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On multiplication lattice modules

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Abstract

In this paper we study multiplication lattice modules. Next we characterize hollow lattices modules. We also establish maximal elements in multiplication lattices modules.In [16], we introduced the concept of a multiplication lattice L-module and we characterized it by principal elements.In this paper, we continue study on multiplication lattice L-module.

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1.

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero) . Multiplicative lattices have been studied extensively by E.W.Johnson, C.Jayaram, the current authors, and others, see, for example, $[1-16]$.

An element $a \in L$ is said to be proper if $a < 1$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1$ in L is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. It is easily seen that maximal elements are prime.

If a, b belong to L, $(a :_L b)$ is the join of all $c \in L$ such that $cb \leq a$. An element e of L is called meet principal if $a \bigwedge be = ((a :_L e) \bigwedge b)\, e$ for all $a, b \in L$. An element e of L is called join principal if $((ae \bigvee b) :_L e) = a \bigvee (b :_L e)$ for all $a, b \in L$. $e \in L$ is said to be principal if e is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \bigwedge e = e\,(a :_L e)$ $(a \bigvee (0_L :_L e) = (ea :_L e))$ for all $a \in L$. An element a of a multiplicative lattice L is called compact if $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \bigvee b_{\alpha_2} \bigvee ... \bigvee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. If each element of L is a join of principal (compact) elements of L, then L is called a $PG-$lattice $(CG-\text{lattice})$.

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module in the case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties :

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(i): $(lb)\,B = l\,(bB)$;

(ii): $\left(\bigvee_{\alpha} l_{\alpha}\right)\left(\bigvee_{\beta} B_{\beta}\right) = \bigvee_{\alpha,\beta} l_{\alpha} B_{\beta}$;

(iii): $1_L B = B$;

(iv): $0_L B = 0_M$ for all l, l_{α}, b in L and for all B, B_{β} in M.

Let M be an L-module. If N, K belong to M, $(N :_L K)$ is the join of all $a \in L$ such that $aK \leq N$. If $a \in L$, $(0_L :_M a)$ is the join of all $H \in M$ such that $aH = 0_M$. An element N of M is called meet principal if $(b \bigwedge (B :_L N))\, N = bN \bigwedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called join principal if $b \bigvee (B :_L N) = ((bN \bigvee B) :_L N)$ for all $b \in L$ and for all $N \in M$. N is said to be principal if it is both meet principal and join principal. In special case an element N of M is called weak meet principal (weak join principal) if $(B :_L N)\, N = B \bigwedge N$ $((bN :_L N) = b \bigvee (0_M :_L N))$ for all $B \in M$ and for all $b \in L$. N is said to be weak principal if N is both weak meet principal and weak join principal.

Let M be an L-module. An element N in M is called compact if $N \leq \bigvee_{\alpha} B_{\alpha}$ implies $N \leq B_{\alpha_1} \bigvee B_{\alpha_2} \bigvee ... \bigvee B_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. The greatest element of M will be denoted by 1_M. If each element of M is a join of principal (compact) elements of M, then M is called a $PG-$lattice (CG-lattice).

Let L be a multiplicative lattice and let M be an L-module. If M is CG-lattice, then any weak principal element N of M is compact [14, Corollary 2.2]. Especially, if L is a CG-lattice, then any weak principal element in L is compact [14, Corollary 2.3].

Let M be an L-module. An element $N \in M$ is said to be proper if $N < 1_M$. If $ann\,(1_M) = (0_M :_L 1_M) = 0_L$, M is called a faithful L-module. If $cm = 0_M$ implies $m = 0_M$ or $c = 0_L$ for any $c \in L$ and $m \in M$, then M is called a torsion-free L-module.

For various characterizations of lattice modules, the reader is referred to $[9 - 16]$.

H.M.Nakkar and I.A.Al-Khouja [13, 14] studied multiplicative lattice modules over multiplicative lattices. In [16], we introduced the concept of a multiplication lattice L-module and we characterized it by principal elements. In this study, we continue study on multiplication lattice L-module and we prove that many important theorems like Nakayama Lemma. We also prove that if L is a multiplicative $PG-$lattice and M is a multiplication PG- lattice module, then K is maximal element of M if and only if there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$.

1.1. Definition. Let L be a multiplicative lattice and $c \in L$. c is said to be a multiplication element if for every element a of L such that $a \leq c$ there exists an element $d \in L$ such that $a = cd$.

1.2. Definition. Let L be a multiplicative lattice and M a lattice L-module. $N \in M$ is said to be a multiplication element if for every element K of M such that $K \leq N$ there exists an element $a \in L$ such that $K = aN$.

Note that, $a \in L$ is a multiplication element if and only if a is a weak meet principal element in L and $N \in M$ is a multiplication element if and only if N is a weak meet principal element in M. We say that M is a multiplication lattice L-module if 1_M is a multiplication element in M.

1.3. Theorem. *Let L be a $PG-$lattice and M be a PG-lattice $L-$module. Then M is a multiplication lattice $L-$module if and only if for every maximal element $q \in L$,*

(i): *For every principal element $Y \in M$, there exists a principal element $q_Y \in L$ with $q_Y \nleq q$ such that $q_Y Y = 0_M$ or*

(ii): *There exists a principal element $X \in M$ and a principal element $b \in L$ with $b \nleq q$ such that $b1_M \leq X$.*

Proof. [see 16, Theorem 4] . □

1.4. Theorem. *Let L be a PG-lattice, and M be a faithful multiplication PG-lattice L-module. Then the following conditions are equivalent.*

(i): 1_M is a compact element of M.
(ii): If $a, c \in L$ such that $a1_M \leq c1_M$, then $a \leq c$.
(iii): For each element N of M there exists a unique element a of L such that $N = a1_M$.
(iv): $1_M \neq a1_M$ for any proper element a of L.
(v): $1_M \neq p1_M$ for any maximal element p of L.

Proof. [see 16, Theorem 5] . □

1.5. Proposition. *Let L be a $PG-$lattice and M be a faithful multiplication $PG-$lattice L-module such that 1_M compact. If $a \in L$ is a multiplication element, then $a1_M \in M$ is a multiplication element.*

Proof. Let $K \leq a1_M$. Since M is a multiplication module, $K = b1_M$ for some $b \in L$. Then $K = b1_M \leq a1_M$. Since 1_M is compact, $b \leq a$ by Theorem 2 (ii). Since $a \in L$ is a multiplication element, we have $b = ac$ for some $c \in L$ and so $K = b1_M = (ac)\,1_M = c\,(a1_M)$. Consequently, $a1_M$ is a multiplication element. □

1.6. Proposition. *Let L be a $PG-$lattice and M a faithful multiplication $PG-$lattice L-module such that 1_M is compact.*

(i): N is a multiplication element in M if and only if $(K :_L N)(N :_L 1_M) = (K :_L 1_M)$ for all $K \leq N$.
(ii): $a = (a1_M :_L 1_M)$ for all $a \in L$.
(iii): N is a multiplication element in M if and only if $(N :_L 1_M)$ is a multiplication element in L.
(iv): $a1_M$ is a multiplication element in M if and only if a is a multiplication element in L.

Proof. $(i) \Rightarrow$: Let N be a multiplication element in M and $K \leq N$. Then $K = bN$ for some $b \in L$. Since M is a multiplication lattice $L-$module,

$$K = bN = (bN :_L N)\,N = (bN :_L N)\,(N :_L 1_M)\,1_M = (K :_L 1_M)\,1_M.$$

Therefore $(K :_L N)\,(N :_L 1_M) = (K :_L 1_M)$ by Theorem 2 (ii).

$\Leftarrow$: Since

$$K = (K :_L 1_M)\,1_M = (K :_L N)\,(N :_L 1_M)\,1_M = (K :_L N)\,N$$

for all $K \leq N$, N is a multiplication element.

(ii) Since M is a multiplication lattice module,we have $a1_M = (a1_M :_L 1_M)\,1_M$ and so $a = (a1_M :_L 1_M)$ for all $a \in L$ by Theorem 2 (ii).

$(iii) \Rightarrow$: Let N be a multiplication element. If $a \leq (N :_L 1_M)$, then $a = (a1_M :_L 1_M)$ by (ii) and $a = (a1_M :_L 1_M) = (a1_M :_L N)\,(N :_L 1_M)$ by (i) . Therefore, $a = c\,(N :_L 1_M)$ where $c = (a1_M :_L N)$. Then $(N :_L 1_M)$ is a multiplication element in L.

$\Leftarrow$: Let $(N :_L 1_M)$ be a multiplication element in L. Then $(N :_L 1_M)\,1_M = N$ multiplication elemen in M by Proposition 1.

$(iv) \Rightarrow$: Let $N = a1_M$ be a multiplication element in M. Then $(N :_L 1_M) = (a1_M :_L 1_M) = a$ is a multiplication element in L by (iii) .

$\Leftarrow$: Let $a \in L$ be a multiplication element in L. Then $N = a1_M$ is a multiplication element in M by Proposition 1. □

1.7. Proposition. *Let L be a multiplicative lattice and M a multiplication lattice L-module. If L is a Noetherian (Artinian) latice, then M is a Noetherian (Artinian) $L-$module.*

Proof. Suppose that $N_1 \leq N_2 \leq ....$ and L is Noetherian . Then $(N_1 :_L 1_M) \leq (N_2 :_L 1_M) \leq ...$. Since L is Noetherian, there is a positive integer $k > 0$ such that $(N_k :_L 1_M) = (N_{k+1} :_L 1_M) = ...$ and so $(N_k :_L 1_M) 1_M = (N_{k+1} :_L 1_M) 1_M = ...$. Therefore, $N_k = N_{k+1} = ...$. Similarly, if L is Artinian, then M is Artinian. □

1.8. Definition. Let L be a multiplicative lattice and M be a lattice L-module. Let K be a proper element in M. K is said to be a small element if for every element N of M such that $K \vee N = 1_M$ implies $N = 1_M$.

1.9. Definition. Let L be a multiplicative lattice and M be a lattice L-module. If every proper element of M is small, then M is called a hollow $L-$module.

1.10. Theorem. *Let L be a PG-lattice and M be a faithful multiplication PG$-$lattice L-module with 1_M compact. Then M is a hollow $L-$module if and only if L is a hollow L-module.*

Proof. $\Rightarrow$: Suppose that M is hollow. Let $a < 1_L$ such that $a \vee b = 1_L$ for some $b \in L$. Then $(a \vee b) 1_M = a1_M \vee b1_M = 1_M$. Since $a < 1_L$, $a1_M < 1_M$ by Theorem 2. By hypothesis, $b1_M = 1_M$ and hence $b = 1_L$ by Theorem 2 (ii). Therefore a is a small element in L.

$\Leftarrow$: Suppose that L is a hollow L-module. Let $N < 1_M$ and K be a any element in M such that $N \vee K = 1_M$. Since M is a multiplication L-module, we have $N = (N :_L 1_M) 1_M$ and $K = (K :_L 1_M) 1_M$. Then,

$$1_M = N \vee K = (N :_L 1_M) 1_M \vee (K :_L 1_M) 1_M = [(N :_L 1_M) \vee (K :_L 1_M)] 1_M.$$

Therefore, $(N :_L 1_M) \vee (K :_L 1_M) = 1_L$ by Theorem 2 (ii). Since $N = (N :_L 1_M) 1_M < 1_M$, we have $(N :_L 1_M) < 1_L$ and so $(K :_L 1_M) = 1_L$ by hypothesis. This shows that $K = 1_M$. Consequently, M is hollow. □

1.11. Theorem. *Let L be a PG-lattice and M be a faithful multiplication PG$-$lattice L-module with 1_M compact. Then, N is small if and only if there exists a small element $a \in L$ such that $N = a1_M$.*

Proof. $\Rightarrow$: Suppose that $N \in M$ is small and $N = a1_M$ for some proper element a in L. Suppose that $a \vee b = 1_L$ for some $b \in L$. Then

$$N \vee b1_M = a1_M \vee b1_M = (a \vee b) 1_M = 1_M$$

and so $b1_M = 1_M$ by hypothesis. Hence $b = 1_L$ by Theorem 2. This shows that a is small in L.

$\Leftarrow$: Suppose that $a \in L$ is small such that $N = a1_M$. Let $N \bigvee K = a1_M \vee K = 1_M$ for some $K \in M$. Since M is a multiplication L-module, there is an element $b \in L$ such that $K = b1_M$ and hence $(a \vee b) 1_M = a1_M \vee K = 1_M$. Then $a \vee b = 1_L$ by Theorem 2 (ii) and hence $b = 1_L$ by hypothesis. Therefore, $K = 1_M$. This shows that $a1_M$ is small. □

1.12. Definition. Let M be a L-module. An element $N < 1_M$ in M is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ i.e. $a \leq (N :_L 1_M)$ for every $a \in L, X \in M$.

1.13. Definition. Let M be an L-module. M is said to be prime L-module if 0_M is prime element of M.

It is clear that 0_M is prime element in M if and only if $(0_M :_L 1_M) = (0_M :_L N)$ for all $0_M \neq N \in M$.

1.14. Definition. Let M be an L-module. M is said to be coprime L-module if $(0_M :_L 1_M) = (N :_L 1_M)$ for all $N \in M$ such that $N < 1_M$.

Recall that a lattice L-module M is called simple if $M = \{0_M, 1_M\}$.

1.15. Proposition. *If M is a multiplication and coprime L-module, then M is simple.*

Proof. Let $N \in M$ such that $N < 1_M$. Since M is a coprime L-module, we have $(0_M :_L 1_M) = (N :_L 1_M)$. Since M is a multiplication L-module, it follows that $N = (N :_L 1_M) 1_M = (0_M :_L 1_M) 1_M = 0_M$. Then M is simple. □

1.16. Definition. Let L be a $PG-$lattice and M a $PG-$lattice L-module. Let p be a maximal element of L. M is called $p-$torsion provided for each principal element $X \in M$ there exists a principal element $q_X \in L$, $q_X \nleq p$ such that $q_X X = 0_M$.

1.17. Definition. Let L be a $PG-$lattice and M be a $PG-$lattice L-module. M is called p-cyclic provided there exists a principal element $Z \in M$ and a principal element $q \in L$, $q \nleq p$ such that $q1_M \leq Z$.

Let M be an L-module. Let N and K be elements of M such that $N \leq K$. Define $[N, K] = \{A \in M : N \leq A \leq K\}$.Then $[N, K]$ is an L-module. It is clear that N is a multiplication element if and only if $[0_M, N]$ is a multiplication lattice L-module. Recall that $ann(X) = (0_M :_L X)$ for any $X \in M$.

1.18. Theorem. *Let L be a PG-lattice and M be a PG-lattice L-module. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of M such that $N = \bigvee_{\lambda\in\Lambda} N_\lambda$ and $a = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N)$.*

i) N is a multiplication element in M.
ii) $H = aH$ for all elements $H \leq N$.
iii) $1_L = a \bigvee ann(X)$ for all principal elements $X \leq N$.
iv) $(N :_L K) \bigvee ann(X) = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann(X)$ for all principal elements $X \leq N$ and for all elements K in M.

Then, $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. If all N_λ are multiplication elements, then the conditions are equivalent.

Proof. $(i) \Rightarrow (ii)$. Let $a = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N)$. Then $aN = \bigvee_{\lambda\in\Lambda} (N_\lambda : N) N = N$. Since N is a multiplication element, there exist an element $h \in L$ such that $H = hN$ for $H \leq N$. Therefore, $aH = ahN = h(aN) = hN = H$.

$(ii) \Rightarrow (iii)$.Suppose that $1_L \neq a \bigvee ann(X)$ for all principal elements $X \leq N$. There exists a maximal element $p \in L$ such that $a \bigvee ann(X) \leq p$ for each principal element $X \leq N$. Since $X = aX \leq pX \leq X$, we have $X = pX$. Then, $1_L = (pX :_L X) = p \bigvee ann(X)$. Since $ann(X) \leq a \bigvee ann(X) \leq p$, we get a contradiction.

$(iii) \Rightarrow (ii)$. Since $X = (a \bigvee ann(X)) X = aX$ for all principal elements $X \leq N$, it follows that $H = aH$ for every $H \leq N$.

$(iii) \Rightarrow (iv)$. Since $\bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) K \leq \bigvee_{\lambda\in\Lambda} N_\lambda = N$, we have $\bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \leq \left(\bigvee_{\lambda\in\Lambda} N_\lambda :_L K\right) = (N :_L K)$. Therefore, $\bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann(X) \leq (N :_L K) \bigvee ann(X)$ for all principal elements $X \leq N$. Conversely, $w_1$and w_2 be principal elements such that $w_1 \bigvee w_2 \leq (N : K) \bigvee ann(X)$ where $w_1 \leq (N : K)$ and $w_2 \leq ann(X)$. Then $1_L = a \bigvee ann(X) = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N) \bigvee ann(X)$. Hence $w_1 = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N) w_1 \bigvee ann(X) w_1$. Since $w_1 K \leq N$, we have $(N_\lambda :_L N) w_1 K \leq (N_\lambda :_L N) N \leq N_\lambda$ and so $(N_\lambda :_L N) w_1 \leq (N_\lambda :_L K)$. Therefore,

$$w_1 \bigvee w_2 = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N)\, w_1 \bigvee ann\,(X)\, w_1 \bigvee w_2$$
$$\leq \bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann\,(X)\,.$$

Hence, $(N :_L K) \bigvee ann\,(X) \leq \bigvee_{\lambda\in\Lambda} (N_\lambda :_L K) \bigvee ann\,(X)\,.$

$(iv) \Rightarrow (iii)$. If we take $K = N$, then $1_L = (N :_L N) \bigvee ann\,(X) = a \bigvee ann\,(X)$

$(iii) \Rightarrow (i)$. Suppose that $1_L = a \bigvee ann\,(X)$ for all principal elements $X \leq N$. Suppose that N is not p-torsion. There exists a principal element $X \leq N$ such that $ann\,(X) = (0_M :_L X) \leq p$. Since $1_L = a \bigvee ann\,(X)$, we have $a = \bigvee_{\lambda\in\Lambda} (N_\lambda :_L N) \nleq p$. There exists $\lambda \in \Lambda$ such that $(N_\lambda :_L N) \nleq p$. There exists a principal element $b \nleq p$ such that $b \leq (N_\lambda :_L N)$ .Since N_λ is a multiplication element and not $p-$torsion, it follows that N_λ is p-cyclic by Theorem 1. Indeed, if N_λ is a p-torsion, then $cN_\lambda = 0_M$ for some principal elemebt $c \nleq p$ and so $bN \leq N_\lambda$ implies $bcN \leq cN_\lambda = 0_M$. Then $bc \leq (0_M :_L N)$. Therefore $bcY = 0_M$ for all principal elements $Y \leq N$ and principal element $bc \nleq p$. Since N is not $p-$torsion, this is a contradiction. Hence N_λ is p-cyclic. Therefore, $dN_\lambda \leq Y_\lambda$ for some principal element $Y_\lambda \leq N_\lambda$ and principal element $d \nleq p$.Therefore, $bN \leq N_\lambda$ and so $bdN \leq dN_\lambda \leq Y_\lambda$ and $bd \nleq p$. Consequently, N is p-cyclic. □

1.19. Theorem. *(Nakayama Lemma) Let M be a non-zero multiplication $PG-$lattice $L-$module. Let $a \in L$ such that for all maximal element $q \in L$, $a \leq q$. Then $a1_M < 1_M$.*

Proof. Let $a \in L$ such that $a \leq q$ for all maximal element $q \in L$ and suppose that $a1_M = 1_M$. Let consider a principal element $0_M \neq X \in M$.Since M is a multiplication $L-$module, we have $X = b1_M$ for some $b \in L$. Hence $X = b1_M = ab1_M = aX$. Thus $1_L = (aX : X) = a \vee (0_M :_L X)$ for all principal elements $X \in M$. Since $(0_M :_L X) < 1_L$, there exists a maximal element $p \in L$ such that $(0_M :_L X) \leq p$. By hypothesis $a \leq p$, hence $1_L = (aX : X) = a \vee (0_M :_L X) \leq p$ and we obtain a contradiction. □

1.20. Proposition. *Let L be a multiplicative $PG-$lattice. Let M be a multiplication PG-lattice L-module and $(0_M : 1_M) \leq p$ for some prime element $p \in L$. If $a1_M \leq p1_M$ for some $a \in L$, then $a \leq p$ or $p1_M = 1_M$.*

Proof. If $a1_M \leq p1_M$ for some $a \in L$, then $aX \leq p1_M$ for all principal element $X \in M$. Hence $a \leq p$ or $X \leq p1_M$ [see 16, Theorem 6]. If $a \nleq p$, then $X \leq p1_M$ for all principal element $X \in M$, hence it is clear that $p1_M = 1_M$. □

1.21. Proposition. *Let L be a multiplicative $PG-$lattice. Let M be a multiplication PG-lattice L-module. Then K is maximal element of M if and only if there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$.*

Proof. $\Leftarrow$: If there exist a maximal element $p \in L$ such that $K = p1_M < 1_M$, then K is maximal [see16, Proposition 4].

$\Rightarrow$: Let K be a maximal element of M and $(K :_L 1_M) = q$. Since M is a multiplication lattice module, we have $K = q1_M$. We show that $(K :_L 1_M) = q$ is a maximal element of L. If q is not a maximal element, there exists a maximal element p such that $q < p$. Then $p1_M \nleq K$. Indeed, if $p1_M \leq K$, then $p \leq (K :_L 1_M) = q$. This is a contradiction. Therefore, $1_M = K \vee p1_M = (q \vee p)1_M = p1_M$. Hence $X = pX$ for all principal elements X and so $1_L = (pX :_L X) = p \vee (0_M :_L X)$. This implies that $(0_M :_L X) \nleq p$. Therefore, there exists a principal element $p_X \in L$ such that $p_X \leq (0_M :_L X)$ and $p_X \nleq p$. If we take a principal element X such that $X \nleq K$, then $X \vee K = 1_M$. Hence $p_X X \vee p_X K = p_X 1_M$ and so $p_X K = p_X 1_M \leq K$. Therefore, $p_X \leq (K :_L 1_M) = q < p$. This is a contradiction. □

1.22. Theorem. *Let L be a $CG-$lattice and M be a $PG-$lattice $L-$module. Then M is a multiplication lattice $L-$module if and only if for every maximal element $q \in L$,*

(i) For every principal element $Y \in M$,there exists a compact element $q_Y \in L$ with $q_Y \nleq q$ such that $q_Y Y = 0_M$ or

(ii) There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \nleq q$ such that $b1_M \leqslant X$.

Proof. $\Longrightarrow$: Let M be a multiplication lattice $L-$module. We have two cases.

Case 1. Let $q1_M = 1_M$ where q is a maximal element of L. For every principal element $Y \in M$, there exists an element $a \in L$ such that $Y = a1_M$. Then $Y = a1_M = aq1_M = qY$. Therefore, $1_L = (qY :_L Y) = q \vee (0_M :_L Y)$.Hence $(0_M :_L Y) \nleq q$. There exists a compact element q_Y such that $q_Y \leqslant (0_M :_L Y)$ and $q_Y \nleq q$.

Case 2. Let $q1_M < 1_M$.There exists a principal element $X \in M$ such that $X = j1_M \nleq q1_M$, with $j \in L, j \nleq q$.There exists a compact element $b \in L$ with $b \leq j$ and $b \nleq q$.We obtain $b1_M \leq j1_M = X$.

$\Longleftarrow$:Let $N \in M$. Put $a = (N :_L 1_M)$. Clearly $a1_M = (N :_L 1_M)1_M \leq N$. Take any principal element $Y \leq N$. We will show that $(a1_M :_L Y) = 1_L$. Suppose there exists a maximal element $q \in L$ such that $(a1_M :_L Y) \leq q$.We have two case.

Case 1. Suppose that (i) is satisfied. There exists a compact element $q_Y \in L$ with $q_Y \nleq q$ such that $q_Y Y = 0_M$ for every principal element $Y \in M$. Then $q_Y \leqslant (0_M : Y) \leqslant (a1_M : Y) \leq q$.This is a contradiction.

Case 2. Suppose that (ii) is satisfied.There exists a principal element $X \in M$ and a compact element $b \in L$ with $b \nleq q$ such that $b1_M \leqslant X$. Then $bN \leq b1_M \leqslant X$ for any $N \in M$. Since X is a principal element of $M, bN = (bN :_L X)X$. Then $b(bN :_L X)1_M \leq (bN :_L X)X = bN \leq N$ and so $b(bN :_L X) \leq a = (N :_L 1_M)$. Therefore $b^2Y \leq b^2N = b(bN :_L X)X \leq aX \leq a1_M$ imply $b^2 \leq (a1_M :_L Y) \leq q$. Since q is maximal (and so prime) element of L, we have $b \leq q$.This is a contradiction. □

1.23. Definition. Let M be an L-module. An element $N < 1_M$ in M is said to be primary, if $aX \leq N$ and $X \nleq N$ implies $a^k 1_M \leq N$, for some $k \geq 0$ *i.e* $a^k \leq (N : 1_M)$ for every $a \in L, X \in M$.

If a is an element of a multiplicative lattice L, we define $\sqrt{a} = \bigvee \{b \in L : b^n \leq a \text{ for some natural number } n\}$.

1.24. Theorem. *Let L be a CG-lattice and M be a multiplication PG-lattice L-module. Suppose that p is a primary element in L with $(0_M :_L 1_M) \leq p$. If $aX \leq p1_M$, where $a \in L, X \in M$, then $X \leq p1_M$ or $a \leq \sqrt{p}$.*

Proof. We may suppose that X is principal in M. Suppose that $aX \leq p1_M$ with $a \nleq \sqrt{p}$. We will show that $(p1_M :_L X) = 1_L$. Suppose that there exists a maximal element $q \in L$ such that $(p1_M :_L X) \leq q$. By theorem 7, we have two cases.

Case 1. If there exists a compact element $q_x \in L$ with $q_x \nleq q$ such that $0_M = q_x X$, then $q_x \leq (0_M :_L X) \leq (p1_M :_L X) \leq q$. This is a contradiction.

Case 2. If there exists a principal element $Y \in M$ and a compact element $b \in L$ with $b \nleq q$ such that $b1_M \leq Y$, then $bX \leq b1_M \leq Y$. Since Y is principal, we have $bX = (bX :_L Y)Y$. Put $(bX :_L Y) = s$. Then $abX = asY$. Since Y is join principal, it follows that $(asY :_L Y) = as \vee (0_M :_L Y)$. Since Y is meet principal, we have $abX = (abX :_L Y)Y$. Put $c = (abX :_L Y)$. Since $cY = abX \leq bp1_M \leq pY$, it follows that $c \vee (0_M :_L Y) = (cY :_L Y) \leq (pY :_L Y) = p \vee (0_M :_L Y)$. Since $b(0_M :_L Y)1_M = (0_M :_L Y)b1_M \leq (0_M :_L Y)Y = 0_M$, we have $b(0_M :_L Y) \leq (0_M :_L 1_M) \leq p$. Hence $bc \vee b(0_M :_L Y) \leq bp \vee b(0_M :_L Y) \leq p$. Therefore, $bc \leq p$. On the other hand, $c = (abX :_L Y) = (asY :_L Y) = as \vee (0_M :_L Y)$ and so $abs \leq abs \vee b(0_M :_L Y) = bc \leq p$. If $b \leq \sqrt{p}$, since b is compact ,we obtain $b \leq a_1 \vee a_2 \vee ... \vee a_m$ such that $a_i^{n_i} \leq p$

for each $i = 1, ..m$. For $k = n_1 + n_2 + ... + n_m$, we have $b^k \leq a_1^k \vee a_2^k \vee ... \vee a_m^k \leq p$. Then $b^k \leq p \leq (p1_M :_L X) \leq q$. Since q is maximal, we obtain $b \leq q$. This is a contradiction. Therefore, $b \nleq \sqrt{p}$. Since $a \nleq \sqrt{p}, b \nleq \sqrt{p}$ and p is primary, we have $s \leq p$. So, $bX = sY \leq pY \leq p1_M$ and therefore $b \leq (p1_M :_L X) \leq q$. This is a contradiction.

1.25. Corollary. *Let L be a CG-lattice and M be a PG-lattice L-module. Let M be a multiplication lattice L-module and $N < 1_M$. Then the following condition are equivalent.*

i) N is a primary element in M.

ii) $(N :_L 1_M)$ is a primary element in L.

iii) There exists a primary element p in L with $(0_M :_L 1_M) \leq p$ such that $N = p1_M$.

□

Proof. $i) \Longrightarrow ii) \Rightarrow iii)$: Clear.

$iii) \Longrightarrow i)$: Let $aX \leq N$ and $X \nleq N$ for $a \in L, X \in M$. Since there exists a primary element p in L with $(0_M :_L 1_M) \leq p$ such that $N = p1_M$, we have $aX \leq p1_M$ and $X \nleq p1_M$. By theorem 8, $a \leq \sqrt{p}$ and so $a^k \leq p$ for some $k > 0$. Hence $a^k \leq (p1_M :_L 1_M) = (N :_L 1_M)$. □

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