Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 561 – 569

Edelstein-type fixed point theorems in compact TVS-cone metric spaces

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Abstract

In this paper we prove two fixed point theorems in compact cone metric spaces over normal cones. The first theorem generalizes Edelstein theorem [8] and is different from the generalization obtained in [11]. The second theorem generalizes the main result in [10] and the first theorem. However, the two theorems fail in different categories. Moreover, different versions of the two theorems are proved in TVS-cone metric spaces by making use of the nonlinear scalarization function used very recently by Wei-Shih Du in [A note on cone metric fixed point theory and its equivalence, Nonlinear Analysis,72(5),2259-2261 (2010).] to prove the equivalence of the Banach contraction principle in cone metric spaces and usual metric spaces.

Received 01/07/2010 : Accepted 18/06/2013

2000 AMS Classification: 34A30, 34G10.

Keywords: cone metric, normal cone, strictly normal cone, fixed point, contractive condition. TVS-cone metric space.

1. Introduction and Preliminaries

There are thousands of fixed points theorems in complete or compact metric spaces. However, these theorems can be categorized into the following four types. Let $T : (X, d) \to (X, d)$ be a mapping on a metric space not assumed to be convex or normed, then

(T1) Leader-type [1]: The mapping T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator as in [2].

(T2) Unnamed-type: The mapping T has a unique fixed point and $\{T^n x\}$ does not necessary converge to the fixed point.

(T3) Subrahmanyam-type [3]: The mapping T may have more than one fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator as in [4] and [5].

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(T4) Caristi-type ([6], [7]): The mapping T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

The following Edelstein fixed point theorem [8] belongs to (T1) because it can be proved using Meir-Keeler theorem [9].

1.1. Theorem. *Let (X,d) be a compact metric space and let T be a mapping on X. Assume $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point.*

Motivated by the above, T. Suzuki in [10] generalized Theorem 1.1 by a theorem belongs to (T2). Indeed, he obtained the following theorem

1.2. Theorem. *Let (X,d) be a compact metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) < d(x,y) \quad \text{for } x,y \in X. \tag{1.1}$$

Then T has a unique fixed point.

Recently, and in [11], the authors there rather implied that metric spaces do not provide enough space for fixed point theory. They introduced the notion of cone metric space, where they gave an example of a function which is contraction in the category of cone metric spaces but not contraction if considered over metric spaces and hence, by proving a fixed point theorem in cone metric spaces, ensured that this map must have a unique fixed point. They also there generalized Theorem 1.1 to compact cone metric spaces over regular cones. Namely, they proved

1.3. Theorem. *Let (X,d) be a compact cone metric space over a regular cone and let T be a mapping on X. Assume $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \neq y$. Then T has a unique fixed point.*

After that series of articles about cone metric spaces started to appear. Some of those articles dealt with fixed point theorems in those spaces and some other with the structure of the spaces themselves. For example we name [12], [13], [14], [15], [16], [17], [18], [19] and [20]. For more recent articles about cone metric spaces we refer to [26, 27, 28, 29, 30].

In this article, we shall generalize Theorem 1.2 to compact cone metric spaces over normal cones. Also, recalling that there exist normal cones with constant 1 which are not regular, we shall prove another generalization to Theorem 1.3. Moreover, by the help of the nonlinear scalarization, we prove other different versions of Edelstein fixed point theorem in TVS-cone metric spaces without any normality or regularity assumptions.

Throughout this paper, (E,S) stands for real Hausdorff locally convex topological vector space (t.v.s.) with S its generating system of seminorms. A non-empty subset P of E is called cone if $P+P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed and has nonempty interior as well. For a given cone P, one can define a partial ordering (denoted by $\leq$: or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x << y$ will show $y - x \in int(P)$, where $int(P)$ denotes the interior of P. Continuity of the algebric operations in a topological vector space and the properties of the cone imply the relations:

$$int(P) + int(P) \subseteq int(P) \text{ and } \lambda int(P) \subseteq int(P) \ (\lambda > 0).$$

We appeal to these operations in the following.

1.4. Definition. [23] A cone P of a topological vector space (X,τ) is said to be normal whenever τ has a base of zero consisting of $P-$ full sets. Where a subset of A of an order vector space via a cone P is said to be $P-$full if for each $x,y \in A$ we have $\{a \in E : x \leq a \leq y\} \subset A$.

1.5. Theorem. [23] *(a) A cone P of a topological vector space (X, τ) is normal if and only if whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \to 0$, then $x_\alpha \to 0$.*

(b) The cone of an ordered locally convex space (X, τ) is normal if and only if τ is generated by a family of monotone $\tau-$ continuous seminorms. Where a seminorm q on X is called monotone if $q(x) \leq q(y)$ for all $x, y \in X$ with $0 \leq x \leq y$.

1.6. Definition. A cone P of a locally convex topological vector space (E, S) is said to be (sequentially) regular if every sequence in E which is increasing and bounded above must be convergent in (E, S). That is, if $\{a_n\}$ is a sequence in E such that $a_1 \leq a_2 \leq ... \leq a_n \leq ... \leq b$ for some $b \in E$, then there is $a \in E$ such that $q(a_n) \to 0$ as $n \to \infty$, for all $q \in S$. Equivalently, if every sequence in E which is decreasing and bounded below is convergent.

It is well-known that if E is a real Banach space then every (sequentially) regular cone is normal.

In particular, if P is a cone of a real Banach space E, then it is called *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$: $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of P. Also, P is said to be *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. For more details about cones in locally convex topological vector spaces we may refer the reader to [23].

It is worthwhile to mention here, that there are certain real Banach spaces whose positive cones have empty interior such as the sequences spaces l^p, $1 \leq p < \infty$ and the Lebesgue integrable spaces L_p, $1 \leq p < \infty$ [21]. On the other hand the positive cone in the Euclidean space $\mathbb{R}^n$ does not have empty interior. For example in $\mathbb{R}^2$ the interior of the positive cone $P = \{(x, y) : x \geq 0, y \geq 0\}$ is $\{(x, y) : x > 0, y > 0\}$ which is nonempty. For the infinite dimensional case the positive cones of $AM - spaces$ can have nonempty interiors [22]. For the purposes in defining convergence [11] and other topological concepts in cone metric spaces [15], the cones under consideration are always assumed to have nonempty interiors.

The cone $[0, \infty)$ in $(\mathbb{R}, |.|)$ and the cone $P = \{(x, y) : x \geq 0, y \geq 0\}$ in $\mathbb{R}^2$ are normal cones with constant $K = 1$. However, there are examples of non-normal cones.

1.7. Example. [24] Let $E = C^1[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone $P = \{f \in E : f \geq 0\}$.

For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then, $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k + 1$. Since $k\|f\| < \|g\|$, k is not normal constant of P and hence P is a non-normal cone.

Normal cones are very important when some fixed point theorems are generalized to cone metric spaces. Most of normal cones have normal constant 1. Hence, it is still of great interest when some fixed point theorems are generalized to cone metric spaces over normal cones with normal constant 1. Here are some examples of normal cones with normal constant 1.

1.8. Example. Let $E = \mathbb{R}^n$, $n \geq 1$ and $P = \{(x_1, x_2, ..., x_n) : x_i \geq 0, i = 1, 2, ..., n\}$. Then, P is a normal cone with normal constant $K = 1$.

1.9. Example. Let $E = C[0, 1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $K = 1$ which is not regular. This is clear, since the sequence x^n is monotonically decreasing, but not uniformly convergent to 0.

A cone P of real Banach space E is called strictly normal with normal constant K, if K is the least number such that

$$0 \leq a < b \text{ implies } \|a\| < K\|b\|, \text{ for all } a, b \in E.$$

The cone given in Example 1.8 is clearly strictly normal. Also, the positive cones of the L_p-spaces with $1 \leq p < \infty$ are strictly normal with $K = 1$. Actually, the norms of those spaces are strictly monotone ([22]). However, the cone given in Example 1.9 is not strictly normal.

1.10. Definition. (See [25], [30]) For $e \in int(P)$, the nonlinear scalarization function $\xi_e : E \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{t \in \mathbb{R} : y \in tc - P\}, \text{ for all } y \in E.$$

1.11. Lemma. *(See* [25],[30]*) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

(i) $\xi_e(y) \leq t \Leftrightarrow y \in te - P$,
(ii) $\xi_e(y) > t \Leftrightarrow y \notin te - P$,
(iii) $\xi_e(y) \geq t \Leftrightarrow y \notin te - int(P)$,
(iv) $\xi_e(y) < t \Leftrightarrow y \in te - int(P)$,
(v) $\xi_e(y)$ *is positively homogeneous and continuous on* E,
(vi) *if* $y_1 \in y_2 + P$, *then* $\xi_e(y_2) \leq \xi_e(y_1)$,
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, *for all* $y_1, y_2 \in E$,
(viii) *if* $y_1 \in y_2 + int(P)$, *then* $\xi_e(y_2) < \xi_e(y_1)$.

Lemma 1.11 (vi) and Lemma 1.11 (viii) mean that the nonlinear scalarization function ξ_e is monotone and strictly monotone. However, it is not strongly monotone (see [30]).

1.12. Definition. Let X be a non-empty set. and E as usual a Hausdorff locally convex topological space. Suppose a vector-valued function $p : X \times X \to E$ satisfies:

$(M1)$ $0 \leq p(x, y)$ for all $x, y \in X$,
$(M2)$ $p(x, y) = 0$ if and only if $x = y$,
$(M3)$ $p(x, y) = p(y, x)$ for all $x, y \in X$
$(M4)$ $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Note that in [11], the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang [11] is a special case of TVS-CMS.

1.13. Lemma. *(See* [25]*) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ p$ is a metric.*

1.14. Remark. Since a cone metric space (X, d) in the sense of Huang and Zhang [11], is a special case of TVS-CMS, then $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ d$ is also a metric.

1.15. Definition. (See [25]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.

(i) $\{x_n\}_{n=1}^{\infty}$ TVS-cone converges to $x \in X$ whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x) << c$ for all $n \geq M$ and denoted by $cone - \lim_{n\to\infty} x_n = x$ (or $x_n \overset{cone}{\to} x$ as $n \to \infty$),
(ii) $\{x_n\}_{n=1}^{\infty}$ TVS-cone Cauchy sequence in (X, p) whenever for every $0 << c \in E$, there is a natural number M such that $p(x_n, x_m) << c$ for all $n, m \geq M$,

(iii) (X, p) is TVS-cone complete if every sequence TVS-cone Cauchy sequence in X is a TVS-cone convergent.

1.16. Lemma. *(See [25]) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(i) If $\{x_n\}_{n=1}^{\infty}$ converges to x in TVS-CMS (X, p), then $d_p(x_n, x) \to 0$ as $n \to \infty$,
(ii) If $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in TVS-CMS (X, p), then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in usual sense) in (X, d_p),
(iii) If (X, p) is complete TVS-CMS, then (X, d_p) is a complete metric space.

1.17. Remark. (see also [30, 31]) Note that the implications in (i) and (ii) of Lemma 1.16 are also conversely true. Regarding (i) we prove that if $x_n \to x$ in (X, d_p) then $x_n \to x$ in (X, p). To this end, let $c >> 0$ be given, then find $q \in S$ and $\delta > 0$ such that $q(b) < \delta$ implies that $b << c$. Since $\frac{e}{n} \to 0$ in (E, S) find $\epsilon = \frac{1}{n_0}$ such that $\epsilon q(e) = q(\epsilon e) < \delta$ and hence $\epsilon e << c$. Now, find n_0 such that $d_p(x_n, x) = \xi_e \circ p(x_n, x) < \epsilon$ for all $n \geq n_0$. Hence, by Lemma 1.11 (iv) $p(x_n, x) << \epsilon e << c$ for all $n \geq n_0$. The proof of the converse of implication (ii) is similar. Now it is possible to say that (iii) of Lemma 1.16 is immediate from (i) and (ii). This remark clearly is also valid for any topological vector space since neighborhood elements are symmetric.

The following lemma characterizes convergence and Cauchyness in cone metric spaces by means of the scalar norm of the Banach space E under the normality assumption for the cone P.

1.18. Lemma. [11] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then,*
(i) $\{x_n\}$ converges to x if and only if $lim_{n\to\infty} d_s(x_n, x) = 0$,
(ii) $\{x_n\}$ is Cauchy if and only if $lim_{m,n\to\infty} d_s(x_n, x_m) = 0$,
where $d_s(x, y) = \|d(x, y)\|$, for $x, y \in X$.

Regarding the above Lemma, It is always worthwhile to mention that normality is only used in proving the necessity (for example see [24]).

1.19. Lemma. [11] *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $lim_{n\to\infty} x_n = x$ and $lim_{n\to\infty} y_n = y$. Then*

$$lim_{n\to\infty} \|d(x_n, y_n) - d(x, y)\| = 0$$

In case $Int(P) \neq \emptyset$ it was proved in [15] that cone metric spaces are topological spaces and compactness and sequential compactness coincide (see also [15]).

2. The Main Results

2.1. Theorem. *Let (X, d) be a (sequentially) compact cone metric space over a normal cone P with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

(2.1) $\quad d_s(Tx, Ty) < d_s(x, y),$ *for all* $x, y \in X,\ x \neq y.$

Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0,\ x_2 = Tx_2, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$ If for some n, $x_{n+1} = x_n$, then $\{x_n\}$ is a fixed point of T, the proof is complete. So we assume that for all n, $x_{n+1} \neq x_n$. Set $d_n = d_s(x_n, x_{n+1})$, then the contractive condition (2.1) implies

$$d_{n+1} = d_s(x_{n+1}, x_{n+2}) = d_s(Tx_n, Tx_{n+1}) < d_s(x_n, x_{n+1}).$$

Therefore, the sequence $\{d_n\}$ is a decreasing sequence of positive real numbers and hence convergent. Assume $d_n \to d^*$. From the compactness of X, we may assume that $x_n \to x^*$ in (X, d). Then, by the contractive condition (2.1) we have $Tx_n \to Tx^*$ and $T^2x_n \to T^2x^*$. By using Lemma 1.19 and uniform continuity of the norm $\|.\|$, we have $d_s(Tx_n, x_n) \to d_s(Tx^*, x^*)$ and $d_s(T^2x_n, Tx_n) \to d_s(T^2x^*, Tx^*)$. Hence, we have $d_s(Tx_n, x_n) \to d^* = d_s(Tx^*, x^*)$. Finally, we show that $Tx^* = x^*$. If not, then $d^* \neq 0$ and hence

$$d^* = d_s(Tx^*, x^*) > d_s(T^2x^*, Tx^*) = \lim_{n\to\infty} d_s(T^2x_n, Tx_n) = \lim_{n\to\infty} d_{n+1} = d^*.$$

Which is a contradiction, so $Tx^* = x^*$. The uniqueness of the fixed point is obvious from the contractive condition (2.1). □

It is desired to mention that Theorem 2.1 above can be applied for cone metric spaces over the cone defined in Example 1.9. However, Theorem 2 in [11] can not be applied for such kind of cone metric spaces.

2.2. Remark. If the cone P is strictly normal with constant $K = 1$ then the contractive condition (2.1) can be replaced by

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, \ x \neq y.$$

In this case the proof of Theorem 2.1 can be easily also achieved by applying Edelstein Theorem to the metric space (X, d_s).

2.3. Theorem. *Let (X, d) be a (sequentially) compact cone metric space over a normal cone P with normal constant $K \geq 1$ and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_s(x, Tx) < Kd_s(x, y) \text{ implies } d_s(Tx, Ty) < d_s(x, y) \text{ for } x, y \in X. \tag{2.2}$$

Then, T has a unique fixed point.

Proof. Set $\alpha = \inf\{d_s(x, Tx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha$. Since X is compact, without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements v and w, respectively. We shall show that $\alpha = 0$. If not then $\alpha > 0$. By Lemma 1.19 we have

$$\lim_{n\to\infty} d_s(x_n, w) = d_s(v, w) = \lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha$$

We can choose $n_0 \in \mathbb{N}$ such that

$$\frac{2}{3}\alpha < d_s(x_n, w) \text{ and } d_s(x_n, Tx_n) < \frac{4}{3}\alpha$$

for all $n \geq n_0$. Thus $\frac{1}{2}d_s(x_n, Tx_n) < d_s(x_n, w)$ for $n \geq n_0$. By (3.4), $d_s(Tx_n, Tw) < d_s(x_n, w)$ holds for $n \geq n_0$. This implies, again by means of Lemma 1.19, that

$$d_s(w, Tw) = \lim_{n\to\infty} d_s(Tx_n, Tw) \leq \lim_{n\to\infty} d_s(x_n, w) = \alpha.$$

From the definition of α, we obtain $d_s(w, Tw) = \alpha$. Since $\frac{1}{2}d_s(w, Tw) < d_s(w, Tw)$, then by (3.4), we have

$$d_s(Tw, T^2w) < d_s(w, Tw) = \alpha.$$

Which contradicts the definition of α. Therefore, we must have $\alpha = 0$. We next show that T has a fixed point. Arguing by contradiction, we assume that T does not have a fixed point. Hence, $0 < \frac{1}{2}d_s(x_n, Tx_n) < d_s(x_n, Tx_n)$ for all $n \in \mathbb{N}$ and so by (3.4) we have

$$d_s(Tx_n, T^2x_n) < d_s(x_n, Tx_n) \tag{2.3}$$

for all $n \in \mathbb{N}$. Also by Lemma 1.19 we have

$$\lim_{n\to\infty} d_s(v, Tx_n) = d_s(v, w) = \lim_{n\to\infty} d_s(x_n, Tx_n) = \alpha = 0,$$

which implies that $Tx_n \to v$. We also have

$$\lim_{n\to\infty} d_s(v, T^2x_n) \leq$$

$$K \lim_{n\to\infty}(d_s(v, Tx_n) + d_s(Tx_n, T^2x_n)) \leq K \lim_{n\to\infty}(d_s(v, Tx_n) + d_s(x_n, Tx_n)) = 0. \tag{2.4}$$

Thus, $T^2x_n \to v$. If we accept

$$\frac{1}{2}d_s(x_n, Tx_n) \geq Kd_s(x_n, v) \text{ and } \frac{1}{2}d_s(Tx_n, T^2x_n) \geq Kd_s(Tx_n, v),$$

then by the triangle inequality and (2.3) we have

$$d_s(x_n, Tx_n) \leq Kd_s(x_n, v) + Kd_s(Tx_n, v) \leq$$

$$\frac{1}{2}d_s(x_n, Tx_n) + \frac{1}{2}d_s(Tx_n, T^2x_n) < \frac{1}{2}d_s(x_n, Tx_n) + \frac{1}{2}d_s(x_n, Tx_n) = d_s(x_n, Tx_n), \tag{2.5}$$

which is a contradiction. Hence for every $n \in \mathbb{N}$, either

$$\frac{1}{2}d_s(x_n, Tx_n) < Kd_s(x_n, v) \text{ or } \frac{1}{2}d_s(Tx_n, T^2x_n) < Kd_s(Tx_n, v)$$

holds. Then by assumption (3.4), either

$$d_s(Tx_n, Tv) < d_s(x_n, v) \text{ or } d_s(T^2x_n, Tv) < d_s(Tx_n, v)$$

holds. Hence one of the following holds:

(i) There exists an infinite subset I of $\mathbb{N}$ such that

$$d_s(Tx_n, Tv) < d_s(x_n, v) \text{ for all } n \in I.$$

(ii) There exists an infinite subset J of $\mathbb{N}$ such that

$$d_s(T^2x_n, Tv) < d_s(Tx_n, v) \text{ for all } n \in J.$$

In the first case, we obtain

$$d_s(v, Tv) = \lim_{n\in I, n\to\infty} d_s(Tx_n, Tv) \leq \lim n \in I, n \to \infty d_s(x_n, v) = 0,$$

which implies that $Tv = v$. Also, in the second case, we obtain

$$d_s(v, Tv) = \lim_{n\in J, n\to\infty} d_s(T^2x_n, Tv) \leq \lim_{n\in J, n\to\infty} d_s(Tx_n, v) = 0,$$

which implies that $Tv = v$. That is v is a fixed point in both cases. This is a contradiction. Therefore there exists $z \in X$ such that $Tz = z$. To prove uniqueness, fix $y \in X$ with $y \neq z$. Then since $\frac{1}{2}d_s(z, Tz) = 0 < Kd_s(z, y)$, we have

$$d_s(z, Ty) = d_s(Tz, Ty) < d_s(z, y)$$

and hence y is not a fixed point. □

Note that also, if in Theorem 2.3 the cone P is strictly normal with $K = 1$ then the assumption (3.4) can be replaced by the condition

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) \quad \text{for } x, y \in X.$$

As a matter of great interest in Theorem 2.3, is that the author in [10] proved that the constant $\frac{1}{2}$ is the best constant. Namely, he proved that for every $\eta \in (\frac{1}{2}, \infty)$, there exists a compact metric space (X, d) and a mapping T on X satisfying the following:

(i) T has no fixed point.

(ii) $\eta d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x,\ y \in X$.

3. Other Edelstein -type fixed point theorem in TVS-cone metric spaces

3.1. Lemma. *If (X, p) is a (sequentially) compact TVS-metric space then (X, d_p) is a (sequentially) compact metric space, where $d_p = \xi_e \circ p$.*

The proof followed by Lemma 1.13 , Lemma 1.16 (ii).

3.2. Theorem. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_p(x, Tx) < d_p(x, y) \text{ implies } d(Tx, Ty) \ll d(x, y) \quad \text{for } x, y \in X. \tag{3.1}$$

Then, T has a unique fixed point.

The proof followed by Lemma 1.11 (viii), Lemma 3.1 and the classical Edelstein fixed point theorem.

3.3. Corollary. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$d(Tx, Ty) \ll d(x, y) \quad \text{for } x, y \in X. \tag{3.2}$$

Then, T has a unique fixed point.

Note that no regularity assumption is not required in Lemma 3.1 and hence in Corollary 3.3. Therefore, Corollary 3.3 improves Theorem 2 in [11] by replacing " $<$ " by " $\ll$ " and removing regularity.

3.4. Theorem. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d_p(x, Tx) < d_p(x, y) \text{ implies } d_p(Tx, Ty) < d_p(x, y) \quad \text{for } x, y \in X. \tag{3.3}$$

Then, T has a unique fixed point.

The proof followed by Lemma 3.1 and the classical Edelstein fixed point theorem.

3.5. Corollary. *Let (X, p) be a (sequentially) compact TVS-cone metric space and let T be a mapping on X. Assume that*

$$d_p(Tx, Ty) < d_p(x, y) \quad \text{for } x, y \in X. \tag{3.4}$$

Then, T has a unique fixed point.

Note that no normality assumptions are needed in Theorem 3.4 and Corollary 3.5.

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