1,0}(\mathcal{M}/\mathcal{F},E) \to \Omega^{1,1}(\mathcal{M}/\mathcal{F},E)$ defines a homomorphism

$$\overline{H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})} \stackrel{\overline{\partial}^*_{b,E}}{\longrightarrow} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E) \tag{2.19}$$

which is entirely similar to (2.17).

Following step by step the proof of Theorem 4.7. from [18], we obtain an analogous result for basic cohomologies, which allows to compute the basic Bott-Chern cohomology classes in terms of 0–th basic Dolbeault cohomology and basic Lichnerowicz cohomology.

2.12. Theorem. *Let $(\mathcal{M},\mathcal{F})$ be a transversally holomorphic foliation and $E_{\mathbb{R}}$ a trivial real foliated line bundle with flat basic connection $d_b-\theta$, where θ is a real closed basic 1-form. Denote by E its complexification, and let $\mathcal{E}$ be the underlying holomorphic bundle. Then there is an exact sequence*

$$H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E}) \oplus \overline{H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})} \stackrel{\partial^*_{b,E}+\overline{\partial}^*_{b,E}}{\longrightarrow} H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E) \stackrel{\nu}{\longrightarrow} H^2_\theta(\mathcal{M}/\mathcal{F}), \tag{2.20}$$

where $H^2_\theta(\mathcal{M}/\mathcal{F})$ is the basic Lichnerowicz cohomology, ν a tautological map, and the first arrow is obtained as a direct sum of (2.17) and (2.19).

From the above theorem, we immediately obtain

2.13. Corollary. *Let $(\mathcal{M},\mathcal{F},\omega,\theta)$ be a transversally LCK foliation, E the corresponding flat foliated weight bundle, and $\mathcal{E}$ the underlying holomorphic foliated bundle. Asume that $H^{0,1}(\mathcal{M}/\mathcal{F},\mathcal{E})=0$ and $H^2_\theta(\mathcal{M}/\mathcal{F})=0$. Then $H^{1,1}_{BC}(\mathcal{M}/\mathcal{F},E)=0$.*

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