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# On Involutes of Order k of a Space-like Curve in Minkowski 4-space $IE_1^4$

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**Abstract**

The orthogonal trajectories of the first tangents of a curve x are called the involutes of *x*. In this study, we give a characterization of involutes of order k of a space-like curve x with time-like principal normal in Minkowski 4-space $IE_1^4$.

# $IE_1^4$ Minkowski 4-uzayında bir Space-like Eğrinin k'yinci Mertebeden İnvolütleri Üzerine

***Anahtar kelimeler***
İnvolüt; Space-like Eğri; W-eğrisi; Helis.

**Özet**

Bir x eğrisinin birinci teğetlerinin dik yörüngelerine eğrinin involütleri adı verilir. Bu çalışmada, $IE_1^4$ Minkowski 4-uzayında time-like asli normalli bir space-like eğrinin k'yinci mertebeden involütlerinin bir karakterizasyonunu verdik.



## 1. Introduction

$IE_1^4$ Minkowski space-time $IE_1^4$ is a pseudo-Euclidean space $IE^4$ provided with the standart flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \qquad (1)$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system in $IE_1^4$. Since g is an indefinite metric, recall that a vector $v \in IE_1^4$ can have one of the three causal characters; it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$, and null (light-like) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $x = x(s)$ in $IE_1^4$ can be locally space-like, time-like or null if all of its velocity vectors $x'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in $IE_1^4$ are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $x(s)$ is given by $\|x'(s)\|$. Space-like or time-like curve $x(s)$ is said to be parametrized by arc-length function s, if $g(x'(s), x'(s)) = \pm 1$ (O'Neill, 1983).

Let $x(s)$ be a space-like curve with a time-like principal normal in the space-time $IE_1^4$, parametrized by arc-length function s. Then we have the following Frenet equations (Walfare, 1995):

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \qquad (2)$$

where $V_1$, $V_2$, $V_3$ and $V_4$ are the Frenet vectors satisfy the equations:

$$g(V_1,V_1) = g(V_3,V_3) = g(V_4,V_4) = 1, \; g(V_2,V_2) = -1.$$

Here $k_1$, $k_2$, $k_3$ are respectively, the first, the second and the third curvatures of the curve $x(s)$.

**Definition 1.** (Yılmaz and Turgut, 2008) Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $IE_1^4$. The vector product in Minkowski space-time $IE_1^4$ is defined by the determinant

$$a \wedge b \wedge c = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \qquad (3)$$

where $e_1$, $e_2$, $e_3$ and $e_4$ are mutually orthogonal vectors satisfying the equations

$$e_1 \wedge e_2 \wedge e_3 = e_4,\ e_2 \wedge e_3 \wedge e_4 = e_1,$$
$$e_3 \wedge e_4 \wedge e_1 = e_2,\ e_4 \wedge e_1 \wedge e_2 = -e_3.$$

Let $x(s)$ be a space-like curve in $IE_1^4$. The Frenet frame vectors $V_1$, $V_2$, $V_3$, $V_4$ and Frenet curvatures $k_1$, $k_2$, $k_3$ are given by

$$V_1(s) = \frac{x'(s)}{\|x'(s)\|},$$
$$V_4(s) = \frac{x'(s) \wedge x''(s) \wedge x'''(s)}{\|x'(s) \wedge x''(s) \wedge x'''(s)\|},$$
$$V_3(s) = \frac{V_4 \wedge x'(s) \wedge x''(s)}{\|V_4 \wedge x'(s) \wedge x''(s)\|}, \qquad (4)$$
$$V_2(s) = \frac{V_3 \wedge V_4 \wedge x'(s)}{\|V_3 \wedge V_4 \wedge x'(s)\|}$$

and

$$k_1(s) = \frac{g(V_2(s), x''(s))}{\|x'(s)\|^2},$$
$$k_2(s) = \frac{g(V_3(s), x'''(s))}{\|x'(s)\|^3 k_1(s)}, \qquad (5)$$
$$k_3(s) = \frac{g(V_4(s), x^{(iv)}(s))}{\|x'(s)\|^4 k_1(s) k_2(s)},$$

respectively, where $\wedge$ is vector product in $IE_1^4$ (Gluck, 1966).

A curve which has constant first Frenet curvature $IE_1^4$ is called a Salkowski curve (Salkowski, 1909). ( or T.C-curve (Kılıç et al. 2008)). An arbitrary curve is called W-curve or (circular) helix if it has constant Frenet curvatures (Klein and Lie, 1871). Meanwhile, a curve with constant curvature ratios $IE_1^4$ is called a ccr-curve (Monterde, 2007), (Öztürk et al. 2008).

In (Öztürk et al.) (2016), the authors gave a characterization of involutes of order k of a given curve in $IE^n$. They obtain some results about the involutes of order 1, 2, 3 of a given curve in $IE^3$, $IE^4$ , respectively.

In the present study, we give a characterization of involutes of order k of a space-like curve x in Minkowski space-time $IE_1^4$ .

## 2. Involute curves of order k

**Definition 2.** Let $x(s)$ be a regular space-like curve in $IE_1^4$ given with arc-length parameter s. Then the curves which are orthogonal to the system of k-dimensional osculating hyperplanes of x are called the involutes of k (or $k^{th}$ involute) of the curve x (Balazenka and Zeljka 1999). For simplicity, we call the involutes of order 1, the involute of the given curve.

In order to find the parametrization of involutes $\bar{x}(s)$ of order k of the curve x in $IE_1^4$, we put

$$\bar{x}(s) = x(s) + \sum_{i=1}^{k} \lambda_i(s) V_i(s), \quad k \le 3, \qquad (6)$$

where $\lambda_i$ is a differentiable function and s, which is not necessarily an arc-length parameter, is the parameter of $\bar{x}(s)$.

Furthermore, the involutes $\bar{x}$ of order k of the curve x in $IE_1^4$ are detemined by

$$g(\bar{x}'(s), V_i(s)) = 0, \quad 1 \le i \le k \le 3. \qquad (7)$$

### *2.1. Involute curves of order 1*

**Theorem 1.** Let $x(s)$ be a space-like curve with time-like principal normal in $IE_1^4$ given with the

Frenet curvatures $k_1$, $k_2$, $k_3$. Then, the involute $\bar{x}$ of the curve x is a time-like curve with the Frenet frame vectors $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$, $\bar{V}_4$ and Frenet curvatures $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$ which are given by

$$\bar{V}_1(s)=V_2,$$
$$\bar{V}_2(s)=\frac{k_1V_1+k_2V_3}{\sqrt{k_1^2+k_2^2}},$$
$$\bar{V}_3(s)=\frac{1}{W\sqrt{k_1^2+k_2^2}}\begin{pmatrix}-k_2(k_2A-k_1C)V_1\\+k_1(k_2A-k_1C)V_3\\-D(k_1^2+k_2^2)V_4\end{pmatrix}, \quad (8)$$
$$\bar{V}_4(s)=\frac{1}{W}\left(-k_2DV_1+k_1DV_3+(k_2A-k_1C)V_4\right),$$

and

$$\bar{k}_1(s)=\frac{\sqrt{k_1^2+k_2^2}}{\varphi},$$
$$\bar{k}_2(s)=-\frac{W}{\varphi^2(k_1^2+k_2^2)}, \quad (9)$$
$$\bar{k}_3(s)=-\frac{\sqrt{k_1^2+k_2^2}}{W^2\varphi}\begin{pmatrix}(k_2A-k_1C)(k_3C+D')\\+D(k_1C'-k_2A')-k_1k_3D^2\end{pmatrix}$$

respectively, where

$$\varphi=(c-s)k_1,$$
$$A=k_1'\varphi+2k_1\varphi',$$
$$B=k_1^2\varphi+\varphi''+k_2^2\varphi,$$
$$C=k_2'\varphi+2k_2\varphi',$$
$$D=k_2k_3\varphi,$$

and

$$W=\sqrt{D^2(k_1^2+k_2^2)+(k_2A-k_1C)^2}=|\varphi|\sqrt{k_2^2k_3^2(k_1^2+k_2^2)+(k_1'k_2-k_1k_2')^2}. \quad (10)$$

**Proof.** Let $\bar{x}(s)$ be the involute of a space-like curve x with time-like principal normal in $IE_1^4$. Then by the use of (6) with (7), we get $1+\lambda_1'(s)=0$, and furthermore $\lambda_1(s)=(c-s)$ for some constant c. We have the following parametrization

$$\bar{x}(s)=x(s)+(c-s)V_1(s) \quad (11)$$

Further, differentiating the equation (11), we find

$$\bar{x}'(s)=\varphi V_2,$$
$$\bar{x}''(s)=\varphi k_1V_1+\varphi'V_2+\varphi k_2V_3, \quad (12)$$
$$\bar{x}'''(s)=(k_1'\varphi+2k_1\varphi')V_1+(k_1^2\varphi+\varphi''+k_2^2\varphi)V_2+(k_2'\varphi+2k_2\varphi')V_3+k_2k_3V_4$$

where $\varphi(s)=\lambda_1(s)k_1(s)$ is a differentiable function. Substituting

$$A=k_1'\varphi+2k_1\varphi',$$
$$B=k_1^2\varphi+\varphi''+k_2^2\varphi,$$
$$C=k_2'\varphi+2k_2\varphi',$$
$$D=k_2k_3\varphi.$$

in the last equation, we obtain

$$\bar{x}'''(s)=AV_1+BV_2+CV_3+DV_4.$$

Furthermore, differentiating $\bar{x}'''(s)$ with respect to s, we get

$$\bar{x}^{(iv)}(s)=(A'+k_1B)V_1+(B'+k_1A+k_2C)V_2+(C'+k_2B-k_3D)V_3+(k_3C+D')V_3$$

By the use of (12), we find

$$\bar{V}_1(s)=V_2.$$

While $g(V_2,V_2)=-1$, we can write $g(\bar{V}_1,\bar{V}_1)=-1$ which implies that involute $\bar{x}$ is a time-like curve.

Then we can compute the vector form $\bar{x}'(s)\wedge\bar{x}''(s)\wedge\bar{x}'''(s)$ and $\bar{V}_4$ of $\bar{x}$ as in the following:

$$\bar{x}'(s)\wedge\bar{x}''(s)\wedge\bar{x}'''(s)=\varphi^2\begin{pmatrix}-k_2DV_1+k_1DV_3\\+(k_2A-k_1C)V_4\end{pmatrix}$$

and

$$\bar{V}_4(s)=\frac{\bar{x}'(s)\wedge\bar{x}''(s)\wedge\bar{x}'''(s)}{\|\bar{x}'(s)\wedge\bar{x}''(s)\wedge\bar{x}'''(s)\|}=\frac{1}{W}\left(-k_2DV_1+k_1DV_3+(k_2A-k_1C)V_4\right)$$

where

$$W=\sqrt{D^2(k_1^2+k_2^2)+(k_2A-k_1C)^2}.$$

Similarly, we can compute

$$\bar{V}_4\wedge\bar{x}'(s)\wedge\bar{x}''(s)=\frac{\varphi^2}{W}\begin{pmatrix}-k_2(k_2A-k_1C)V_1\\+k_1(k_2A-k_1C)V_3\\-D(k_1^2+k_2^2)V_4\end{pmatrix}$$

and

$$\overline{V}_3=\frac{\overline{V}_4\wedge\overline{x}'(s)\wedge\overline{x}''(s)}{\left\|\overline{V}_4\wedge\overline{x}'(s)\wedge\overline{x}''(s)\right\|}$$

$$=\frac{1}{W\sqrt{k_1^2+k_2^2}}\begin{pmatrix}-k_2(k_2A-k_1C)V_1\\+k_1(k_2A-k_1C)V_3\\-D(k_1^2+k_2^2)V_4\end{pmatrix}$$

Finally, if we calculate $\overline{V}_3\wedge\overline{V}_4\wedge\overline{x}'(s)$ and substitute in (4), we get

$$\overline{V}_2(s)=\frac{k_1V_1+k_2V_3}{\sqrt{k_1^2+k_2^2}}.$$

Consequently, an easy calculation gives

$$g(\overline{V}_2(s),\overline{x}''(s))=\varphi\sqrt{k_1^2+k_2^2},$$

$$g(\overline{V}_3(s),\overline{x}'''(s))=-\frac{W}{\sqrt{k_1^2+k_2^2}},\tag{13}$$

$$g(\overline{V}_4(s),\overline{x}^{(iv)}(s))=\frac{1}{W}\begin{pmatrix}(k_2A-k_1C)(k_3C+D')\\+D(k_1C'-k_2A')-k_1k_3D^2\end{pmatrix}.$$

Hence, from equations (13) and (5), we get (9), which completes the proof.

For the case x is a W-curve, one can get the following results.

**Corollary 1** Let $x(s)$ be a space-like curve with time-like principal normal in $IE_1^4$ given with the Frenet curvatures $k_1$, $k_2$, $k_3$. If x is a W-curve, then the Frenet frame vectors $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, $\overline{V}_4$ and Frenet curvatures $\overline{k}_1$, $\overline{k}_2$, $\overline{k}_3$ of the involute $\overline{x}$ of the curve x are given by

$$\overline{V}_1(s)=V_2,$$

$$\overline{V}_2(s)=\frac{k_1V_1+k_2V_3}{\sqrt{k_1^2+k_2^2}},$$

$$\overline{V}_3(s)=V_4,\tag{14}$$

$$\overline{V}_4(s)=\frac{-k_2V_1+k_1V_3}{\sqrt{k_1^2+k_2^2}},$$

and

$$\overline{k}_1(s)=\frac{\sqrt{k_1^2+k_2^2}}{(c-s)k_1},$$

$$\overline{k}_2(s)=\frac{k_2k_3}{(c-s)k_1(k_1^2+k_2^2)},\tag{15}$$

$$\overline{k}_3(s)=\frac{k_3}{(c-s)\sqrt{k_1^2+k_2^2}},$$

respectively (Turgut et al. 2010).

**Corollary 2.** Let $\overline{x}(s)$ be an involute of a space-like curve x with time-like principal normal in $IE_1^4$ given with the Frenet curvatures $\overline{k}_1$, $\overline{k}_2$, $\overline{k}_3$. If x is a W-curve, then $\overline{x}$ becomes a ccr-curve.

### *2.2. Involute of order 2*

An involute of order 2 of a space-like curve x in $IE_1^4$ has the parametrization

$$\overline{x}(s)=x(s)+\lambda_1(s)V_1(s)+\lambda_2(s)V_2(s)\tag{16}$$

where $\lambda_1$, $\lambda_2$ are differential functions satisfying

$$\lambda_1'(s)=-1-\lambda_2(s)k_1(s)$$
$$\lambda_2'(s)=-\lambda_1(s)k_1(s).\tag{17}$$

From the differentiable equation system (17), we get the following result.

**Corollary 3.** Let $x=x(s)$ be a space-like Salkowski curve with time-like principal normal in $IE_1^4$. Then the involute $\overline{x}$ of order 2 of the curve x has the parametrization (16) given with the coefficient functions

$$\lambda_1(s)=c_1\cosh(k_1s)+c_2\sinh(k_1s)$$

$$\lambda_2(s)=-c_1\sinh(k_1s)-c_2\cosh(k_1s)-\frac{1}{k_1}$$

where $c_1$ and $c_2$ are real constants.

**Theorem 2.** Let $x=x(s)$ be a space-like curve with time-like principal normal in $IE_1^4$ given with Frenet curvatures $k_1$, $k_2$, $k_3$. Then the involute $\overline{x}$ of order 2 of the curve x is a space-like curve with the Frenet frame vectors $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, $\overline{V}_4$ and Frenet curvatures $\overline{k}_1$, $\overline{k}_2$, $\overline{k}_3$ which are given by

$$\overline{V}_1(s)=V_3,$$
$$\overline{V}_2(s)=\frac{k_2V_2+k_3V_4}{\sqrt{|k_3^2-k_2^2|}}, \tag{18}$$
$$\overline{V}_3(s)=\frac{1}{\overline{W}\sqrt{|k_3^2-k_2^2|}}\begin{pmatrix}K(k_3^2-k_2^2)V_1-k_3(k_2N-k_3L)V_2\\ -k_2(k_2N-k_3L)V_4\end{pmatrix}$$
$$\overline{V}_4(s)=\frac{1}{\overline{W}}\left((k_2N-k_3L)V_1-k_3KV_2-k_2KV_4\right),$$

and

$$\overline{k}_1(s)=\frac{\sqrt{|k_3^2-k_2^2|}}{\phi},$$
$$\overline{k}_2(s)=-\frac{\overline{W}}{\phi^2(k_3^2-k_2^2)}, \tag{19}$$
$$\overline{k}_3(s)=-\frac{\sqrt{|k_3^2-k_2^2|}}{\overline{W}^2\phi}\begin{pmatrix}(k_2N-k_3L)(K'+k_1L)\\ +K(k_3L'-k_2N')\\ +k_1k_3K^2\end{pmatrix},$$

where

$$\phi=\lambda_2k_2,$$
$$K=k_1k_2\phi,$$
$$L=k_2'\phi+2k_2\phi',$$
$$N=k_3'\phi+2k_3\phi',$$

and

$$\overline{W}=\sqrt{|(k_2N-k_3L)^2-K^2(k_3^2-k_2^2)|}$$
$$=|\phi|\sqrt{|(k_2k_3'-k_2'k_3)^2-k_1^2k_2^2(k_3^2-k_2^2)|}.$$

**Proof.** Let $\overline{x}=\overline{x}(s)$ be the involute of order 2 of a space-like curve with time-like principal normal in $IE_1^4$. Then by the use of (16) with (7), we get

$$\overline{x}'(s)=\phi V_3, \tag{20}$$

where $\phi(s)=\lambda_2(s)k_2(s)$ is a differentiable function.

By the use of (20), we find

$$\overline{V}_1(s)=V_3.$$

While $g(V_3,V_3)=1$, we can write $g(\overline{V}_1,\overline{V}_1)=1$ which implies that involute $\overline{x}$ of order 2 is a space-like curve.

Further, the differentiation of (20) implies that

$$\overline{x}''(s)=\phi k_2V_2+\phi'V_3+\phi k_4V_4, \tag{21}$$

$$\overline{x}'''(s)=k_1k_2\phi V_1+(k_2'\phi+2k_2\phi')V_2$$
$$+(k_2^2\phi+\phi''-k_3^2\phi)V_3+(k_3'\phi+2k_3\phi')V_4.$$

Consequently, substituting

$$K=k_1k_2\phi,$$
$$L=k_2'\phi+2k_2\phi',$$
$$M=k_2^2\phi+\phi''-k_3^2\phi,$$
$$N=\phi k_3'\phi+2k_3\phi',$$

in the last vector, we obtain

$$\overline{x}'''(s)=KV_1+LV_2+MV_3+NV_4. \tag{22}$$

Furthermore, differentiating $\overline{x}'''$ with respect to s, we get

$$\overline{x}^{(iv)}(s)=(K'+k_1L)V_1+(L'+k_1K+k_2M)V_2$$
$$+(M'+k_2L-k_3N)V_3+(N'+k_3M)V_4 \tag{23}$$

Hence substituting (20)-(23) into (4) and (5), after making some calculations as in the previous theorem, we obtain the result.

For the case x is a W-curve, one can get the following result.

**Corollary 4.** Let $\overline{x}$ be an involute of order 2 of a space-like curve with time-like principal normal in $IE_1^4$ given with the Frenet curvatures $\overline{k}_1, \overline{k}_2, \overline{k}_3$. If x is a W-curve, then the Frenet frame vectors $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, $\overline{V}_4$ and Frenet curvatures $\overline{k}_1$, $\overline{k}_2$, $\overline{k}_3$ of the involute $\overline{x}$ of order 2 of the curve x are given by

$$\overline{V}_1(s)=V_3,$$
$$\overline{V}_2(s)=\frac{k_2V_2+k_3V_4}{\sqrt{|k_3^2-k_2^2|}},$$
$$\overline{V}_3(s)=V_1, \tag{24}$$
$$\overline{V}_4(s)=\frac{k_3V_2+k_2V_4}{\sqrt{|k_3^2-k_2^2|}},$$

and

$$\overline{k}_1(s)=\frac{\sqrt{|k_3^2-k_2^2|}}{\phi},$$
$$\overline{k}_2(s)=-\frac{k_1k_2}{\phi\sqrt{|k_3^2-k_2^2|}}, \tag{25}$$

$$\bar{k}_3(s) = -\frac{k_1 k_3}{\phi\sqrt{\left|k_3^2 - k_2^2\right|}},$$

where $\phi(s) = \lambda_2(s)k_2(s)$.

**Corollary 5.** Let $\bar{x}$ be an involute of order 2 of a space-like curve x with time-like principal normal in $IE_1^4$ given with the Frenet curvatures $\bar{k}_1, \bar{k}_2, \bar{k}_3$. If x is a W-curve, then $\bar{x}$ becomes a ccr-curve.

### *2.3. Involute of order 3*

An involute of order 3 of a space-like curve x in $IE_1^4$ has the parametrization

$$\bar{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) + \lambda_3(s)V_3(s) \quad (26)$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are differentiable functions satisfying

$$\begin{aligned} \lambda_1'(s) &= -1 - \lambda_2(s)k_1(s), \\ \lambda_2'(s) &= -\lambda_1(s)k_1(s) - \lambda_3(s)k_2(s), \\ \lambda_3'(s) &= -\lambda_2(s)k_2(s). \end{aligned} \quad (27)$$

By solving the differential equation system (27), we get the following result.

**Corollary 6.** Let $x = x(s)$ be a space-like W-curve with time-like principal normal in $IE_1^4$. Then the involute $\bar{x}$ of order 3 of the curve x has the parametrization (26) given with the coefficient functions

$$\lambda_1(s) = \frac{k_1(c_3\cos(ks) - c_2\sin(ks))}{k} - \frac{k_2^2 s}{k^2} + c_1,$$

$$\lambda_2(s) = c_2\cos(ks) + c_3\sin(ks) - \frac{k_1}{k^2},$$

$$\lambda_3(s) = \frac{k_2(c_3\cos(ks) - c_2\sin(ks))}{k} + \frac{k_1 k_2 s}{k^2} - \frac{c_1 k_1}{k_2},$$

where $k = \sqrt{k_1 + k_2}$, $c_1$, $c_2$ and $c_3$ are real constants.

**Theorem 3.** Let $x = x(s)$ be a space-like curve with time-like principal normal in $IE_1^4$ given with Frenet curvatures $k_1$, $k_2$ and $k_3$. Then the involute $\bar{x}$ of order 3 of the curve x is a space-like curve with the Frenet frame vectors $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$, $\bar{V}_4$ and Frenet curvatures $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$ which are given by

$$\begin{aligned} \bar{V}_1(s) &= V_4, \\ \bar{V}_2(s) &= -V_3, \\ \bar{V}_3(s) &= V_2, \\ \bar{V}_4(s) &= -V_1 \end{aligned} \quad (28)$$

and

$$\begin{aligned} \bar{k}_1(s) &= \frac{k_3}{\psi}, \\ \bar{k}_2(s) &= -\frac{k_2}{\psi}, \\ \bar{k}_3(s) &= -\frac{k_1}{\psi}, \end{aligned} \quad (29)$$

where $\psi(s) = \lambda_3(s)k_3(s)$.

**Proof.** Let $\bar{x} = \bar{x}(s)$ be the involute of order 3 of a space-like curve with time-like principal normal in $IE_1^4$. Then by the use of (26) with (7), we get

$$\bar{x}'(s) = \psi V_4 \quad (30)$$

where $\psi(s) = \lambda_3(s)k_3(s)$ is a differentiable function. By the use of (30), we find

$$\bar{V}_1(s) = V_4.$$

While $g(V_4, V_4) = 1$, we can write $g(\bar{V}_1, \bar{V}_1) = 1$ which implies that involute $\bar{x}$ of order 3 is a space-like curve.

Further, the differentiation of (29) implies that

$$\begin{aligned} \bar{x}''(s) &= -k_3\psi V_3 + \psi' V_4, \\ \bar{x}'''(s) &= -k_2 k_3 \psi V_2 - (k_3'\psi + 2k_3\psi')V_3 \\ &\quad + (\psi'' - k_3^2\psi)V_4. \end{aligned} \quad (31)$$

Consequently, substituting

$$E = -k_2 k_3 \psi$$

$$F = -(k_3'\psi + 2k_3\psi')$$

$$G = \psi'' - k_3^2\psi$$

in the last vector, we obtain

$$\bar{x}'''(s) = EV_2 + FV_3 + GV_4. \quad (32)$$

Furthermore, differentiating $\bar{x}'''$ with respect to s, we get

$$\begin{aligned} \bar{x}^{(iv)}(s) &= k_1 EV_1 + (E' + k_2 F)V_2 \\ &\quad + (F' + k_2 E - k_3 G)V_3 + (G' + k_3 F)V_4 \end{aligned} \quad (33)$$

Hence substituting (30)-(33) into (4) and (5), after some calculations as in the previous theorem, we obtain the result.

**Corollary 7.** The ivolute $\bar{x}$ of order 3 of a space-like ccr-curve x with time-like principal normal in $IE_1^4$ is also a ccr-curve in $IE_1^4$.

**3. Conclusion**

In recent years, many authors have studied with the involute-evolute curve couples in many paper. Turgut et al. (2010) gave the characterization of the involute of order 1 (involute) of a W-curve in $IE_1^4$

In this paper, we study involute curves of order k of a space-like curve x with time-like principal normal in Minkowski 4-space $IE_1^4$. First, we investigate an involute curve of order 1 of a given curve. Furthermore, we give the characterizations of the involutes of order 2 and 3. We obtain the Frenet Frame and Frenet curvatures of the involutes of order k of the curve with respect to the Frenet Frame and Frenet curvatures of the given curve.

Nowadays, as known W-curve (or helix) is very important topic in curve theory, we characterize the involutes of order k of a W-curve in $IE_1^4$.

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