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# Finite difference method with different high order approximations for solving complex equation

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**Abstract:** In this paper, we take finite difference method with different high order approximations for solving Hirota Equation is presented. The stability analysis using Von-Neumann technique shows schemes are unconditionally stable. To test accuracy the error norms $L_2$, $L_\infty$ are computed. We compute local truncation error for different schemes. We make comparison between these approximations through the results that we are get it. These results show that the approximation of O$(k^2+h^4)$ introduced here is more accurate than others and easy to apply.

**Keywords:** Finite difference method with different high order approximations, Hirota equation, stability, local truncation error.

## 1 Introduction

The purpose of this paper is to apply finite difference method with different high order approximations to the Hirota equation. The Hirota equation in the form [1]

$$u_t+3\alpha\,|u|^2\,u_x+\gamma u_{xxx}=0, \tag{1}$$

where $u$ is a complex valued function of the spatial coordinate $x$ and the time $t$ and $\alpha$, $\gamma$ are positive real constants. Boundary conditions

$$u(x,t)=u_x(x,t)=0, |x|\to\infty, 0\le t\le T. \tag{2}$$

And initial conditions.

$$u(x,0)=f(x), -\infty<x<\infty. \tag{3}$$

The exact solution of Hitora equation (1) is

$$\begin{aligned} &u(x,t)=\beta\,\mathrm{sec}\,h\left[\kappa\left(x-s-vt\right)\right]\exp\left(i\varphi\right),\\ &\beta=\sqrt{\tfrac{2\gamma}{\alpha}}\kappa, \varphi=a\left(x-bt-s\right),\\ &v=\gamma\left(\kappa^2-3a^2\right), b=\gamma\left(3\kappa^3-a^2\right), \end{aligned} \tag{4}$$

where $\beta$ is the amplitude of the wave, $\kappa$ is related to the width of the wave envelope and $v$ is the velocity. The parameter $a$ is the wave number of the phase and $b$ is related to the frequency of the phase. This equation is an integrable equation and it is very important because it has many physical applications, such as the propagation of optical pluses in nematic liquid crystal waveguides. The Hirota equation is closely related to both the nonlinear Schrodinger equation and modified Korteweg-de Vries (mKdV) equations, as it is complex generalization of the mKdV equation and it is a part of the nonlinear Schrodinger equation hierarchy of the integrable equation. Also, its soliton solution has a very similar form


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to the nonlinear Schrodinger equation soliton. The Eq. (1) has two-parameter soliton family, with amplitude and velocity.

The Hirota equation has been solved numerically by Hoseini S. M. and Marchant T. R. [1] and the Eq. (1) has been studied by W. G. Al.Harbi [2]. The numerical solution of nonlinear wave equations has been the subject of many studies in recent years. Such as the Korteweg-de Vries (KdV) equation [3, 4, 5, 6] and the nonlinear Schrodinger equation has been solved by [7, 8]. Numerical solution of coupled partial differential equations, as an example, the coupled nonlinear Schrodinger equation admits soliton solution and it has many applications in communication, this system has been solved numerically by Ismail [9,10,11,12] and the coupled Korteweg-de Vries equation has been solved numerically [13, 14, 15, 16]. The complex nonlinear partial differential equations have been solved in [17, 18, 19, 20, 21]. The nonintegrable variant of Hirota equation in which the nonlinear term in (1) is replaced by $\left(|u|^2 u\right)_x$, is solved numerically by [17, 19].

The paper is organized as follows. In Section 2, we convert the Eq. (1) from complex equation to system of nonlinear equations that have real functions. In Section 3, we have four different approximation of finite difference method and we introduce dissection of stability and local truncation error for different schemes. In section 4, numerical results for problem and some related figures are given in order to show the efficiency as well as the accuracy of the proposed method and we introduced the interaction of two and three solitary waves. Finally, conclusions are followed in Section 5.

## 2 The Hirota equation

In this section we convert the complex Eq. (1) to system of nonlinear equations that have real functions. We assume that [19, 20].

$$u(x,t) = u_1(x,t) + iu_2(x,t), \; i^2 = -1, \tag{5}$$

where $u_1(x,t)$and$u_2(x,t)$are real functions.

By substituting in Eq. (1) we will reduce Hirota equation to the coupled system in this form

$$\begin{aligned}(u_1)_t + 3\alpha\left(u_1^2+u_2^2\right)(u_1)_x + \gamma(u_1)_{xxx} = 0,\\ (u_2)_t + 3\alpha\left(u_1^2+u_2^2\right)(u_2)_x + \gamma(u_2)_{xxx} = 0.\end{aligned} \tag{6}$$

We can write this system in this form

$$(u)_t + 3\alpha z(u)(u)_x + \gamma(u)_{xxx} = 0, \tag{7}$$

where $z(u) = \left(u_1^2+u_2^2\right), u = [u_1,u_2]^T$.

## 3 Derivation of the numerical method

In this section we given theoretically discussed for the numerical method using finite difference method with different high order approximations.

### 3.1 The first approximation

In this section we will prove that the method is second order in space and time. We take approximations for space derivatives and time derivatives as:

$$\begin{aligned}(u_t)_j^n &= \frac{u_j^{n+1}-u_j^n}{k} + \mathrm{O}(k^2),\\ (u_x)_j^n &= \frac{u_{j+2}^n-u_{j-2}^n}{4h} + \mathrm{O}(h^2),\\ (u_{xxx})_j^n &= \frac{u_{j+2}^n-2u_{j+1}^n+2u_{j-1}^n-u_{j-2}^n}{2h^3} + \mathrm{O}(h^2).\end{aligned} \tag{8}$$

Now, we assume that $(u)_j^n$ is the exact solution ate the grid point $(x_j,t_n)$ and $(U)_j^n$ is the numerical solution at the same point.

Substituting (8) into (6) and using Crank-Nicolson formula [22] we get

$$\begin{aligned}&(U_1)_j^{n+1}+r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j+2}^{n+1}-(U_1)_{j-2}^{n+1}\right)+r_2\left((U_1)_{j+2}^{n+1}-2(U_1)_{j+1}^{n+1}+2(U_1)_{j-1}^{n+1}-(U_1)_{j-2}^{n+1}\right)\\&=(U_1)_j^n-r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j+2}^n-(U_1)_{j-2}^n\right)-r_2\left((U_1)_{j+2}^n-2(U_1)_{j+1}^n+2(U_1)_{j-1}^n-(U_1)_{j-2}^n\right),\end{aligned} \tag{9}$$

$$\begin{aligned}&(U_2)_j^{n+1}+r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_2)_{j+2}^{n+1}-(U_2)_{j-2}^{n+1}\right)+r_2\left((U_2)_{j+2}^{n+1}-2(U_2)_{j+1}^{n+1}+2(U_2)_{j-1}^{n+1}-(U_2)_{j-2}^{n+1}\right)\\&=(U_2)_j^n-r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_2)_{j+2}^n-(U_2)_{j-2}^n\right)-r_2\left((U_2)_{j+2}^n-2(U_2)_{j+1}^n+2(U_2)_{j-1}^n-(U_2)_{j-2}^n\right),\end{aligned} \tag{10}$$

where $z=(U_1)^2+(U_2)^2, r_1=\frac{3\alpha k}{16h}, r_2=\frac{\gamma k}{4h^3}$. This system can be solved by many methods.

### 3.1.1 Stability analysis of first scheme

In this section, the standard Von-Neumann concept is applied to investigate the stability analysis of the schemes. At first, we must linearize the nonlinear term of the Hirota equation by making $\left((z)_j^{n+1}+(z)_j^n\right)$ as a local constant $\lambda_1$. According to the Von- Neumann concept, we get

$$\begin{aligned}(U_1)_j^n&=A\zeta^n\exp(ij\phi),\\(U_2)_j^n&=B\zeta^n\exp(ij\phi),\end{aligned} \tag{11}$$

$$g=\frac{\zeta^{n+1}}{\zeta^n},$$

where $A$ and $B$ are the harmonics amplitude, $\phi=kh$ , $k$ is the mode number, $i=\sqrt{-1}$ and $g$ is the amplification factor of the schemes. Substituting (11) into the difference (9), we get

$$\zeta^{n+1}A\left[1+2i\left((r_1\lambda_1+r_2)\sin 2\phi-2r_2\sin\phi\right)\right]=\zeta^nA\left[1-2i\left((r_1\lambda_1+r_2)\sin 2\phi-2r_2\sin\phi\right)\right]$$

we get

$$g=\frac{1-2i\left((r_1\lambda_1+r_2)\sin 2\phi-2r_2\sin\phi\right)}{1+2i\left((r_1\lambda_1+r_2)\sin 2\phi-2r_2\sin\phi\right)}, \tag{12}$$

from (12) we get $|g|\le 1$, hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on $h$ and $\Delta t$, but we should choose them in such a way that the accuracy of the scheme is not degraded. Similar results can be obtained from the difference (10).

### 3.1.2 Local truncation error of first scheme

To study the accuracy of (10) we replace $(U)_j^n$by$(u)_j^n$ first, then from Taylor's series expansion for all terms in (10) about the point $(x_j,t_n)$ we get

$$\begin{aligned}T_j^n=&\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n+\frac{k}{2}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n\\&+\left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}+2\alpha h^2 z(u)\frac{\partial^3 u}{\partial x^3}+\alpha k h^2 z(u)\frac{\partial^4 u}{\partial x^3\partial t}+\frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x\partial t^2}+\\\frac{\gamma h^2}{4}\frac{\partial^5 u}{\partial x^5}+\frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3\partial t^2}+\ldots\end{array}\right],\end{aligned}$$

the first and second terms is zero by the Hirota equation, and so we end with local truncation error

$$T_j^n = \begin{bmatrix} \frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + 2\alpha h^2 z(u)\frac{\partial^3 u}{\partial x^3} + \alpha k h^2 z(u)\frac{\partial^4 u}{\partial x^3 \partial t} + \frac{3\alpha k^2}{4} z(u)\frac{\partial^3 u}{\partial x \partial t^2} + \\ \frac{\gamma h^2}{4}\frac{\partial^5 u}{\partial x^5} + \frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3 \partial t^2} + \ldots \end{bmatrix}.$$

This means that Crank-Nicolson scheme is second order accuracy in space and second order in time, i.e. $O(k^2+h^2)$.

## 3.2 *The second approximation*

In this section we will prove that the method is fourth order in first derivative space and second order in third derivative space and time. We take approximations for space derivatives and time derivatives as:

$$\begin{aligned} (u_t)_j^n &= \frac{u_j^{n+1}-u_j^n}{k} + O(k^2), \\ (u_x)_j^n &= \frac{u_{j-2}^n - 8u_{j-1}^n + 8u_{j+1}^n - u_{j+2}^n}{12h} + O(h^4), \\ (u_{xxx})_j^n &= \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2h^3} + O(h^2). \end{aligned} \tag{13}$$

Now, we assume that$(u)_j^n$is the exact solution ate the grid point $(x_j, t_n)$and $(U)_j^n$is the numerical solution at the same point.

Substituting (13) into (6) and using Crank-Nicolson formula we get

$$\begin{aligned} &(U_1)_j^{n+1} + r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_1)_{j-2}^{n+1} - 8(U_1)_{j-1}^{n+1} + 8(U_1)_{j+1}^{n+1} - (U_1)_{j+2}^{n+1}\right) \\ &\quad + r_2\left((U_1)_{j+2}^{n+1} - 2(U_1)_{j+1}^{n+1} + 2(U_1)_{j-1}^{n+1} - (U_1)_{j-2}^{n+1}\right) \\ &\quad = (U_1)_j^n - r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_1)_{j-2}^n - 8(U_1)_{j-1}^n + 8(U_1)_{j+1}^n - (U_1)_{j+2}^n\right) \\ &\quad - r_2\left((U_1)_{j+2}^n - 2(U_1)_{j+1}^n + 2(U_1)_{j-1}^n - (U_1)_{j-2}^n\right), \end{aligned} \tag{14}$$

$$\begin{aligned} &(U_2)_j^{n+1} + r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_2)_{j-2}^{n+1} - 8(U_2)_{j-1}^{n+1} + 8(U_2)_{j+1}^{n+1} - (U_2)_{j+2}^{n+1}\right) \\ &\quad + r_2\left((U_2)_{j+2}^{n+1} - 2(U_2)_{j+1}^{n+1} + 2(U_2)_{j-1}^{n+1} - (U_2)_{j-2}^{n+1}\right) \\ &\quad = (U_2)_j^n - r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_2)_{j-2}^n - 8(U_2)_{j-1}^n + 8(U_2)_{j+1}^n - (U_2)_{j+2}^n\right) \\ &\quad - r_2\left((U_2)_{j+2}^n - 2(U_2)_{j+1}^n + 2(U_2)_{j-1}^n - (U_2)_{j-2}^n\right), \end{aligned} \tag{15}$$

where $z = (U_1)^2 + (U_2)^2, r_1 = \frac{3\alpha k}{48h}, r_2 = \frac{\gamma k}{4h^3}$. This system can be solved by many methods.

### 3.2.1 Stability analysis of second scheme

In this section, the standard Von-Neumann concept is applied to investigate the stability analysis of the schemes. At first, we must linearize the nonlinear term of the Hirota equation by making $\left((z)_j^{n+1} + (z)_j^n\right)$ as a local constant $\lambda_1$. According to the Von- Neumann concept, we get

$$\begin{aligned} (U_1)_j^n &= A\zeta^n \exp(ij\phi), \\ (U_2)_j^n &= B\zeta^n \exp(ij\phi), \end{aligned} \tag{16}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where $A$ and $B$ are the harmonics amplitude, $\phi = kh$ , $k$ is the mode number, $i = \sqrt{-1}$and $g$ is the amplification factor of the schemes. Substituting (16) into the difference (**??**), we get

$$\zeta^{n+1}A\left[1+2i\left((-r_1\lambda_1+r_2)\sin 2\phi-2(r_2-4r_1\lambda_1)\sin\phi\right)\right]$$
$$=\zeta^{n}A\left[1-2i\left((-r_1\lambda_1+r_2)\sin 2\phi-2(r_2-4r_1\lambda_1)\sin\phi\right)\right]$$

we get

$$g=\frac{1-2i\left((-r_1\lambda_1+r_2)\sin 2\phi-2(r_2-4r_1\lambda_1)\sin\phi\right)}{1+2i\left((-r_1\lambda_1+r_2)\sin 2\phi-2(r_2-4r_1\lambda_1)\sin\phi\right)}, \tag{17}$$

from (17) we get $|g| \leq 1$, hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on $h$ and $\Delta t$, but we should choose them in such a way that the accuracy of the scheme is not degraded. Similar results can be obtained from the difference (15).

#### 3.2.2 Local truncation error of second scheme

To study the accuracy of (14) we replace $(U)_j^n$by$(u)_j^n$ first, then from Taylor's series expansion for all terms in (14) about the point $(x_j,t_n)$we get

$$T_j^n=\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n+\frac{k}{2}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n$$
$$+\left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x\partial t^2}+\frac{3\alpha k^3}{12}z(u)\frac{\partial^4 u}{\partial x\partial t^3}-\frac{3\alpha h^4}{30}z(u)\frac{\partial^5 u}{\partial x^5}+\\ \frac{3\alpha k^4}{48}z(u)\frac{\partial^5 u}{\partial x\partial t^4}-\frac{3\alpha h^4 k}{60}z(u)\frac{\partial^6 u}{\partial x^5\partial t}+\frac{\gamma h^2}{4}\frac{\partial^5 u}{\partial x^5}+\frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3\partial t^2}+\ldots\end{array}\right],$$

the first and second terms is zero by the Hirota equation, and so we end with local truncation error

$$T_j^n=\left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x\partial t^2}+\frac{3\alpha k^3}{12}z(u)\frac{\partial^4 u}{\partial x\partial t^3}-\frac{3\alpha h^4}{30}z(u)\frac{\partial^5 u}{\partial x^5}+\\ \frac{3\alpha k^4}{48}z(u)\frac{\partial^5 u}{\partial x\partial t^4}-\frac{3\alpha h^4 k}{60}z(u)\frac{\partial^6 u}{\partial x^5\partial t}+\frac{\gamma h^2}{4}\frac{\partial^5 u}{\partial x^5}+\frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3\partial t^2}+\ldots\end{array}\right].$$

This means that Crank-Nicolson scheme is second order accuracy in space and second order in time, i.e. $O(k^2+h^2)$.

### *3.3 The third approximation*

In this section we will prove that the method is fourth order in third derivative space and second order in first derivative space and time. We take approximations for space derivatives and time derivatives as:

$$\begin{array}{l}(u_t)_j^n=\frac{u_j^{n+1}-u_j^n}{k}+O(k^2),\\ (u_x)_j^n=\frac{u_{j+2}^n-u_{j-2}^n}{4h}+O(h^2),\\ (u_{xxx})_j^n=\frac{u_{j-3}^n-8u_{j-2}^n+13u_{j-1}^n-13u_{j+1}^n+8u_{j+2}^n-u_{j+3}^n}{8h^3}+O(h^4).\end{array} \tag{18}$$

Now, we assume that $(u)_j^n$ is the exact solution ate the grid point $(x_j,t_n)$ and $(U)_j^n$ is the numerical solution at the same point. Substituting (18) into (6) and using Crank-Nicolson formula we get

$$\begin{array}{l}(U_1)_j^{n+1}+r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j+2}^{n+1}-(U_1)_{j-2}^{n+1}\right)\\ +r_2\left((U_1)_{j-3}^{n+1}-8(U_1)_{j-2}^{n+1}+13(U_1)_{j-1}^{n+1}-13(U_1)_{j+1}^{n+1}+8(U_1)_{j+2}^{n+1}-(U_1)_{j+3}^{n+1}\right)\\ =(U_1)_j^n-r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j+2}^n-(U_1)_{j-2}^n\right)\\ -r_2\left((U_1)_{j-3}^n-8(U_1)_{j-2}^n+13(U_1)_{j-1}^n-13(U_1)_{j+1}^n+8(U_1)_{j+2}^n-(U_1)_{j+3}^n\right),\end{array} \tag{19}$$

$$\begin{aligned}
&(U_2)_j^{n+1} + r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_2)_{j+2}^{n+1} - (U_2)_{j-2}^{n+1}\right) \\
&+ r_2\left((U_2)_{j-3}^{n+1} - 8(U_2)_{j-2}^{n+1} + 13(U_2)_{j-1}^{n+1} - 13(U_2)_{j+1}^{n+1} + 8(U_2)_{j+2}^{n+1} - (U_2)_{j+3}^{n+1}\right) \\
&= (U_2)_j^n - r_1\left((z)_j^{n+1} + (z)_j^n\right)\left((U_2)_{j+2}^n - (U_2)_{j-2}^n\right) \\
&- r_2\left((U_2)_{j-3}^n - 8(U_2)_{j-2}^n + 13(U_2)_{j-1}^n - 13(U_2)_{j+1}^n + 8(U_2)_{j+2}^n - (U_2)_{j+3}^n\right),
\end{aligned} \tag{20}$$

where $z = (U_1)^2 + (U_2)^2, r_1 = \frac{3\alpha k}{16h}, r_2 = \frac{\gamma k}{16h^3}$. This system can be solved by many methods.

### 3.3.1 Stability analysis of third scheme

In this section, the standard Von-Neumann concept is applied to investigate the stability analysis of the schemes. At first, we must linearize the nonlinear term of the Hirota equation by making $\left((z)_j^{n+1} + (z)_j^n\right)$ as a local constant $\lambda_1$. According to the Von- Neumann concept, we get

$$\begin{aligned}
(U_1)_j^n &= A\zeta^n \exp(ij\phi), \\
(U_2)_j^n &= B\zeta^n \exp(ij\phi),
\end{aligned} \tag{21}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where $A$and$B$are the harmonics amplitude, $\phi = kh$ , $k$ is the mode number, $i = \sqrt{-1}$ and $g$ is the amplification factor of the schemes. Substituting (21) into the difference (19), we get

$$\begin{aligned}
&\zeta^{n+1}A[1 + 2i((r_1\lambda_1 + 8r_2)\sin 2\phi - 13r_2\sin\phi - r_2\sin 3\phi)] \\
&= \zeta^n A[1 - 2i((r_1\lambda_1 + 8r_2)\sin 2\phi - 13r_2\sin\phi - r_2\sin 3\phi)]
\end{aligned}$$

we get

$$g = \frac{1 - 2i((r_1\lambda_1 + 8r_2)\sin 2\phi - 13r_2\sin\phi - r_2\sin 3\phi)}{1 + 2i((r_1\lambda_1 + 8r_2)\sin 2\phi - 13r_2\sin\phi - r_2\sin 3\phi)}, \tag{22}$$

from (22) we get $|g| \leq 1$, hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on $h$ and $\Delta t$, but we should choose them in such a way that the accuracy of the scheme is not degraded. Similar results can be obtained from the difference (20)

### 3.3.2 Local truncation error of third scheme

To study the accuracy of (19) we replace $(U)_j^n$ by $(u)_j^n$ first, then from Taylor's series expansion for all terms in (19) about the point $(x_j, t_n)$ we get

$$\begin{aligned}
T_j^n &= \left[\frac{\partial u}{\partial t} + 3\alpha z(u)\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n + \frac{k}{2}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t} + 3\alpha z(u)\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n \\
&+ \left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + 2\alpha h^2 z(u)\frac{\partial^3 u}{\partial x^3} + \alpha k h^2 z(u)\frac{\partial^4 u}{\partial x^3 \partial t} + \frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x \partial t^2} + \\ \frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3 \partial t^2} + \frac{\gamma k^3}{12}\frac{\partial^6 u}{\partial x^3 \partial t^3} - \frac{7\gamma h^4}{120}\frac{\partial^7 u}{\partial x^7} + \frac{\gamma k^4}{48}\frac{\partial^7 u}{\partial x^3 \partial t^4} + \ldots\end{array}\right],
\end{aligned}$$

the first and second terms is zero by the Hirota equation, and so we end with local truncation error

$$T_j^n = \left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3} + 2\alpha h^2 z(u)\frac{\partial^3 u}{\partial x^3} + \alpha k h^2 z(u)\frac{\partial^4 u}{\partial x^3 \partial t} + \frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x \partial t^2} + \\ \frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3 \partial t^2} + \frac{\gamma k^3}{12}\frac{\partial^6 u}{\partial x^3 \partial t^3} - \frac{7\gamma h^4}{120}\frac{\partial^7 u}{\partial x^7} + \frac{\gamma k^4}{48}\frac{\partial^7 u}{\partial x^3 \partial t^4} + \ldots\end{array}\right].$$

This means that Crank-Nicolson scheme is second order accuracy in space and second order in time, i.e. $O(k^2 + h^2)$.

### 3.4 The fourth approximation

In this section we will prove that the method is fourth order in space and second order in time. We take approximations for space derivatives and time derivatives as:

$$\begin{aligned}(u_t)_j^n &= \frac{u_j^{n+1}-u_j^n}{k}+\mathrm{O}(k^2),\\ (u_x)_j^n &= \frac{u_{j-2}^n-8u_{j-1}^n+8u_{j+1}^n-u_{j+2}^n}{12h}+\mathrm{O}(h^4),\\ (u_{xxx})_j^n &= \frac{u_{j-3}^n-8u_{j-2}^n+13u_{j-1}^n-13u_{j+1}^n+8u_{j+2}^n-u_{j+3}^n}{8h^3}+\mathrm{O}(h^4).\end{aligned} \tag{23}$$

Now, we assume that $(u)_j^n$ is the exact solution ate the grid point $(x_j,t_n)$ and $(U)_j^n$ is the numerical solution at the same point. Substituting (23) into (6) and using Crank-Nicolson formula we get

$$\begin{aligned}&(U_1)_j^{n+1}+r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j-2}^{n+1}-8(U_1)_{j-1}^{n+1}+8(U_1)_{j+1}^{n+1}-(U_1)_{j+2}^{n+1}\right)\\ &+r_2\left((U_1)_{j-3}^{n+1}-8(U_1)_{j-2}^{n+1}+13(U_1)_{j-1}^{n+1}-13(U_1)_{j+1}^{n+1}+8(U_1)_{j+2}^{n+1}-(U_1)_{j+3}^{n+1}\right)\\ &=(U_1)_j^n-r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_1)_{j-2}^{n}-8(U_1)_{j-1}^{n}+8(U_1)_{j+1}^{n}-(U_1)_{j+2}^{n}\right)\\ &-r_2\left((U_1)_{j-3}^{n}-8(U_1)_{j-2}^{n}+13(U_1)_{j-1}^{n}-13(U_1)_{j+1}^{n}+8(U_1)_{j+2}^{n}-(U_1)_{j+3}^{n}\right),\end{aligned} \tag{24}$$

$$\begin{aligned}&(U_2)_j^{n+1}+r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_2)_{j-2}^{n+1}-8(U_2)_{j-1}^{n+1}+8(U_2)_{j+1}^{n+1}-(U_2)_{j+2}^{n+1}\right)\\ &+r_2\left((U_2)_{j-3}^{n+1}-8(U_2)_{j-2}^{n+1}+13(U_2)_{j-1}^{n+1}-13(U_2)_{j+1}^{n+1}+8(U_2)_{j+2}^{n+1}-(U_2)_{j+3}^{n+1}\right)\\ &=(U_2)_j^n-r_1\left((z)_j^{n+1}+(z)_j^n\right)\left((U_2)_{j-2}^{n}-8(U_2)_{j-1}^{n}+8(U_2)_{j+1}^{n}-(U_2)_{j+2}^{n}\right)\\ &-r_2\left((U_2)_{j-3}^{n}-8(U_2)_{j-2}^{n}+13(U_2)_{j-1}^{n}-13(U_2)_{j+1}^{n}+8(U_2)_{j+2}^{n}-(U_2)_{j+3}^{n}\right),\end{aligned} \tag{25}$$

where $z=(U_1)^2+(U_2)^2, r_1=\frac{3\alpha k}{48h}, r_2=\frac{\gamma k}{16h^3}$. This system can be solved by many methods.

#### 3.4.1 Stability analysis of fourth scheme

In this section, the standard Von-Neumann concept is applied to investigate the stability analysis of the schemes. At first, we must linearize the nonlinear term of the Hirota equation by making $\left((z)_j^{n+1}+(z)_j^n\right)$ as a local constant $\lambda_1$. According to the Von- Neumann concept, we get

$$\begin{aligned}(U_1)_j^n &= A\zeta^n\exp(ij\phi),\\ (U_2)_j^n &= B\zeta^n\exp(ij\phi),\end{aligned} \tag{26}$$

$$g=\frac{\zeta^{n+1}}{\zeta^n},$$

where $A$ and $B$ are the harmonics amplitude, $\phi=kh$ , $k$is the mode number, $i=\sqrt{-1}$and $g$ is the amplification factor of the schemes. Substituting (26) into the difference (24), we get

$$\begin{aligned}&\zeta^{n+1}A[(r_1\lambda_1+8r_2)\sin 2\phi+(8r_1\lambda_1-13r_2)\sin\phi-r_2\sin 3\phi]\\ &=\zeta^nA[1-2i((r_1\lambda_1+8r_2)\sin 2\phi+(8r_1\lambda_1-13r_2)\sin\phi-r_2\sin 3\phi)]\end{aligned}$$

we get

$$g=\frac{1-2i((r_1\lambda_1+8r_2)\sin 2\phi+(8r_1\lambda_1-13r_2)\sin\phi-r_2\sin 3\phi)}{1+2i((r_1\lambda_1+8r_2)\sin 2\phi+(8r_1\lambda_1-13r_2)\sin\phi-r_2\sin 3\phi)}, \tag{27}$$

from (27) we get $|g|\leq 1$, hence the scheme is unconditionally stable. It means that there is no restriction on the grid size, i.e. on $h$ and $\Delta t$, but we should choose them in such a way that the accuracy of the scheme is not degraded.

Similar results can be obtained from the difference (25).

3.4.2 Local truncation error of fourth scheme

To study the accuracy of (25) we replace $(U)_j^n$by$(u)_j^n$ first, then from Taylor's series expansion for all terms in (25) about the point $(x_j,t_n)$ we get

$$T_j^n=\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n+\frac{k}{2}\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}+3\alpha z(u)\frac{\partial u}{\partial x}+\gamma\frac{\partial^3 u}{\partial x^3}\right]_j^n+\left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x\partial t^2}+\frac{3\alpha k^3}{12}z(u)\frac{\partial^4 u}{\partial x\partial t^3}-\frac{3\alpha h^4}{30}z(u)\frac{\partial^5 u}{\partial x^5}+\\ \frac{3\alpha k^4}{48}z(u)\frac{\partial^5 u}{\partial x\partial t^4}-\frac{3\alpha h^4 k}{60}z(u)\frac{\partial^6 u}{\partial x^5\partial t}+\frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3\partial t^2}+\frac{\gamma k^3}{12}\frac{\partial^6 u}{\partial x^3\partial t^3}-\\ \frac{7\gamma h^4}{120}\frac{\partial^7 u}{\partial x^7}+\frac{\gamma k^4}{48}\frac{\partial^7 u}{\partial x^3\partial t^4}+\cdots\end{array}\right],$$

the first and second terms is zero by the Hirota equation, and so we end with local truncation error

$$T_j^n=\left[\begin{array}{l}\frac{k^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{3\alpha k^2}{4}z(u)\frac{\partial^3 u}{\partial x\partial t^2}+\frac{3\alpha k^3}{12}z(u)\frac{\partial^4 u}{\partial x\partial t^3}-\frac{3\alpha h^4}{30}z(u)\frac{\partial^5 u}{\partial x^5}+\\ \frac{3\alpha k^4}{48}z(u)\frac{\partial^5 u}{\partial x\partial t^4}-\frac{3\alpha h^4 k}{60}z(u)\frac{\partial^6 u}{\partial x^5\partial t}+\frac{\gamma k^2}{4}\frac{\partial^5 u}{\partial x^3\partial t^2}+\frac{\gamma k^3}{12}\frac{\partial^6 u}{\partial x^3\partial t^3}-\\ \frac{7\gamma h^4}{120}\frac{\partial^7 u}{\partial x^7}+\frac{\gamma k^4}{48}\frac{\partial^7 u}{\partial x^3\partial t^4}+\cdots\end{array}\right].$$

This means that Crank-Nicolson scheme is second order accuracy in space and second order in time, i.e. $O(k^2+h^4)$.

## 4 Numerical tests and results of Hirota equation

In this section, we present numerical example to test validity of our scheme for solving Hirota equation. The norms $L_2$-norm and $L_\infty$-norm are used to compare the numerical solution with the analytical solution [23].

$$\begin{array}{l}L_2=\left\|u^E-u^N\right\|=\sqrt{h\sum_{i=0}^{N}(u_j^E-u_j^N)^2},\\ L_\infty=\max_j\left|u_j^E-u_j^N\right|,j=0,1,\cdots,N.\end{array}\tag{28}$$

Where $u^E$ is the exact solution $u$ and $u^N$ is the approximation solution$U_N$. And the quantities $I_1$ and $I_2$ are shown to measure conservation for the schemes.

$$\left.\begin{array}{l}I_1=\int_{-\infty}^{\infty}|u(x,t)|^2\,dx\cong h\sum_{j=0}^{N}\left|U_j^n\right|^2,\\ I_2=\int_{-\infty}^{\infty}(\frac{\alpha}{2}|u(x,t)|^4-|u_x(x,t)|^2)dx\cong h\sum_{j=0}^{N}(\frac{\alpha}{2}(|U|^4)_j^n-(|U_x|^2)_j^n)\end{array}\right\},\tag{29}$$

where $u(x,t)=u_1(x,t)+iu_2(x,t),(U)_j^n=(U_1)_j^n+i(U_2)_j^n$. Now we consider this test problem.

## Test problem

We assume that the solution of the Hirota equation is negligible outside the interval $[x_L,x_R]$, together with all its $x$ derivatives tend to zero at the boundaries. Therefore, in our numerical study we replace Eq. (1) by

$$u_t+3\alpha|u|^2u_x+\gamma u_{xxx}=0,x_L<x<x_R.\tag{30}$$

Where $u$ is a complex valued function of the spatial coordinate $x$and the time $t$ and $\alpha$, $\gamma$ are positive real constants. Boundary conditions

$$\begin{array}{l}u(x_L,t)=u(x_R,t)=0\\ u_x(x_L,t)=u_x(x_R,t)=0,0\le t\le T.\end{array}\tag{31}$$

And initial conditions.

$$u(x,0)=f(x),x_L<x<x_R.\tag{32}$$

For our numerical work, we decompose the complex function $u$ into their real and imaginary parts by writing

$$u(x,t) = u_1(x,t) + iu_2(x,t), \; i^2 = -1, \tag{33}$$

where $u_1(x,t)$and$u_2(x,t)$are real functions. This will reduce Hirota equation to the coupled system

$$\begin{aligned} (u_1)_t + 3\alpha\left(u_1^2+u_2^2\right)(u_1)_x + \gamma(u_1)_{xxx} = 0, \\ (u_2)_t + 3\alpha\left(u_1^2+u_2^2\right)(u_2)_x + \gamma(u_2)_{xxx} = 0. \end{aligned} \tag{34}$$

Then the exact solutions of system (34) is

$$\begin{aligned} &u_1(x,t) = \beta \operatorname{sec}h\left[\kappa(x-s-vt)\right]\cos(\varphi), \\ &u_2(x,t) = \beta \operatorname{sec}h\left[\kappa(x-s-vt)\right]\sin(\varphi), \\ &\beta = \sqrt{\tfrac{2\gamma}{\alpha}}\kappa, \varphi = a(x-bt-s), \\ &v = \gamma\left(\kappa^2-3a^2\right), b = \gamma\left(3\kappa^3-a^2\right), \end{aligned} \tag{35}$$

$\beta$ is the amplitude of the wave, $\kappa$ is related to the width of the wave envelope and $v$ is the velocity. The parameter $a$ is the wave number of the phase and $b$ is related to the frequency of the phase. Also the solution is at $x = s$ at $t = 0$. In order to derive a numerical method for solving the system given in (34). The region $R = [x_L < x < x_R] \times [t > 0]$with its boundary consisting of the ordinates $x_0 = x_L, x_N = x_R$ and the axis $t = 0$ is covered with a rectangular mesh of points with coordinates

$$\begin{aligned} &x = x_j = x_L + jh, j = 0,1,2,\ldots,N. \\ &t = t_n = nk, n = 0,1,2,\ldots. \end{aligned}$$

where $h$ and $k$ are the space and time increments, respectively. To investigate the performance of the proposed schemes we consider solving the following problem.

### *4.1 Single soliton*

In previous section, we have provided four finite difference schemes for the Hirota equation, and we can take the following as an initial condition.

$$\begin{aligned} &u(x,0) = \beta \operatorname{sec}h\left[\kappa(x-s)\right]\exp(i\varphi), \\ &\beta = \sqrt{\tfrac{2\gamma}{\alpha}}\kappa, \varphi = a(x-bt-s), \\ &v = \gamma\left(\kappa^2-3a^2\right), b = \gamma\left(3\kappa^3-a^2\right). \end{aligned} \tag{36}$$

The norms $L_2$ and $L_\infty$ are used to compare the numerical results with the analytical values and the quantities $I_1$ and $I_2$ are shown to measure conservation for the schemes.

Now, we consider two different cases to study the motion of single soliton.

**Case 1.** Now, for comparison, we consider a test problem where, $k = 0.05$, $\alpha = 2, \gamma = 1$, $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$. The simulations are done up to$t = 5$. The invariant $I_1$ approach to zero and $I_2$ changed by less than $2.76\times10^{-4}$ in the computer program for the first scheme. The invariant $I_1$ approach to zero and $I_2$ changed by less than $8.53\times10^{-4}$ for the second scheme. The invariant $I_1$ approach to zero and $I_2$ changed by less than $3.46\times10^{-4}$for third scheme. The invariants $I_1$ and $I_2$are approach to zero for fourth scheme. Errors, also, at time 5 are satisfactorily small $L_2$-error $=8.07047 \times10^{-4}$and $L_\infty$-error $= 3.61238 \times10^{-4}$ for the first scheme, are satisfactorily small $L_2$-error $=1.39931\times10^{-3}$ and $L_\infty$-error $= 5.54282 \times10^{-4}$ for the second scheme. And are satisfactorily small $L_2$-error $= 5.74713\times10^{-4}$and $L_\infty$-error $=2.72002\times10^{-4}$for third scheme. And are satisfactorily small $L_2$-error $= 1.42121\times10^{-5}$and $L_\infty$-error $=7.83610\times10^{-6}$for fourth scheme. Our results are recorded in Table 1. The motion of solitary wave using fourth scheme is plotted at times$t = 0,3,5$ in Fig.1. These results illustrate that the fourth scheme has a highest accuracy and best conservation than other three schemes. So we use it to study the motion of single solitary waves and interaction between two and three solitons.

**Table 1:** Invariants and errors for single solitary wave $k = 0.05, \alpha = 2, \gamma = 1, a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$.

| **Schemes** | T | $I_1$ | $I_2$ | $L_2 - norm$ | $L_\infty - norm$ |
|---|---|---|---|---|---|
| First Scheme $h = 0.05$ | 0.0 | 1.0000 | -0.166526 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.166205 | 1.40351E-4 | 7.23214E-5 |
| | 2.0 | 1.0000 | -0.166102 | 2.90824E-4 | 1.09485E-4 |
| | 3.0 | 0.9999 | -0.166989 | 4.71530E-4 | 1.74162E-4 |
| | 4.0 | 0.9999 | -0.166885 | 6.44279E-4 | 3.09531E-4 |
| | 5.0 | 0.9999 | -0.166802 | 8.07047E-4 | 3.61238E-4 |
| Second scheme $h = 0.05$ | 0.0 | 1.0000 | -0.166649 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.166813 | 3.21264E-4 | 1.40132E-4 |
| | 2.0 | 1.0000 | -0.167015 | 5.94272E-4 | 2.46535E-4 |
| | 3.0 | 1.0000 | -0.167209 | 8.53845E-4 | 4.39482E-4 |
| | 4.0 | 1.0000 | -0.167378 | 1.14821E-3 | 4.59472E-4 |
| | 5.0 | 1.0000 | -0.167502 | 1.39931E-3 | 5.54282E-4 |
| Third Scheme $h = 0.05$ | 0.0 | 1.0000 | -0.166551 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.166494 | 1.17149E-5 | 6.55307E-5 |
| | 2.0 | 1.0000 | -0.166414 | 2.30553E-4 | 1.23419E-4 |
| | 3.0 | 1.0000 | -0.166332 | 3.48658E-4 | 1.75536E-4 |
| | 4.0 | 1.0000 | -0.166259 | 4.64827E-4 | 2.22999E-4 |
| | 5.0 | 1.0000 | -0.166205 | 5.74713E-4 | 2.72002E-4 |
| fourth scheme $h = 0.1$ | 0.0 | 1.0000 | -0.166648 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.166648 | 4.25079E-6 | 1.58619E-6 |
| | 2.0 | 1.0000 | -0.166646 | 6.74977E-6 | 3.89436E-6 |
| | 3.0 | 1.0000 | -0.166644 | 1.07949E-5 | 6.05488E-6 |
| | 4.0 | 1.0000 | -0.166641 | 1.68464E-5 | 6.19336E-6 |
| | 5.0 | 1.0000 | -0.166641 | 1.42121E-5 | 7.83610E-6 |

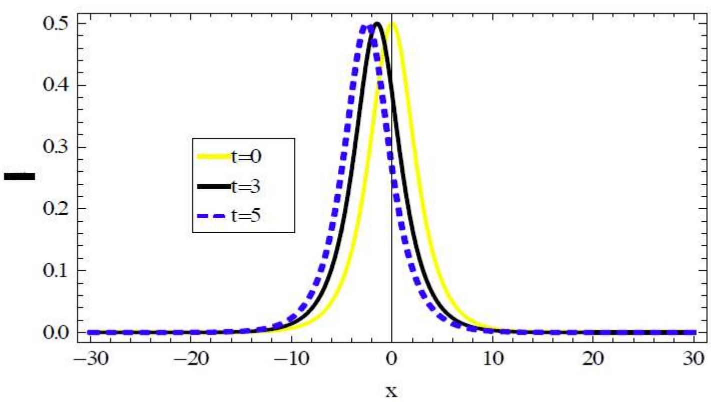


**Fig. 1:** Single solitary wave with $h = 0.1, k = 0.05, \alpha = 2, \gamma = 1$, and $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30, t = 0, 3, 5$ respectively.

**Case 2.** In this case we study the motion of single soliton by fourth scheme. In this case, we choose $k = 0.1, \alpha = 2, \gamma = 1, a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$, with different values of $h = 0.05, 0.1$ the simulations are done up to $t = 5$. The invariants $I_1$ and $I_2$approach to zero percent, respectively at $h = 0.05, 0.5$. Errors, also, are satisfactorily small $L_2$-error $=1.31781 \times 10^{-3}$ and $L_\infty$-error $=6.24576 \times 10^{-4}$, percent, respectively at$h = 0.05$. The invariants $I_1$ and $I_2$approach to zero percent, respectively. Errors, also, are satisfactorily small $L_2$-error =5.63371 $\times 10^{-3}$and $L_\infty$-error =2.68608 $\times 10^{-3}$, percent, respectively at$h = 0.1$. Our results are recorded in Table 2 and the motion of solitary wave is plotted at different time levels in Fig 2.

**Table 2:** Invariants and errors for single solitary wave by fourth scheme$k = 0.1, \alpha = 2, \gamma = 1, a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$.

| $h$ | T | $I_1$ | $I_2$ | $L_2 - norm$ | $L_\infty - norm$ |
|---|---|---|---|---|---|
| $h = 0.05$ | 0.0 | 1.0000 | -0.166597 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.166757 | 3.08344E-4 | 1.77451E-4 |
| | 2.0 | 1.0000 | -0.166944 | 5.62022E-4 | 2.93477E-4 |
| | 3.0 | 1.0000 | -0.167123 | 8.24336E-4 | 4.17184E-4 |
| | 4.0 | 1.0000 | -0.167276 | 1.08278E-3 | 5.28448E-4 |
| | 5.0 | 1.0000 | -0.167389 | 1.31781E-3 | 6.24576E-4 |
| $h = 0.1$ | 0.0 | 1.0000 | -0.166605 | 0.0 | 0.0 |
| | 1.0 | 1.0000 | -0.167278 | 1.28535E-3 | 7.41369E-4 |
| | 2.0 | 1.0000 | -0.168093 | 2.37453E-3 | 1.26919E-3 |
| | 3.0 | 1.0000 | -0.168888 | 3.50933E-3 | 1.83964E-3 |
| | 4.0 | 1.0000 | -0.169562 | 4.60728E-3 | 2.32914E-3 |
| | 5.0 | 1.0000 | -0.170062 | 5.63371E-3 | 2.68608E-3 |

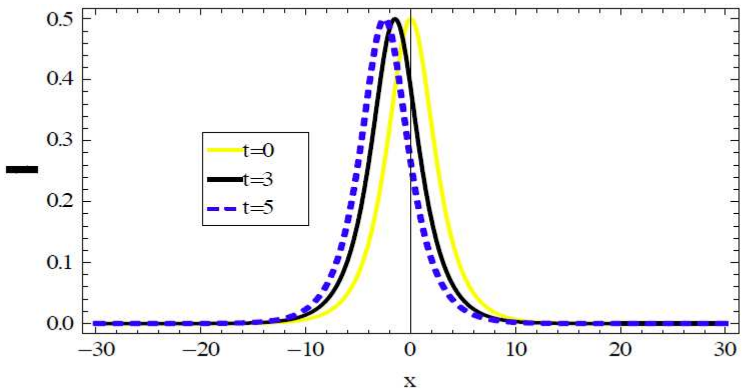


**Fig. 2:** Single solitary wave by fourth scheme with $h = 0.05, k = 0.1, \alpha = 2, \gamma = 1$, and $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30$, $t = 0, 3, 5$ respectively.

In the next table we make comparison between the results of fourth scheme and the results have been published in Search [2]. The results of our scheme are accurate than the results in [2].

**Table 3:** Invariants and errors for single solitary wave $k = 0.05, \alpha = 2, \gamma = 1$, $a = 0.5, \kappa = 0.5, x_L = -30, x_R = 30, t = 4$.

| **Method** | $I_1$ | $I_2$ | $L_2 - norm$ | $L_\infty - norm$ |
|---|---|---|---|---|
| **Analytical**$h = 0.1$ | 1.0000 | -0.166648 | 0.0 | 0.0 |
| **Our scheme**$h = 0.1$ | 1.0000 | -0.166648 | 1.68464E-5 | 0.000006 |
| **[2]**a $h = 0.05$ | 1.0000 | -0.166648 | - | 0.00001 |
| **[2]**b $h = 0.05$ | 1.0000 | -0.166648 | - | 0.00001 |
| **[2]**c $h = 0.05$ | 1.0000 | -0.166648 | - | 0.00014 |

### 4.2 Interaction of two solitary waves

The interaction of two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider Hipota equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$\begin{aligned} &u(x,0) = \beta \sec h\left[\kappa_j\left(x-s_j\right)\right]\exp\left(i\varphi\right), \\ &\beta = \sqrt{\tfrac{2\gamma}{\alpha}}\kappa_j, \varphi = a_j\left(x-bt-s_j\right), \\ &\nu = \gamma\left(\kappa_j^2-3a_j^2\right), b = \gamma\left(3\kappa_j^3-a_j^2\right). \end{aligned} \tag{37}$$

where, $j = 1,2,\ \ a_j$ and $s_j, \kappa_j$are arbitrary constants. In our computational work.

Now, we choose $s_1 = -10, s_2 = 10, a_1 = 0.1, a_2 = 1, \kappa_1 = 0.4, \kappa_2 = 0.7, \gamma = 1,\ \alpha = 2,\ h = 0.05, k = 0.05$with interval [-30, 30]. In Fig. 3, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the two invariants for this case are shown in Table 4. The invariants $I_1$ and $I_2$are changed by less than$2.3\times10^{-4}$and$3\times10^{-4}$, respectively for the scheme.

**Table 4:** Invariants of interaction two solitary waves of Hirota equation $s_1 = -10, s_2 = 10,\ a_1 = 0.1,\ a_2 = 1, \kappa_1 = 0.4, \kappa_2 = 0.7, \gamma = 1,\ \alpha = 2, -30 \le x \le 30,\ h = 0.05, k = 0.05$.

| T | $I_1$ | $I_2$ |
|---|---|---|
| 0 | 2.19957 | -0.539631 |
| 2 | 2.19957 | -0.539631 |
| 4 | 2.19954 | -0.539628 |
| 6 | 2.19951 | -0.539629 |
| 8 | 2.19924 | -0.539627 |
| 10 | 2.19934 | -0.539601 |

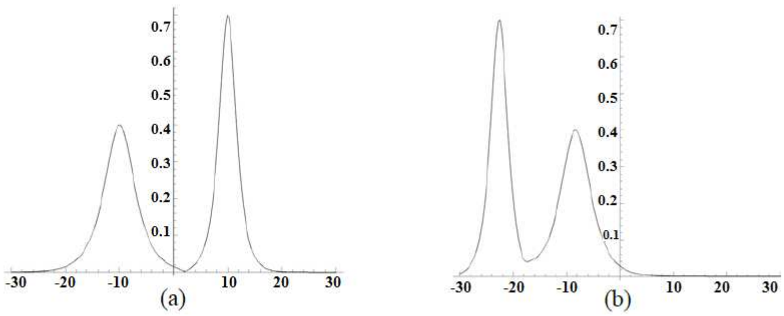


**Fig. 3:** interaction two solitary waves with $s_1 = -10, s_2 = 10, a_1 = 0.1, a_2 = 1, \kappa_1 = 0.4, \kappa_2 = 0.7, \gamma = 1,\ \alpha = 2, h = 0.05, k = 0.05, -30 \le x \le 30$ at time $t = 0, 14$ respectively.

### 4.3 Interaction of three solitary waves

The interaction of three solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the Hirota equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes.

$$\begin{aligned} &u(x,0) = \beta \sec h\left[\kappa_j\left(x-s_j\right)\right]\exp\left(i\varphi\right), \\ &\beta = \sqrt{\tfrac{2\gamma}{\alpha}}\kappa_j, \varphi = a_j\left(x-bt-s_j\right), \\ &\nu = \gamma\left(\kappa_j^2-3a_j^2\right), b = \gamma\left(3\kappa_j^3-a_j^2\right), \end{aligned} \tag{38}$$

where, $j = 1,2,\ 3,\ a_j$ and $s_j, \kappa_j$ are arbitrary constants. In our computational work. Now, we choose $s_1 = -5, s_2 = 5, s_3 = 15, a_1 = 0.1, a_2 = 0.5, a_3 = 1, \kappa_1 = 0.3, \kappa_2 = 0.4, \kappa_3 = 0.8, \gamma = 1,\ \alpha = 2,\ h = 0.05,\ k = 0.05$ with interval [-30, 30]. In Fig. 4. The interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the two invariants for this case are shown in Table 5. The invariants $I_1$ and $I_2$ are changed by less than $1\times10^{-5}$ and $2.56\times10^{-4}$, respectively for the scheme.

**Table 5:** Invariants of interaction three solitary waves of Hirota equation. $s_1 = -5, s_2 = 5, s_3 = 15, a_1 = 0.1, a_2 = 0.5, a_3 = 1, \kappa_1 = 0.3, \kappa_2 = 0.4, \kappa_3 = 0.8,\ \gamma = 1, \alpha = 2, h = 0.05, k = 0.05, -30 \le x \le 30$.

| T | $I_1$ | $I_2$ |
|---|---|---|
| 0 | 2.98713 | -0.776279 |
| 2 | 2.98701 | -0.776245 |
| 4 | 2.98645 | -0.776211 |
| 6 | 2.98624 | -0.776198 |
| 8 | 2.98589 | -0.776011 |
| 10 | 2.98512 | -0.776023 |

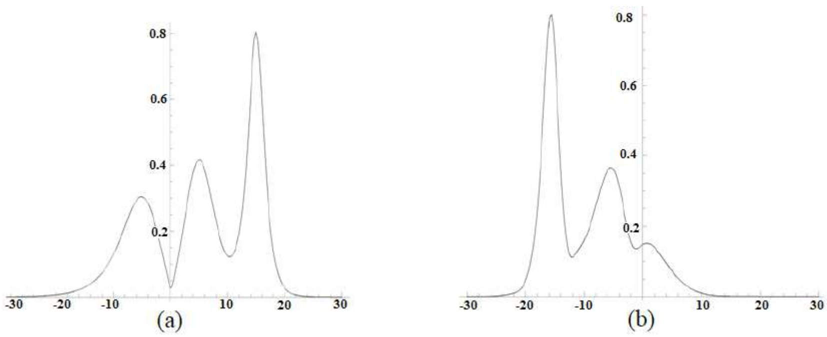


**Fig. 4:** interaction three solitary waves with$s_1 = -5, s_2 = 5, s_3 = 15, a_1 = 0.1, a_2 = 0.5, a_3 = 1, \kappa_1 = 0.3, \kappa_2 = 0.4, \kappa_3 = 0.8, \gamma = 1,\ \alpha = 2, h = 0.05, k = 0.05, -30 \le x \le 30$ at times $t = 0$, 14 respectively.

## 5 Conclusions

In this paper, we applied the finite difference method with different high order approximations to develop a numerical method for solving Hirota equation and shown that the schemes are unconditionally stable. We tested our schemes through a single solitary wave in which the analytic solution is known, then extend it to study the interaction of solitons where no analytic solution is known during the interaction and its accuracy was shown by calculating error norms $L_2$ and $L_\infty$.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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