Araştırma Makalesi / *Research Article*
Matematik / *Mathematics*

Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 9(1): 389-396, 2019
***Journal of the Institute of Science and Technology*, 9(1): 389-396, 2019**

DOI: 10.21597/jist.462637

ISSN: 2146-0574, eISSN: 2536-4618

Kotanjant Demette Modified Riemannian Extension'a Göre Projektif Vektör Alanları

Lokman BİLEN[1*]

ÖZET: $T^{*}M$, $n-$boyutlu M Riemannian manifoldunun kotanjant demeti olsun. Bu çalışmadaki amacımız kotanjant demette modifiye edilmiş Riemann genişlemesine göre fibre koruyan projektif vektör alanlarının karakterizasyonunu yapmaktır.

Anahtar kelimeler: Fibre-koruyan vektör alanları, infinitesimal projektif dönüşümler, Riemannian metriği, modified Riemannian extension, adapte olmuş çatı

Projective Vector Fields on the Cotangent Bundle with Modified Riemannian Extension

ABSTRACT: Let $T^{*}M$ be the cotangent bundle of an $n-$dimensional Riemannian manifold M. The purpose of the present paper is give a characterization of fibre-preserving projective vector fields with respect to modified Riemannian extension.

Keywords: Fibre-preserving vector fields, infinitesimal projective transformations, Riemannian metric, modified Riemannian extension, adapted frame.

[1] Lokman BİLEN (**Orcid ID:** 0000-0001-8240-5359), Iğdır Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Iğdır, Türkiye
*Sorumlu yazar/Corresponding Author: Lokman BİLEN, lokman.bilen@igdir.edu.tr

Geliş tarihi / *Received:*21.09.2018
Kabul tarihi / *Accepted:*25.10.2018

INTRODUCTION

Let $T^*(M)$ be the cotangent bundle over M and Φ be a transformation of $T^*(M)$. If the transformation Φ preserves the fibres, it is called a fibre-preserving transformation. Consider a vector field $\tilde{X}$ on $T^*(M)$ and the local one-parameter group $\{\Phi_t\}$ of local transformations of $T^*(M)$ generated by $\tilde{X}$. The vector field $\tilde{X}$ is called an infinitesimal fibre-preserving transformation if each Φ_t is a local fibre-preserving transformation of $T^*(M)$. A transformation Φ of M is called a projective transformation if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore, Φ is called an affine transformation if it preserves the connection.

We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with the geodesics. An infinitesimal fibre-preserving transformation $\tilde{X}$ on $T^*(M)$ is called an infinitesimal fibre-preserving projective transformation if each Φ_t is a local fibre-preserving projective transformation of $T^*(M)$. Let $\tilde{g}$ be a Riemannian or a pseudo-Riemannian metric on $T^*(M)$. It is well known that $\tilde{X}$ is an infinitesimal projective transformation of $T^*(M)$ if and only if there exist a 1-form θ such that $(L_{\tilde{X}}\nabla)(\tilde{Y},\tilde{Z}) = \theta(\tilde{Y})\tilde{Z} + \theta(\tilde{Z})\tilde{Y}$ for any $\tilde{X},\tilde{Y} \in \mathfrak{T}_0^1(T^*M)$. Where $L_{\tilde{X}}$ denotes the Lie derivation with respect to $\tilde{X}$.

Infinitesimal projective transformations on tangent and cotangent bundles have been researched by some authors {for example see (Gezer, 2011; Hasegawa and Yamauchi, 2003; Yamauchi, 1998; Yamauchi, 1999.)}. In this paper, we aim to research infinitesimal projective transformations on the cotangent bundle with modified Riemann extension over Riemannian manifolds.

MATERIALS AND METHODS

Let M be an n-dimensional smooth manifold and denote by $\pi: T^*M \to M$ its cotangent bundle whose fibres are cotangent spaces to M. Then T^*M is a $2n$- dimensional smooth manifold and some local charts induced naturally from local charts on M can be used. Namely, a system of local coordinates $(U, x^i), i = 1,\ldots,n$ in M induces on T^*M a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = p_i), \bar{\imath} = n + i = n + 1,\ldots,2n$, where $x^{\bar{\imath}} = p_i$ are the components of covectors p in each cotangent space $T_x{}^*M, x \in U$ with respect to the natural coframe $\{dx^i\}$. Let $\tilde{X} = X^i \frac{\partial}{\partial x^i}$ and $w = w_i dx^i$ be the local expressions in $U$of a vector field $\tilde{X}$ and a covector (1-form) field w on M, respectively. Then the vertical lift ${}^V w$ of w, the horizontal lift ${}^H\tilde{X}$ and the complete lift ${}^C\tilde{X}$ of $\tilde{X}$ are given, with respect to the induced coordinates, by (Yano and Ishihara, 1973).

$$ {}^V w = w_i\,\partial_{\bar{\imath}}, \tag{2.1}$$

$$ {}^H\tilde{X} = X^i\,\partial_i + p_h \Gamma_{ij}^h X^j\,\partial_{\bar{\imath}}, \tag{2.2}$$

and ${}^C\tilde{X} = X^j\,\partial_i - p_h\,\partial_i X^h\,\partial_{\bar{\imath}}$ where $\partial_i = \frac{\partial}{\partial x^i}, \partial_{\bar{\imath}} = \frac{\partial}{\partial x^{\bar{\imath}}}$ and Γ_{ij}^h are the coefficients of a symmetric (torsion-free) affine connection ∇ on M. The Lie bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas: (Yano and Ishihara, 1973).

$$\begin{cases} [{}^H\tilde{X}, {}^H\tilde{Y}] = {}^H[\tilde{X},\tilde{Y}] + {}^V\left(poR(\tilde{X},\tilde{Y})\right) \\ [{}^H\tilde{X}, {}^V w] = {}^V(\nabla_{\tilde{X}} w) \\ [{}^V\theta, {}^V w] = 0 \end{cases} \tag{2.3}$$

for any $\tilde{X},\tilde{Y}\in\Im_0^1(M)$ and $\theta,w\in\Im_1^0(M)$, where R is the curvature tensor of the symmetric connection ∇ defined by $R(\tilde{X},\tilde{Y})=[\nabla_{\tilde{X}},\nabla_{\tilde{Y}}]-\nabla_{[\tilde{X},\tilde{Y}]}$.

The adapted frames $\{E_\alpha\}=\{E_j,E_{\bar{j}}\}$ on each induced coordinate neighbourhood $\pi^{-1}(U)$ of T^*M is given by (Yano and Ishihara, 1973).

$$\left.\begin{aligned} E_j &= {}^H\tilde{X}_{(j)} = \partial_j + p_a\Gamma_{hj}^a\,\partial_{\bar{h}} \\ E_{\bar{j}} &= {}^V\theta^{(j)} = \partial_{\bar{j}}. \end{aligned}\right\} \tag{2.4}$$

The indices $\alpha,\beta,\gamma,\ldots=1,\ldots,2n$ indicate the indices with respect to the adapted frame.

It follows from (2.1), (2.2) and (2.4) that

$${}^V w=\begin{pmatrix}0\\ w_j\end{pmatrix}$$

and

$${}^H\tilde{X}=\begin{pmatrix}X^j\\ 0\end{pmatrix}$$

with respect to the adapted frame $\{E_\alpha\}$. The straightforward calculations give:

Lemma 2.1: (Yano and Ishihara, 1973). The Lie bracket of the adapted frame of T^*M satisfies the following identities:

$$[E_i,E_j]=p_sR_{ijl}^sE_{\bar{l}}$$
$$[E_i,E_{\bar{j}}]=-\Gamma_{il}^jE_{\bar{l}}$$
$$[E_{\bar{l}},E_{\bar{j}}]=0$$

where $R=R_{ijl}^s$ denotes the Riemannian curvature tensor of (M,g) defined by

$$R_{ijl}^s=\partial_i\Gamma_{jl}^s-\partial_j\Gamma_{il}^s+\Gamma_{ik}^s\Gamma_{jl}^k-\Gamma_{jk}^s\Gamma_{il}^k.$$

Let us consider T^*M equipped with the modified Riemannian extension $(\tilde{g}_{\nabla,c})$ for a given torsion-free connection ∇ on M. In adapted frame $\{E_\beta\}$, the modified Riemannian extension $(\tilde{g}_{\nabla,c})_{\beta\gamma}$ and its inverse $(\tilde{g}_{\nabla,c})^{\beta\gamma}$ have in the following forms: (Gezer et al., 2015).

$$(\tilde{g}_{\nabla,c})_{\beta\gamma}=\begin{pmatrix}c_{ij} & \delta_i{}^j\\ \delta_j{}^i & 0\end{pmatrix}$$
$$(\tilde{g}_{\nabla,c})^{\beta\gamma}=\begin{pmatrix}0 & \delta_j{}^i\\ \delta_i{}^j & -c_{ij}\end{pmatrix},$$

where c_{ij} are the components of the symmetric $(0,2)$ tensor field c on (M,∇). Then the quadratic differential form of this metric is $(\tilde{g}_{\nabla,c})=c_{ij}dx^idx^j+2\delta_i{}^jdx^i\delta y_j$.

Lemma 2.2: (Gezer et al., 2015). The Levi-Civita connection $\tilde{\nabla}$ of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ is given by;

i. $\tilde{\nabla}_{E_{\bar{i}}}E_{\bar{j}}=0$
ii. $\tilde{\nabla}_{E_{\bar{i}}}E_j=0$
iii. $\tilde{\nabla}_{E_i}E_{\bar{j}}=-\Gamma_{ih}{}^jE_{\bar{h}}$
iv. $\tilde{\nabla}_{E_i}E_j=\Gamma_{ij}{}^hE_h+\left\{p_sR_{hji}{}^s+\frac{1}{2}(\nabla_ic_{jh}+\nabla_jc_{ih}-\nabla_hc_{ij})\right\}E_{\bar{h}}.$

Lemma 2.3: Let $\tilde{X}$ be a fibre-preserving vector field of T^*M with the components $(v^h,v^{\bar{h}})$. Then, the Lie derivatives of the adapted frame and the dual basis are given as follows:

i. $L_{\tilde{X}}E_i=-(E_iv^k)E_k-\left(v^ap_sR_{iak}{}^s+E_iv^{\bar{k}}-v^{\bar{a}}\Gamma_{ik}^a\right)E_{\bar{k}}$,

ii. $L_{\tilde{X}}E_{\bar{\imath}}=-\left(v^a\Gamma_{ak}^i+E_{\bar{\imath}}v^{\bar{k}}\right)E_{\bar{k}}$,

iii. $L_{\tilde{X}}dx^h=(E_mv^h)dx^m$,

iv. $L_{\tilde{X}}\delta p_h=\left(v^ap_sR_{mah}{}^s-v^{\bar{a}}\Gamma_{mh}^a+\left(E_mv^{\bar{k}}\right)\delta_h{}^k\right)dx^m+\left(v^a\Gamma_{ah}^m+\left(E_{\bar{m}}v^{\bar{k}}\right)\delta_h{}^k\right)\delta p_m.$

{For tangent bundles see (Gezer, 2011; Hasegawa and Yamauchi, 2003; Yamauchi, 1998; Yamauchi, 1999)}.

RESULTS AND DISCUSSION

Theorem 3.1: Let (M, g) be a Riemannian manifold and T^*M its cotangent bundle with the modified Riemannian extension. Then $\tilde{X}$ is an infinitesimal projective transformation with the associated 1- form θ on T^*M if and only if there exist
$A = A_i^j \in \Im_1^1(M)$,
$M = M_{ijk} \in \Im_3^0(M)$,
$B = B_h \in \Im_1^0(M)$ satisfying

1) $\theta = (\theta_i, \theta_{\bar{\imath}}) = \left(\frac{1}{n+1}\nabla_i(\nabla_j v^j), 0\right)$

2) $v^{\bar{k}} = p_s A_k^s + B_k$

3) $v^a R_{iak}{}^j + \nabla_i A_k^j = \theta_i \delta_j^k$

4) $\nabla_i(\theta_j \delta_s^k) + v^h(\nabla_h R_{kji}^s) + (\nabla_j v^h) R_{khi}{}^s + (\nabla_i v^h) R_{kjh}{}^s + A_h^s R_{kji}{}^h - A_k^h R_{hji}{}^s = 0$

5) $v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk} + 2\nabla_i \nabla_j B_k + 2B_h R_{kji}{}^h - A_k^h M_{ijh} = 0$

where $\tilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$, $A_k^j = E_{\bar{j}} v^{\bar{k}}$ and $M_{ijh} = \nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij}$.

Proof:

1) $(L_X \tilde{\nabla})(E_{\bar{\imath}}, E_{\bar{j}}) = L_X(\tilde{\nabla}_{E_{\bar{\imath}}} E_{\bar{j}}) - \tilde{\nabla}_{E_{\bar{\imath}}}(L_X E_{\bar{j}}) - \tilde{\nabla}_{(L_X E_{\bar{\imath}})} E_{\bar{j}} = \theta(E_{\bar{\imath}}) E_{\bar{j}} + \theta(E_{\bar{j}}) E_{\bar{\imath}}$

$$\Rightarrow \tilde{\nabla}_{E_{\bar{\imath}}}\left(v^a \Gamma_{ak}{}^j + E_{\bar{j}} v^{\bar{k}}\right) E_{\bar{k}} - \tilde{\nabla}_{\left(v^a \Gamma_{ak}{}^i + E_{\bar{\imath}} v^{\bar{k}}\right) E_{\bar{k}}} E_{\bar{j}} = \theta_{\bar{\imath}} E_{\bar{j}} + \theta_{\bar{j}} E_{\bar{\imath}}$$

$$\Rightarrow E_{\bar{\imath}}\left(E_{\bar{j}} v^{\bar{k}}\right) E_{\bar{k}} = \left(\theta_{\bar{\imath}} \delta_j^k + \theta_{\bar{j}} \delta_i^k\right) E_{\bar{k}}$$

$$\Rightarrow E_{\bar{\imath}}\left(E_{\bar{j}} v^{\bar{k}}\right) = \theta_{\bar{\imath}} \delta_j^k + \theta_{\bar{j}} \delta_i^k \tag{3.1}$$

2) $(L_X \tilde{\nabla})(E_i, E_{\bar{j}}) = L_X(\tilde{\nabla}_{E_i} E_{\bar{j}}) - \tilde{\nabla}_{E_i}(L_X E_{\bar{j}}) - \tilde{\nabla}_{(L_X E_i)} E_{\bar{j}} = \theta(E_i) E_{\bar{j}} + \theta(E_{\bar{j}}) E_i$

$$\Rightarrow L_X(-\Gamma_{ih}{}^j E_{\bar{h}}) - \tilde{\nabla}_{E_{\bar{\imath}}}\left[-\left(v^a \Gamma_{ak}{}^j + E_{\bar{j}} v^{\bar{k}}\right) E_{\bar{k}}\right] - \tilde{\nabla}_{\left[-(E_i v^k) E_k - \left(v^a p_s R_{iak}{}^s + E_i v^{\bar{k}} - v^{\bar{a}} \Gamma_{ik}{}^a\right) E_{\bar{k}}\right]} E_{\bar{j}} = \theta_i E_{\bar{j}} + \theta_{\bar{j}} E_i$$

$$\Rightarrow \left[-L_X \Gamma_{ik}^j + v^a \Gamma_{ak}^h \Gamma_{ih}^j + \left(E_{\bar{h}} v^{\bar{k}}\right) \Gamma_{ih}^j + (E_i v^a) \Gamma_{ak}^j + v^a\left(E_i \Gamma_{ak}^j\right) + E_i\left(E_{\bar{j}} v^{\bar{k}}\right) - v^a \Gamma_{ah}^j \Gamma_{ik}^h\right.$$

$$\left. -\left(E_{\bar{j}} v^{\bar{h}}\right) \Gamma_{ik}^h - (E_i v^h) \Gamma_{hk}^j\right] E_{\bar{k}} = \left(\theta_i \delta_j^k\right) E_{\bar{k}} + \left(\theta_{\bar{j}} \delta_i^k\right) E_k$$

from which we get

$$\theta_{\bar{j}} \delta_i{}^k = 0 \Rightarrow \theta_{\bar{j}} = 0 \tag{3.2}$$

and

$$-L_X \Gamma_{ik}^j + v^a \Gamma_{ak}^h \Gamma_{ih}^j + \left(E_{\bar{h}} v^{\bar{k}}\right) \Gamma_{ih}^j + v^a\left(E_i \Gamma_{ak}^j\right) + E_i\left(E_{\bar{j}} v^{\bar{k}}\right) - v^a \Gamma_{ah}^j \Gamma_{ik}^h - \left(E_{\bar{j}} v^{\bar{h}}\right) \Gamma_{ik}^h = \theta_i \delta_j{}^k$$

$\Rightarrow -v^a E_a \Gamma_{ik}^j - v^{\bar{a}} E_{\bar{a}} \Gamma_{ik}^j + v^a \Gamma_{ak}^h \Gamma_{ih}^j + \left(E_{\bar{h}} v^{\bar{k}}\right) \Gamma_{ih}^j + v^a \left(E_i \Gamma_{ak}^j\right) + E_i \left(E_{\bar{j}} v^{\bar{k}}\right) - v^a \Gamma_{ah}^j \Gamma_{ik}^h -$

$\left(E_{\bar{j}} v^{\bar{h}}\right) \Gamma_{ik}^h = \theta_i \delta_j{}^k$

$\Rightarrow \left[v^a R_{iak}{}^j + \nabla_i \left(E_{\bar{j}} v^{\bar{k}}\right)\right] E_{\bar{k}} = \left(\theta_i \delta_j{}^k\right) E_{\bar{k}}$

$\Rightarrow v^a R_{iak}{}^j + \nabla_i \left(E_{\bar{j}} v^{\bar{k}}\right) = \theta_i \delta_j{}^k$

where $E_{\bar{j}} v^{\bar{k}} = A_k{}^j$. In this case,

$$v^a R_{iak}{}^j + \nabla_i A_k{}^j = \theta_i \delta_j{}^k \tag{3.3}$$

substituting the equation (3.2) into the equation (3.1) it follows that,

$$E_{\bar{i}}\left(E_{\bar{j}} v^{\bar{k}}\right) = 0$$

$$v^{\bar{k}} = p_s A_k{}^s + B_k \tag{3.4}$$

substituting the equation (3.4) into the equation (3.3), we have;

$$v^a R_{iak}{}^j + \nabla_i \left(E_{\bar{j}}(p_s A_k{}^s + B_k)\right) = \theta_i \delta_j{}^k$$

$$v^a R_{iak}{}^j + \nabla_i A_k{}^j = \theta_i \delta_j{}^k \tag{3.5}$$

contracting j and k in (3.5),

$$v^a R_{iak}{}^k + \nabla_i A_j{}^j = \theta_i \delta_j{}^j$$

from which, we obtain;

$$\theta_i = \frac{1}{n}\left(\nabla_i A_j{}^j\right) \tag{3.6}$$

3) $\left(L_X \widetilde{\nabla}\right)\left(E_i, E_j\right) = L_X\left(\widetilde{\nabla}_{E_i} E_j\right) - \widetilde{\nabla}_{E_i}\left(L_X E_j\right) - \widetilde{\nabla}_{(L_X E_i)} E_j = \theta(E_i) E_j + \theta\left(E_j\right) E_i$

$\Rightarrow L_X \left[\Gamma_{ij}^h E_h + \left\{p_s R_{hji}{}^s + \frac{1}{2}\left(\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij}\right)\right\} E_{\bar{h}}\right] - \widetilde{\nabla}_{E_i} \left[-\left(E_j v^k\right) E_k - \left(v^a p_s R_{jak}{}^s + E_j v^{\bar{k}} -\right.\right.$

$\left.\left. v^{\bar{a}} \Gamma_{jk}^a\right) E_{\bar{k}}\right] - \widetilde{\nabla}_{\left[-(E_i v^k) E_k - \left(v^a p_s R_{iak}{}^s + E_i v^{\bar{k}} - v^{\bar{a}} \Gamma_{ik}^a\right) E_{\bar{k}}\right]} E_j = \theta_i E_j + \theta_j E_i$

from which; if writing,

$$M_{ijh} = \nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij}$$

if necessary actions are taken,

$$v^h E_h \Gamma_{ij}^k - (E_h v^k) \Gamma_{ij}^h + E_i\left(E_j v^k\right) + \left(E_j v^h\right) \Gamma_{ih}^k + (E_i v^h) \Gamma_{hj}^k = \theta_i \delta_j^k + \theta_j \delta_i^k \tag{3.7}$$

and

$$-v^a p_s R_{hak}{}^s \Gamma_{ij}^h - E_h v^{\bar{k}} \Gamma_{ij}^h + v^{\bar{a}} \Gamma_{hk}^a \Gamma_{ij}^h + v^h E_h\left(p_s R_{kji}{}^s\right) + \frac{1}{2} v^h E_h M_{ijk} + v^{\bar{h}} R_{kji}{}^h - v^a p_s R_{hji}{}^s \Gamma_{ak}^h -$$

$$\left(E_{\bar{h}} v^{\bar{k}}\right) p_s R_{hji}{}^s - \frac{1}{2} v^a \Gamma_{ak}^h M_{ijh} - \frac{1}{2}\left(E_{\bar{h}} v^{\bar{k}}\right) M_{ijh} + \left(E_j v^h\right) p_s R_{khi}{}^s + \frac{1}{2}\left(E_j v^h\right) M_{ihk} + \left(E_i v^a\right) p_s R_{jak}{}^s +$$

$$v^a p_s E_i R_{jak}{}^s + E_i\left(E_j v^{\bar{k}}\right) - \left(E_i v^{\bar{a}}\right) \Gamma_{jk}^a - v^{\bar{a}}\left(E_i \Gamma_{jk}^a\right) - v^a p_s R_{jah}{}^s \Gamma_{ik}^h - \left(E_j v^{\bar{h}}\right) \Gamma_{ik}^h + v^{\bar{a}} \Gamma_{jh}^a \Gamma_{ik}^h +$$

$$\left(E_i v^h\right) p_s R_{kjh}{}^s + \frac{1}{2}\left(E_i v^h\right) M_{hjk} = 0 \tag{3.8}$$

From this formula: $E_k v^h = \nabla_k v^h - \Gamma_{ks}^h v^s$, we can write in (3.7),

$$v^h E_h \Gamma_{ij}^k - \Gamma_{ij}^h\left(\nabla_h v^k - \Gamma_{hs}^k v^s\right) + E_i\left(\nabla_j v^k - \Gamma_{js}^k v^s\right) + \Gamma_{ih}^k\left(\nabla_j v^h - \Gamma_{js}^h v^s\right) + \Gamma_{hj}^k\left(\nabla_i v^h - \Gamma_{is}^h v^s\right) =$$

$$\theta_i \delta_j^k + \theta_j \delta_i^k$$

$$\Rightarrow v^h E_h \Gamma_{ij}^k - \Gamma_{ij}^h\left(\nabla_h v^k\right) + \Gamma_{ij}^h\left(\Gamma_{hs}^k v^s\right) + E_i\left(\nabla_j v^k\right) - v^s E_i \Gamma_{js}^k - \Gamma_{js}^k\left(E_i v^s\right) + \Gamma_{ih}^k\left(\nabla_j v^h\right) - \Gamma_{ih}^k \Gamma_{js}^h v^s$$

$$+\Gamma_{hj}^k\left(\nabla_i v^h\right) - \Gamma_{hj}^k \Gamma_{is}^h v^s = \theta_i \delta_j^k + \theta_j \delta_i^k$$

$$\Rightarrow \underbrace{\left[v^h E_h \Gamma_{ij}^k + \Gamma_{ij}^h \Gamma_{hs}^k v^s - v^s E_i \Gamma_{js}^k - \Gamma_{ih}^k \Gamma_{js}^h v^s\right]}_{v^h R_{hij}{}^k} + \underbrace{\left[E_i\left(\nabla_j v^k\right) - \Gamma_{ij}^h\left(\nabla_h v^k\right) + \Gamma_{ih}^k\left(\nabla_j v^h\right)\right]}_{\nabla_i\left(\nabla_j v^k\right)} = \theta_i \delta_j^k + \theta_j \delta_i^k$$

$$\Rightarrow v^h R_{hij}{}^k + \nabla_i\left(\nabla_j v^k\right) = \theta_i \delta_j^k + \theta_j \delta_i^k$$

where, we contracting both sides with δ_k^j,

$$\Rightarrow \underbrace{v^h R_{hij}{}^j}_{0} + \nabla_i\left(\nabla_j v^j\right) = \theta_i n + \underbrace{\theta_j \delta_i^j}_{\theta_i}$$

$$\Rightarrow \nabla_i\left(\nabla_j v^j\right) = (n+1) \theta_i$$

$$\Rightarrow \theta_i = \frac{1}{(n+1)} \nabla_i\left(\nabla_j v^j\right) \tag{3.9}$$

from (3.6) and (3.9), we have

$$\frac{1}{n}\left(\nabla_i A_j^j\right) = \frac{1}{n+1} \nabla_i\left(\nabla_j v^j\right)$$

$$\nabla_i A_j^j = \frac{n}{n+1} \nabla_i\left(\nabla_j v^j\right) \tag{3.10}$$

from (3.8) we obtain,

$$\underbrace{-v^a p_s R_{hak}{}^s \Gamma_{ij}^h}_{2} \underbrace{-E_h v^{\bar{k}} \Gamma_{ij}^h + v^{\bar{a}} \Gamma_{hk}^a \Gamma_{ij}^h}_{1} + v^h E_h (p_s R_{kji}{}^s) + \frac{1}{2} v^h E_h M_{ijk} + v^{\bar{h}} R_{kji}{}^h - v^a p_s R_{hji}{}^s \Gamma_{ak}^h -$$

$$(E_{\bar{h}} v^{\bar{k}}) p_s R_{hji}{}^s - \frac{1}{2} v^a \Gamma_{ak}^h M_{ijh} - \frac{1}{2} (E_{\bar{h}} v^{\bar{k}}) M_{ijh} + (E_j v^h) p_s R_{khi}{}^s + \frac{1}{2} (E_j v^h) M_{ihk} +$$

$$(E_i v^a) p_s R_{jak}{}^s \underbrace{+v^a p_s E_i R_{jak}{}^s}_{2} \underbrace{+E_i (E_j v^{\bar{k}}) - (E_i v^{\bar{a}}) \Gamma_{jk}^a - v^{\bar{a}} (E_i \Gamma_{jk}^a)}_{1}$$

$$\underbrace{-v^a p_s R_{jah}{}^s \Gamma_{ik}^h}_{2} \underbrace{-(E_j v^{\bar{h}}) \Gamma_{ik}^h + v^{\bar{a}} \Gamma_{jh}^a \Gamma_{ik}^h}_{1} + (E_i v^h) p_s R_{kjh}{}^s + \frac{1}{2} (E_i v^h) M_{hjk} = 0$$

if these expressions are used; $1 \to \nabla_i(\nabla_j v^{\bar{k}})$, $2 \to v^a p_s(\nabla_i R_{jak}^s)$

$$\nabla_i(\nabla_j v^{\bar{k}}) + v^a p_s(\nabla_i R_{jak}^s) + v^h E_h(p_s R_{kji}{}^s) + \frac{1}{2} v^h E_h M_{ijk} + v^{\bar{h}} R_{kji}{}^h - v^a p_s R_{hji}{}^s \Gamma_{ak}^h -$$

$$(E_{\bar{h}} v^{\bar{k}}) p_s R_{hji}{}^s - \frac{1}{2} v^a \Gamma_{ak}^h M_{ijh} - \frac{1}{2} (E_{\bar{h}} v^{\bar{k}}) M_{ijh} + (E_j v^h) p_s R_{khi}{}^s$$

$$+ \frac{1}{2} (E_j v^h) M_{ihk} + (E_i v^a) p_s R_{jak}{}^s + (E_i v^h) p_s R_{kjh}{}^s + \frac{1}{2} (E_i v^h) M_{hjk} = 0$$

From this formula: $E_k v^h = \nabla_k v^h - \Gamma_{ks}^h v^s$

$$\nabla_i(\nabla_j v^{\bar{k}}) + v^a p_s(\nabla_i R_{jak}^s) \underbrace{+v^h E_h(p_s R_{kji}{}^s)}_{4} \underbrace{+\frac{1}{2} v^h E_h M_{ijk}}_{3} + v^{\bar{h}} R_{kji}{}^h \underbrace{-v^a p_s R_{hji}{}^s \Gamma_{ak}^h}_{4} -$$

$$(E_{\bar{h}} v^{\bar{k}}) p_s R_{hji}{}^s \underbrace{-\frac{1}{2} v^a \Gamma_{ak}^h M_{ijh}}_{3} - \frac{1}{2} (E_{\bar{h}} v^{\bar{k}}) M_{ijh} + (\nabla_j v^h) p_s R_{khi}{}^s \underbrace{-(\Gamma_{ja}^h v^a) p_s R_{khi}{}^s}_{4}$$

$$\underbrace{+\frac{1}{2} (\nabla_j v^h) M_{ihk} - \frac{1}{2} (\Gamma_{js}^h v^s) M_{ihk}}_{3} + (\nabla_i v^a) p_s R_{jak}{}^s - (\Gamma_{it}^a v^t) p_s R_{jak}{}^s +$$

$$(\nabla_i v^h) p_s R_{kjh}{}^s \underbrace{-(\Gamma_{ia}^h v^a) p_s R_{kjh}{}^s}_{4} \underbrace{+\frac{1}{2} (\nabla_i v^h) M_{hjk}}_{3} - \frac{1}{2} (\Gamma_{is}^h v^s) M_{hjk} = 0$$

hen necessary index changes are made and these expressions are used;

$3 \to \frac{1}{2}\left(v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk}\right)$, $\quad 4 \to v^h p_s(\nabla_h R_{kji}^s)$

$$\nabla_i(\nabla_j v^{\bar{k}}) + v^a p_s(\nabla_i R^s_{jak}) + v^h p_s(\nabla_h R^s_{kji}) + \frac{1}{2}\left[v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk}\right] + (\nabla_j v^h) p_s R_{khi}{}^s + (\nabla_i v^a) p_s R_{jak}{}^s + (\nabla_i v^h) p_s R_{kjh}{}^s - (\Gamma^a_{ih} v^h) p_s R_{jak}{}^s + v^{\bar{h}} R_{kji}{}^h - (E_{\bar{h}} v^{\bar{k}}) p_s R_{hji}{}^s - \frac{1}{2}(E_{\bar{h}} v^{\bar{k}}) M_{ijh} = 0$$

where equation (3.4) is used and if necessary actions are taken,

$$p_s \nabla_i \nabla_j A^s_k + \nabla_i \nabla_j B_k + v^a p_s(\nabla_i R^s_{jak}) + v^h p_s(\nabla_h R^s_{kji}) + \frac{1}{2}\left[v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk}\right] + (\nabla_j v^h) p_s R_{khi}{}^s + (\nabla_i v^a) p_s R_{jak}{}^s + (\nabla_i v^h) p_s R_{kjh}{}^s - (\Gamma^a_{ih} v^h) p_s R_{jak}{}^s + p_s A^s_h R_{kji}{}^h + B_h R_{kji}{}^h - A^h_k p_s R_{hji}{}^s - \frac{1}{2} A^h_k M_{ijh} = 0$$

$$\Rightarrow p_s\left[\nabla_i(\nabla_j A^s_k + v^a R_{jak}{}^s) + v^h(\nabla_h R_{kji}{}^s) + (\nabla_j v^h) R_{khi}{}^s + (\nabla_i v^h) R_{kjh}{}^s - (\Gamma^a_{ih} v^h) R_{jak}{}^s + A^s_h R_{kji}{}^h - A^h_k R_{hji}{}^s\right] + \frac{1}{2}\left[v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk} + 2\nabla_i \nabla_j B_k + 2 B_h R_{kji}{}^h - A^h_k M_{ijh}\right] = 0$$

if we used equation (3.3) we get;

$$\nabla_i(\theta_j \delta^k_s) + v^h(\nabla_h R^s_{kji}) + (\nabla_j v^h) R_{khi}{}^s + (\nabla_i v^h) R_{kjh}{}^s + A^s_h R_{kji}{}^h - A^h_k R_{hji}{}^s = 0$$

and

$$v^h \nabla_h M_{ijk} + (\nabla_j v^h) M_{ihk} + (\nabla_i v^h) M_{hjk} + 2\nabla_i \nabla_j B_k + 2 B_h R_{kji}{}^h - A^h_k M_{ijh} = 0$$

CONCLUSION

Let T^*M be the cotangent bundle of an $n-$dimensional Riemannian manifold M. We give a characterization of fibre-preserving projective vector fields with respect to modified Riemannian extension $\tilde{g}_{\nabla,c}$.

REFERENCES

Gezer A, 2011. On infinitesimal holomorphically projective transformations on the tangent bundles with respect to the Sasaki metric, Proc. Est. Acad. Sci., 60(3): 149–157.

Gezer A, Bilen L, Çakmak A, 2015. Properties of modified Riemannian extensions, Zh. Mat. Fiz. Anal. Geom., 11(2): 159–173.

Hasegawa I, Yamauchi K, 2003. Infinitesimal projective transformations on tangent bundles with lift connection, Scientiae Mathematicae Japonicae., 57(3): 469-483, e7, 489-503.

Yamauchi K, 1998. On infinitesimal projective transformations of the tangent bundles with the complete lift metric over Riemannian manifolds, Ann Rep. Asahikawa. Med. Coll., 19: 49-55.

Yamauchi K, 1999. On infinitesimal projective transformations of the tangent bundles with the metric II+III, Ann Rep. Asahikawa. Med. Coll., (20): 67-72.

Yano K, Ishihara S, 1973. Tangent and Cotangent Bundles, Marcel Dekker, Inc, New York.