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Dual Zariski Topology on Comultiplication Modules

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Abstract

This paper deals with dual Zariski topology on comultiplication modules. We define a subspace topology of dual Zariski topology on comultiplication modules and study some properties of this subspace topology. We prove that X_N^s is an Artinian topological space if and only if M satisfies the SN-condition.

Keywords: Comultiplication Module, Dual Zariski Topology, Second Module

1. INTRODUCTION

A commutative ring with identity will be denoted by R and unital modules on R will be denoted by M.

In the module theory, the notion of second submodule is known as the dual notion of prime submodule. Similarly, Zariski topology on the set of all second submodules of M, denoted by $Spec^s(M)$, is known as dual Zariski topology. These two notions have been studied by many authors ([1], [2], [4], [5], [8], [11]).

Let N be a submodule of M.
If N≠0 and for all r∈R, either rN = 0 or rN = N holds, N is called a second submodule.
Let $V^s(N) = \{P \in Spec^s(M): P \subseteq N\}$ be a set. Then the following hold:
i) $V^s(0) = \emptyset$ and $V^s(M)=Spec^s(M)$.
ii) $\bigcap_{i \in \Lambda} V^s(N_i)=V^s(\bigcap_{i \in \Lambda} N_i)$ for any family of submodules N_i of M (i∈Λ).

iii) $V^s(A) \cup V^s(B) \subseteq V^s(A+B)$, where A and B are submodules of M.

For every submodule N of an R-module M, if there exists an ideal I of R such that $N = (0 :_M I)$, then M is called a comultiplication R-module.

Let K and L be any ideals of R. It is clear that
$$V^s((0 :_M K)) \cup V^s((0 :_M L))=$$
$$V^s((0 :_M K)+ (0 :_M L))=$$
$$V^s((0 :_M K \cap L))= V^s((0 :_M KL)).$$

Let K be an ideal of R.
Then $\Gamma^s(M)=\{V^s((0 :_M K))\}$ satisfies the axioms for closed sets of a topological space on $Spec^s(M)$. If M is a comultiplication R-module, then there exists a topology, the collection of whose all closed sets is $\{V^s(N): N \leq M\}$, on $Spec^s(M)$.

This paper deals with dual Zariski topology on comultiplication modules. With the aim of obtaining some connections between algebraic properties of comultiplication modules and

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topological properties of dual Zariski topology on comultiplication modules, we define a subspace topology of dual Zariski topology on comultiplication modules in Lemma 2.1 and study some properties of this subspace topology in Theorem 2.2, Proposition 2.3, Proposition 2.4, and Theorem 2.6. In order to find a direct relation between the subspace topology and the module, we define a condition in Definition 2.7 and we prove that X_N^s is an Artinian topological space if and only if M satisfies the SN-condition in Theorem 2.9.

Now we recall some definitions and properties related to dual Zariski topology used in this article as follows:

The ring R/Ann(M) is denoted by $\bar{R}$. The map Ψ defined by $\Psi(N)=Ann(N)/Ann(M)$ for every $N \in Spec^s(M)$ is a natural map from $Spec^s(M)$ to $Spec(R/Ann(M))$. Then the following hold:
i) For every ideal I of R, which contains Ann(M), $\Psi^{-1}(V^{\bar{R}}(\bar{I}))=V^s((0:_M I))$.
ii) Ψ is the Zariski topology on Spec(R/Ann(M)).
iii) Let Ψ be surjective. Then Ψ is both closed and open. For every submodule N of M,

$\Psi(V^s(N))= V^{\bar{R}}(Ann(N)/Ann(M))$ and
$$\Psi(Spec^s(M)\backslash V^s(N))=$$
$$Spec(\bar{R})\backslash V^{\bar{R}}(Ann(N)/Ann(M)).$$

Let N be a submodule of M. The second radical N denoted by Sec(N) is defined as the sum of all second submodules in M, which is contained in N ([5]).

Let X be a topological space. If the closed subsets of X satisfy the ascending chain condition, X is called Artinian ([3]).

2. A SUBSPACE OF DUAL ZARISKI TOPOLOGY ON COMULTIPLICATION MODULES

The following lemma defines a subspace topology of dual Zariski topology on comultiplication modules.

Lemma 2.1 Let $N = (0 :_M I)$ be a submodule of a comultiplication R-module M, where I is an ideal of a ring R. Let $X_N^s = Spec^s(M)\backslash V^s((0 :_M I))$ and $\tilde{V}^s((0 :_M J)) = V^s((0 :_M J))\backslash V^s((0 :_M I))$, where J is an ideal of R. Then $\Gamma_N^s = \{ \tilde{V}^s((0 :_M J))\}$ satisfies the axioms for closed sets of a topological space on X_N^s.

Proof i) Let J be an ideal of R. If J=0, $V^s((0 :_M 0))=Spec^s(M)$. Thus
$$\tilde{V}^s((0 :_M 0))= V^s((0 :_M 0))\backslash V^s((0 :_M I))=$$
$$Spec^s(M)\backslash V^s((0 :_M I))= X_N^s$$

and so $X_N^s \in \Gamma_N^s$.

Since $(Spec^s(M), \Gamma^s)$ is a topological space, $V^s((0 :_M R))=\emptyset$ and so
$$\tilde{V}^s((0 :_M R))= V^s((0 :_M R))\backslash V^s((0 :_M I))$$
$$= \emptyset\backslash V^s((0 :_M I))= \emptyset$$

and so $\emptyset \in \Gamma_N^s$.

ii) For $i \in \Lambda$, let $\{\tilde{V}^s((0 :_M B_i))\}$ be any family of Γ_N^s, where B_i is an ideal of R. Since
$$\bigcap_{i \in \Lambda} V^s((0 :_M B_i))=V^s(\bigcap_{i \in \Lambda}(0 :_M B_i)),$$
$$\bigcap_{i \in \Lambda} \tilde{V}^s((0 :_M B_i))= \bigcap_{i \in \Lambda}(V^s((0 :_M B_i))\backslash V^s((0 :_M I)))$$
$$=(\bigcap_{i \in \Lambda} V^s((0 :_M B_i)))\backslash V^s((0 :_M I))$$
$$=V^s(\bigcap_{i \in \Lambda}(0 :_M B_i))\backslash V^s((0 :_M I))$$
$$= \tilde{V}^s(\bigcap_{i \in \Lambda}(0 :_M B_i)) \in \Gamma_N^s.$$

iii) Let $\tilde{V}^s((0 :_M U_1))$ and $\tilde{V}^s((0 :_M U_2))$ be in Γ_N^s, where U_1 and U_2 are ideals of R. Since $(Spec^s(M), \Gamma^s)$ is a topological space,
$$V^s((0 :_M U_1)) \cup V^s((0 :_M U_2))=$$
$$V^s((0 :_M U_1)+(0 :_M U_2)),$$

and it follows that
$$\tilde{V}^s((0 :_M U_1)) \cup \tilde{V}^s((0 :_M U_2))$$
$$=[V^s((0 :_M U_1)) \cup V^s((0 :_M U_2))]\backslash V^s((0 :_M I))$$

$$= [V^s((0 :_M U_1)+(0 :_M U_2))] \backslash V^s((0 :_M I))$$

$$= \tilde{V}^s ((0 :_M U_1)+(0 :_M U_2)) \in \Gamma_N^s .$$

We name this topology as the complement dual Zariski topology of N in M.

We fix the submodule N of M as $N=(0:_M I)$, where I is an ideal of R, and the module M as a comultiplication R-module for the rest of this article.

Before starting the following theorem, note that

$$V^s((0 :_M J)) = V^s((0 :_M \sum_{r_i \in J} Rr_i))$$

$$= V^s(\bigcap_{r_i \in J} (0 :_M Rr_i)) = \bigcap_{r_i \in J} V^s((0 :_M Rr_i)).$$

Theorem 2.2 Let $N=(0:_M I)$ be a submodule of M. For any ideal J of R, the set

$(X_N^s)^{(0:_M J)} = X_N^s \backslash \tilde{V}^s((0 :_M J))$ forms a base for the complement dual Zariski topology of N in M on X_N^s.

Proof If X_N^s is empty, then $(X_N^s)^{(0:_M J)} = \emptyset$, which is the trivial case. Therefore X_N^s can be assumed different from the empty set.

Let $U \subset X_N^s$ be an open set. Then

$U = X_N^s \backslash \tilde{V}^s((0 :_M J))$. Thus

$$U = X_N^s \backslash \tilde{V}^s((0 :_M \sum_{r_i \in J} Rr_i))$$

$$= X_N^s \backslash \bigcap_{r_i \in J} \tilde{V}^s((0 :_M Rr_i)) = \bigcup_{r_i \in J} (X_N^s \backslash \tilde{V}^s((0 :_M Rr_i))$$

$$= \bigcup_{r_i \in J} (X_N^s)^{(0:_M Rr_i)} .$$

We are interested in the properties of this subspace topology.

Proposition 2.3 Let $N=(0:_M I)$ be a submodule of M. Then The following hold:

i) $(X_N^s)^{(0:_M J)} = X_N^s \backslash \tilde{V}^s((0 :_M J)) = X^s \backslash V^s((0:_M IJ))$ for ideal J of R.

ii) $(X_N^s)^{(0:_M J_1)} \bigcap (X_N^s)^{(0:_M J_2)} = (X_N^s)^{(0:_M J_1 J_2)}$ for every ideal J_1, J_2 of R.

Proof i) $(X_N^s)^{(0:_M J)} = X_N^s \backslash \tilde{V}^s((0 :_M J))$

$= Spec^s(M) \backslash (V^s((0:_M I) \cup V^s((0:_M J)))$

$= Spec^s(M) \backslash V^s((0:_M IJ))$.

ii) Let $P \in (X_N^s)^{(0:_M J_1)} \bigcap (X_N^s)^{(0:_M J_2)}$. Then

$P \in (X_N^s)^{(0:_M J_1)}$ and $P \in (X_N^s)^{(0:_M J_1)}$, which means

$P \notin V^s((0:_M IJ_1))$ and $P \notin V^s((0:_M IJ_2))$, implying

$P \notin V^s((0:_M IJ_1)) \cup V^s((0:_M IJ_2)) = V^s((0:_M IJ_1 J_2)$. Since $P \in X^s$, $P \in (X_N^s)^{(0:_M J_1 J_2)}$ and thus

$(X_N^s)^{(0:_M J_1)} \bigcap (X_N^s)^{(0:_M J_2)} \subseteq (X_N^s)^{(0:_M J_1 J_2)}$.

Let $P \in (X_N^s)^{(0:_M J_1 J_2)} = Spec^s(M) \backslash V^s((0:_M IJ_1 J_2))$. Then $P \notin V^s((0:_M IJ_1 J_2)) = V^s((0:_M IJ_1)) \cup V^s((0:_M IJ_2))$. One can observe that $P \notin V^s((0:_M IJ_1))$ and $P \notin V^s((0:_M IJ_2))$. Since $P \in Spec^s(M)$, we have $P \in Spec^s(M) \backslash V^s(0:_M IJ_1)$ and

$P \in Spec^s(M) \backslash V^s(0:_M IJ_2)$. Thus

$P \in (X_N^s)^{(0:_M J_1)} \bigcap (X_N^s)^{(0:_M J_2)}$, which implies

$(X_N^s)^{(0:_M J_1 J_2)} \subseteq (X_N^s)^{(0:_M J_1)} \bigcap (X_N^s)^{(0:_M J_2)}$.

The following proposition reveals some connections between some properties of the complement dual Zariski topology and the second module.

Proposition 2.4 Let $N=(0:_M I)$ be a submodule of M. Then the following hold:

i) $(X_N^s)^{(0:_M J)} = \emptyset$ if and only if

$Sec(M) \subseteq (0 :_M IJ)$ for every ideal J of R.

ii) $(X_N^s)^{(0:_M J_1)} = (X_N^s)^{(0:_M J_2)}$ if and only if

$Sec((0 :_M IJ_1)) = Sec((0 :_M IJ_2))$ for every ideal J_1, J_2 of R.

Proof i) Let $(X_N^s)^{(0:_M J)} = \emptyset$.

Then $\text{Spec}^s(M)= V^s((0:_M IJ))$. Since every second submodule of M is also contained in $(0:_M IJ)$, we have $\text{Sec}(M) \subseteq (0:_M IJ)$.

Let $\text{Sec}(M) \subseteq (0:_M IJ)$.

Then $P \subseteq (0:_M IJ)$ for every second submodule P of M. Thus $\text{Spec}^s(M)=V^s((0:_M IJ))$, which means that $(X_N^s)^{(0:_M J)} = \emptyset$.

ii) Let $(X_N^s)^{(0:_M J_1)} = (X_N^s)^{(0:_M J_2)}$. Then

$V^s(0:_M IJ_1)= V^s(0:_M IJ_2)$, implying

$\text{Sec}((0 :_M IJ_1))= \text{Sec}((0 :_M IJ_2))$.

Let $\text{Sec}((0 :_M IJ_1))= \text{Sec}((0 :_M IJ_2))$. Then

$V^s(0:_M IJ_1)= V^s(0:_M IJ_2)$, implying

$(X_N^s)^{(0:_M J_1)} = (X_N^s)^{(0:_M J_2)}$.

Corollary 2.5 Let $N=(0:_M I)$ be a submodule of M. Then $(X_N^s)^{(0:_M J)} = X_N^s$ if and only if

$\text{Sec}((0 :_M IJ))= \text{Sec}((0 :_M I))$ for every ideal J of R.

Proof Let $(X_N^s)^{(0:_M J)} = X_N^s$. Then

$V^s(0:_M IJ)= V^s(0:_M I)$ and thus

$\text{Sec}((0 :_M IJ))= \text{Sec}((0 :_M I))$.

Let $\text{Sec}((0 :_M IJ))= \text{Sec}((0 :_M I))$. Then

$V^s(0:_M IJ)= V^s(0:_M I)$, implying $(X_N^s)^{(0:_M J)} = X_N^s$.

Theorem 2.6 Let $N=(0:_M I)$ be a proper submodule of M and let Ψ be surjective. Then $(X_N^s)^{(0:_M r)}$ is quasi-compact for every element r of R.

Proof Let $\{A_i: i \in \Lambda\}$ be an open cover of $(X_N^s)^{(0:_M r)}$. Then there exists a family $\{r_i \in R: i \in \Lambda\}$ of elements of R such that

$(X_N^s) \setminus \tilde{V}^s((0 :_M r)) \subseteq \bigcup_{i \in \Lambda} (X_N^s \setminus \tilde{V}^s((0 :_M r_i)))$. One can observe that

$\text{Spec}(\bar{R}) \setminus V((rI)/\text{Ann}(M))= \Psi(X^s \setminus V^s(0:_M rI)) \subseteq$

$\bigcup_{i \in \Lambda} \Psi(X^s \setminus V^s((0 :_M r_iI))=$

$\bigcup_{i \in \Lambda} (\text{Spec}(\bar{R}) \setminus V((r_iI)/\text{Ann}(M)).$

Since $(\text{Spec}(\bar{R}) \setminus V((rI)/\text{Ann}(M))$ is quasi-compact, there exists a finite subset Δ of Λ such that

$(\text{Spec}(\bar{R}) \setminus V((rI)/\text{Ann}(M)) \subseteq$

$\bigcup_{i \in \Delta} (\text{Spec}(\bar{R}) \setminus V((r_iI)/\text{Ann}(M)).$

It follows that

$\Psi^{-1}(\text{Spec}(\bar{R}) \setminus V((rI)/\text{Ann}(M))=$

$X^s \setminus V^s(0:_M rI)) \subseteq$

$\bigcup_{i \in \Delta} \Psi^{-1}((\text{Spec}(\bar{R}) \setminus V((r_iI)/\text{Ann}(M))))=$

$\bigcup_{i \in \Delta} X^s \setminus V^s((0:_M r_iI)).$

Definition 2.7 Let $N=(0:_M I)$ be a proper submodule of a comultiplication R-module M. The set $S_N(T)$ is defined as $\sum \{P \in \text{Spec}^s(M): P \subseteq T \text{ and } P \not\subset N\}$.

The following lemma shows some properties of this new class.

Lemma 2.8 Let $N=(0:_M I)$ be a submodule of M. The following hold:

i) For the submodule $T=(0:_M J)$ of M, where J is an ideal of R, $S_N(T)$ is a submodule of M.

ii) $S_{N/K}(T/K)= S_N(T)/K$, where $K=(0:_M A) \subseteq T=(0:_M J)$ is a submodule of M and A is an ideal of R.

iii) $S_N(M)= S_{\text{Sec}(N)}(M)$.

Proof i) Since the sum of second submodules is also second submodule, it is clear.

ii)

$$S_{N/K}(T/K) = \sum_{\substack{L_i/K \subseteq T/K \\ L_i/K \not\subseteq N/K}} (L_i/K)$$

$$= \left(\sum_{\substack{L_i \subseteq T \\ L_i \not\subseteq N}} L_i \right) / K = S_N(T)/K$$

iii) Since any second submodule of M not contained in N is also not contained in Sec(N), $S_N(M) = S_{Sec(N)}(M)$.

A module M satisfies SN-condition if for any chain $S_N(T_1) \subseteq S_N(T_2) \subseteq S_N(T_3) \subseteq ...$, there exists a positive integer m such that $S_N(T_m) = S_N(T_{m+i})$ for all $i \in \mathbb{Z}^+$, the set of all positive integers, where T_i is a submodule of M such that $T_i = (0:_M U_i)$ and U_i is an ideal of R.

We find a connection between the subspace topology and the module under the above condition as follows:

Theorem 2.9 Let $N=(0:_M I)$ be a proper submodule of M. Then the following are equivalent:

i) M satisfies the SN-condition.

ii) X_N^s is an Artinian topological space.

Proof Suppose that M satisfies the SN-condition. Take the sequence $\tilde{V}^s(A_1) \subseteq \tilde{V}^s(A_2) \subseteq \tilde{V}^s(A_3) \subseteq ...$, where A_i is a submodule of M. Then we have the sequence $S_N(A_1) \subseteq S_N(A_2) \subseteq S_N(A_3) \subseteq ...$ and there exists a positive integer m such that $S_N(A_m) = S_N(A_{m+i})$ for all $i \in \mathbb{Z}^+$ with the hypothesis. Therefore we have $\tilde{V}^s(A_m) = \tilde{V}^s(A_{m+i})$ for all $i \in \mathbb{Z}^+$. Thus X_N^s is an Artinian topological space.

Suppose that X_N^s is an Artinian topological space. Take the sequence $S_N(A_1) \subseteq S_N(A_2) \subseteq S_N(A_3) \subseteq ...$, where A_i is a submodule of M. Then we have the sequence $\tilde{V}^s(A_1) \subseteq \tilde{V}^s(A_2) \subseteq \tilde{V}^s(A_3) \subseteq ...$ and there exists a positive integer m such that $\tilde{V}^s(A_m) = \tilde{V}^s(A_{m+i})$ for all $i \in \mathbb{Z}^+$ with the hypothesis. Therefore we have $S_N(A_m) = S_N(A_{m+i})$ for all $i \in \mathbb{Z}^+$. Thus M satisfies the SN-condition.

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