Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 581 – 594

Properties Q and R for nonlinear contractions in g-metric spaces

Renu Chugh* , B. E. Rhoades†, Madhu Aggarwal‡ and Tamanna Kadian§

Abstract

We prove a general common fixed point theorem for three operators in G-metric spaces satisfying a nonlinear contraction. Also we prove the uniqueness of the fixed point, as well as studying the G-continuity of the fixed point. Moreover, we show that these maps satisfy properties Q and R. Our results extend some recent works. An illustrative example is also discussed.

Received 09/11/2011 : Accepted 13/05/2013

2000 AMS Classification: 47H10, 54H25, 54E35, 54E99.

Keywords: G-metric space, Common fixed point, Nonlinear contraction, G-continuity, Property Q, Property R.

1. Introduction and Preliminaries

The notion of G-metric spaces was introduced by Z. Mustafa and B. Sims [5, 8]. They generalized the concept of a metric space. Then, based on the notion of generalized metric spaces, several authors have obtained some fixed point results for a self-mapping under various contractive conditions, (see [1, 2, 3, 6, 7, 9]). In this paper we prove a general common fixed point theorem for three operators in a complete generalized metric space X involving a nonlinear contraction related to a function $\phi \in \Phi$ where Φ is given by the following:

1.1. Definition. Denote by Φ the set of non-decreasing continuous functions $\phi : R^+ \to R^+$ satisfying:

(1) $\phi(0) = 0$,

(2) $0 < \phi(t) < t$ for all $t > 0$,

(3) the series $\sum_{n\geq 1} \phi^n(t)$converges for all $t > 0$.

*Department of Mathematics, M.D. University,Rohtak-124001, INDIA
†Department of Mathematics, Indiana University,Bloomington, IN47 405-7106(USA)
‡ E-mail: madhumdur@gmail.com
§Department of Mathematics, M.D. University,Rohtak-124001, INDIA

1.2. Definition ([8])**.** Let X be a nonempty set and let $G : X \times X \times X \to R^+$ a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$,
(rectangle inequality).

Then the function G is called a generalized metric, or, simply, a G-metric on X, and the pair (X, G) is called a G-metric space.

1.3. Definition ([8])**.** Let (X, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists a $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$ (throughout this paper we mean by N the set of all natural numbers). We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.

1.4. Definition ([8])**.** Let (X, G) be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x,
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$,
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$, as $n, m \to \infty$.

1.5. Definition ([8])**.** Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-cauchy if for each $\epsilon > 0$, there exists a $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$, that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

1.6. Proposition ([8])**.** Let (X, G) be a G-metric space.Then the following are equivalent:

(1) The sequence $\{x_n\}$ is G-cauchy.
(2) For every $\epsilon > 0$, there exists a $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq k$.

1.7. Proposition ([8])**.** Let (X, G) be a G-metric space.Then $f : X \to X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convegent to x, $f(x_n)$ is G-convergent to $f(x)$.

1.8. Proposition ([8])**.** Let (X, G) be a G-metric space.Then the function $G(x, y, z)$ is jointly continuous in all the three variables.

1.9. Definition ([8])**.** A G-metric space (X, G) is called G-complete if every G-cauchy sequence is G-convergent in (X, G).

1.10. Proposition ([8])**.** Let (X, G) be a G-metric space.Then for any $x, y, z, a \in X$ it follows that:

$$
\begin{aligned}
&(1) \quad G(x,y,z) = 0, \text{ then } x = y = z,\\
&(2) \quad G(x,y,z) \leq G(x,x,y) + G(x,x,z),\\
&(3) \quad G(x,y,y) \leq 2G(y,x,x),\\
&(4) \quad G(x,y,z) \leq G(x,a,z) + G(a,y,z),\\
&(5) \quad G(x,y,z) \leq \frac{2}{3}[G(x,y,a) + G(x,a,z) + G(a,y,z)],\\
&(6) \quad G(x,y,z) \leq [G(x,a,a) + G(y,a,a) + G(z,a,a)].
\end{aligned}
$$

1.11. Definition. Let (X, G) be a G-metric space and $T, S : X \to X$ be two mappings with $F(T) \cap F(S) \neq \varnothing$. Then T and S have property Q if $F(T^n) \cap F(S^n) = F(T) \cap F(S)$, for each $n \in N$.

1.12. Definition. Let $\{T_i\}$ be a sequence of selfmaps of a G-metric space X. We shall say that this family has property R if $\cap_i F(T_i^n) = \cap_i F(T_i)$, for each $n \in N$.

In this paper we prove a general common fixed point theorem for three operators in G-metric spaces satisfying a nonlinear contraction. Also we prove the uniqueness of the fixed point, as well as studying the G-continuity of the fixed point. Moreover, we show that these maps satisfy properties Q and R. An interesting fact about maps satisfying properties Q and R is that they have no nontrivial periodic points. Some papers dealing with properties Q and R are ([4, 10]). Our results extend some recent works. An illustrative example is also discussed.

2. Fixed Point Theorem

2.1. Theorem. *Let (X, G) be a complete G-metric space and let T_1, T_2, T_3 be selfmaps of X satisfying for all $x, y, z \in X$,*

$$G(T_1x, T_2y, T_3z) \leq \phi(M(x, y, z)), \tag{2.1}$$

where

$$
\begin{aligned}
M(x,y,z) = \max\{&G(x,y,z), G(x,x,T_1x), G(y,y,T_2y), G(z,z,T_3z),\\
&G(x,x,T_2y), G(z,z,T_1x), G(y,y,T_1x), G(y,y,T_3z), G(z,z,T_2y)\},
\end{aligned}
$$

and $\phi \in \Phi$. Then T_1, T_2 and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

Proof. We shall first show that, if u is a fixed point of one of the maps, then it is a common fixed point. Suppose that $u \in F(T_1)$. Then, from (2.1),

$$
\begin{aligned}
G(T_1u, T_2u, T_3u) \leq \phi(\max\{&G(u,u,u), G(u,u,T_1u), G(u,u,T_2u),\\
&G(u,u,T_3u), G(u,u,T_2u), G(u,u,T_1u), G(u,u,T_1u),\\
&G(u,u,T_3u), G(u,u,T_2u)\}),
\end{aligned} \tag{2.2}
$$

which implies that

$$G(T_1u, T_2u, T_3u) \leq \phi(\max\{G(u,u,T_3u), G(u,u,T_2u)\}).$$

If $T_2u \neq T_3u$, then, from properties $(G3)$ and $(G4)$, $G(u, u, T_3u) \leq G(T_2u, u, T_3u)$, and $G(u, u, T_2u) \leq G(T_2u, u, T_3u)$.Therefore,$G(T_1u, T_2u, T_3u) \leq \phi(G(T_1u, T_2u, T_3u))$, which leads to a contradiction if $G(T_1u, T_2u, T_3u) > 0$. Therefore, $G(T_1u, T_2u, T_3u) = 0$ and from proposition (1.10), $u = T_1u = T_2u = T_3u$.

Suppose that $T_2u = T_3u$. Then (2.2) becomes

$$G(u, T_2u, T_2u) \le \phi(\max\{0, G(u, T_2u, T_2u)\}),$$

which implies that $u = T_2u$, and therefore u is a common fixed point of T_1, T_2, and T_3.

A similar proof shows that, if u is a fixed point of T_2 or T_3, then it is a common fixed point.

Let x_0 be an arbitrary point in X. Take $x_1 = T_1x_0, x_2 = T_2x_1$ and $x_3 = T_3x_2$. We can then construct a sequence $\{x_n\}$ in X such that for any $n \in N$

$$x_{3n+1} = T_1x_{3n}, \quad x_{3n+2} = T_2x_{3n+1}, \quad x_{3n+3} = T_3x_{3n+2}.$$

If there exists an integer p such that $x_{3p} = x_{3p+1}$, then $x_{3p} \in F(T_1)$, hence $x_{3p} \in F(T_2) \cap F(T_3)$.

Similarly, if there exists an integer p such that $x_{3p+1} = x_{3p+2}$ or $x_{3p+2} = x_{3p+3}$, then we have a common fixed point. Therefore, we shall assume that $x_n \ne x_{n+1}$ for all $n \in N$. Using (2.1) with $x = x_{3n}, y = x_{3n+1}$ and $z = x_{3n+2}$, we get

$$\begin{aligned}
G(x_{3n+1}, x_{3n+2}, x_{3n+3}) &= G(T_1x_{3n}, T_2x_{3n+1}, T_3x_{3n+2}) \\
&\le \phi(\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, T_1x_{3n}), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_2x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_3x_{3n+2}), \\
&\quad G(x_{3n}, x_{3n}, T_2x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_1x_{3n}), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_1x_{3n}), G(x_{3n+1}, x_{3n+1}, T_3x_{3n+2}), \\
&\quad G(x_{3n+2}, x_{3n+2}, T_2x_{3n+1})\}) \\
&= \phi(\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n}, x_{3n+1}), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\
&\quad G(x_{3n}, x_{3n}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+1}), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+1}), G(x_{3n+1}, x_{3n+1}, x_{3n+3}), \\
&\quad G(x_{3n+2}, x_{3n+2}, x_{3n+2})\}).
\end{aligned}$$

Then, using (G1), (G3) and (G4), we get

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \le \phi(\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}).$$

Since the x_i are distinct, the above inequality implies that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \le \phi(G(x_{3n}, x_{3n+1}, x_{3n+2})).$$

In a similar way, we obtain

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \le \phi(G(x_{3n+1}, x_{3n+2}, x_{3n+3})),$$

and

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \le \phi(G(x_{3n+2}, x_{3n+3}, x_{3n+4})).$$

From the above three inequalities, one can assert that

$$G(x_n, x_{n+1}, x_{n+2}) \le \phi(G(x_{n-1}, x_n, x_{n+1})).$$

If we define $t_n = G(x_n, x_{n+1}, x_{n+2})$, then $0 \le t_n \le t_{n-1}$, so that the sequence $\{t_n\}$ is non-increasing, hence convergent to some $r \ge 0$. Letting $n \to \infty$ in (2.4), we have $r \le \phi(r)$, using the continuity of ϕ. If $r \ne 0$, we obtain the contradiction $r < \phi(r) < r$. Hence $r = 0$. We rewrite this as

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0.$$

We next prove that $\{x_n\}$ is a G-Cauchy sequence. From (2.4),

$$G(x_n, x_{n+1}, x_{n+2}) \leq \phi(G(x_{n-1}, x_n, x_{n+1})) \leq ..... \leq \phi^n(G(x_0, x_1, x_2)).$$

Therefore, using conditions (G3), (G4), (G5) and (2.4), we have for any $k \in N$

$$\begin{aligned}
G(x_n, x_{n+k}, x_{n+k}) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + ... \\
&+ G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1}) + G(x_{n+k-1}, x_{n+k}, x_{n+k}) \\
&\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + ... \\
&+ G(x_{n+k-2}, x_{n+k-1}, x_{n+k}) + G(x_{n+k-1}, x_{n+k}, x_{n+k+1}) \\
&\leq \phi^n(G(x_0, x_1, x_2)) + \phi^{n+1}(G(x_0, x_1, x_2)) + ... \\
&+ \phi^{n+k}(G(x_0, x_1, x_2)) \\
&= \sum_{i=n}^{n+k} \phi^i(G(x_0, x_1, x_2)) \\
&\leq \sum_{i=n}^{\infty} \phi^i(G(x_0, x_1, x_2)).
\end{aligned}$$

Using properties of ϕ, $\sum_{i=n}^{\infty} \phi^i(G(x_0, x_1, x_2))$ tends to 0 as $n \to \infty$. Then for any $k \in N$

$$\lim_{n\to\infty} G(x_n, x_{n+k}, x_{n+k}) = 0,$$

and $\{x_n\}$ is a G-Cauchy sequence.Since X is a G-complete space, $\{x_n\}$ is a G-convergent to some $u \in X$; that is

$$\lim_{n\to\infty} G(x_n, x_n, u) = \lim_{n\to\infty} G(x_n, u, u) = 0.$$

To show that u is a common fixed point of the maps $T_i, i = 1, 2, 3$, it will be sufficient to show that $T_1u = u$. From (2.1), we have

$$\begin{aligned}
G(T_1u, x_{3n+2}, x_{3n+3}) &= G(T_1u, T_2x_{3n+1}, T_3x_{3n+2}) \\
&\leq \phi(\max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, T_1u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_2x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_3x_{3n+2}), \\
&\quad G(u, u, T_2x_{3n+1}), G(x_{3n+2}, x_{3n+2}, T_1u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_1u), G(x_{3n+1}, x_{3n+1}, T_3x_{3n+2}), \\
&\quad G(x_{3n+2}, x_{3n+2}, T_2x_{3n+1})\}) \\
&= \phi(\max\{G(u, x_{3n+1}, x_{3n+2}), G(u, u, T_1u), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\
&\quad G(u, u, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, T_1u), \\
&\quad G(x_{3n+1}, x_{3n+1}, T_1u), G(x_{3n+1}, x_{3n+1}, x_{3n+3}), \\
&\quad G(x_{3n+2}, x_{3n+2}, x_{3n+2})\}).
\end{aligned}$$

Thus,

$$\begin{aligned}
G(T_1u, x_{3n+2}, x_{3n+3}) &\leq \phi(\max\{G(u, x_{3n+1}, x_{3n+2}), G(T_1u, u, u), \\
&\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}), G(x_{3n+2}, x_{3n+2}, x_{3n+3}), \\
&\quad G(u, u, x_{3n+2}), G(T_1u, x_{3n+2}, x_{3n+2}), \\
&\quad G(T_1u, x_{3n+1}, x_{3n+1}), G(x_{3n+1}, x_{3n+1}, x_{3n+3})\}).
\end{aligned}$$

Taking the limit as $n \to \infty$, we get $G(T_1u, u, u) \leq \phi(G(T_1u, u, u))$, which implies that $T_1u = u, T_2u = u = T_3u$, and u is a common fixed point of the three maps T_1, T_2 and T_3.

Let v be another common fixed point of each $T_i, i = 1, 2, 3$. Using (2.1)

$$\begin{aligned} G(u,u,v) &= G(T_1u, T_2u, T_3v) \\ &\le \phi(\max\{G(u,u,v), G(u,u,T_1u), G(u,u,T_2u), \\ &\quad G(v,v,T_3v), G(u,u,T_2u), G(v,v,T_1u), \\ &\quad G(u,u,T_1u), G(u,u,T_3v), G(v,v,T_2u)\}) \\ &= \phi(\max\{G(u,u,v), 0, 0, 0, 0, G(v,v,u), 0, \\ &\quad G(u,u,v), G(v,v,u)\}). \end{aligned}$$

This implies that

$$G(u,u,v) \le \phi(\max\{G(u,u,v), G(v,v,u)\}).$$

Then either

$$\begin{aligned} &(1) \quad G(u,u,v) > G(v,v,u), \\ &(2) \quad G(u,u,v) \le G(v,v,u). \end{aligned}$$

If (1) is true, then

$$G(u,u,v) \le \phi(G(u,u,v)) < G(u,u,v),$$

which implies that $u = v$,and u is the unique common fixed point of T_1, T_2 and T_3. If (2) is true, then we have

$$G(u,u,v) \le G(v,v,u).$$

Now, again using (2.1) with $x = v, y = z = u$, we get

$$\begin{aligned} G(v,v,u) &= G(T_1v, T_2v, T_3u) \\ &\le \phi(\max\{G(v,v,u), G(v,v,T_1v), G(v,v,T_2v), \\ &\quad G(u,u,T_3u), G(v,v,T_2v), G(u,u,T_1v), \\ &\quad G(v,v,T_1v), G(v,v,T_3u), G(u,u,T_2v)\}) \\ &= \phi(\max\{G(v,v,u)G(u,u,v)\}). \end{aligned}$$

Then either

$$\begin{aligned} &(3) \quad G(v,v,u) > G(u,u,v), \\ &(4) \quad G(v,v,u) \le G(u,u,v). \end{aligned}$$

If (3) is true, then

$$G(v,v,u) \le \phi(G(v,v,u)) < G(v,v,u),$$

which implies that $u = v$,and u is the unique common fixed point of T_1, T_2 and T_3. If (4) is true, then we have

$$G(v,v,u) \le G(u,u,v).$$

Combining (2.6) and (2.7), we get $G(v,v,u) = G(u,u,v)$. Using it in (2.5), we get that u is the unique common fixed point of T_1, T_2 and T_3.

We shall now show that each $T_i, i = 1, 2, 3$, is G-continuous at u. First we prove that T_1 is G-continuous at u. For this, let $\{u_n\} \subset X$ be a sequence which is G-convergent to

u. Using (2.1), we have

$$\begin{aligned} G(T_1u_n,u,u) &= G(T_1u_n,T_2u,T_3u) \\ &\le \phi(\max\{G(u_n,u,u),G(u_n,u_n,T_1u_n),G(u,u,T_2u), \\ &\quad G(u,u,T_3u),G(u_n,u_n,T_2u),G(u,u,T_1u_n), \\ &\quad G(u,u,T_1u_n),G(u,u,T_3u),G(u,u,T_2u)\}) \\ &= \phi(\max\{G(u_n,u,u),G(u_n,u_n,T_1u_n),0,0, \\ &\quad G(u_n,u_n,u),G(u,u,T_1u_n),0,0\}). \end{aligned}$$

Therefore

$$\begin{aligned} G(T_1u_n,u,u) \le \phi(\max\{&G(u_n,u,u),G(T_1u_n,u_n,u_n), \\ &G(u_n,u_n,u),G(T_1u_n,u,u)\}). \end{aligned}$$

Taking the limit as $n\to\infty$, we get $G(T_1u_n,u,u)=0$, which implies that $\lim_{n\to\infty}T_1u_n = u$ and T_1 is G-continuous at u. Similarly, one can show that each $T_i, i=2,3$, is G-continuous at u. □

We now give some corollaries of Theorem 2.1. The first corresponds to $\phi(t)=kt$ where $0\le k<1$.

2.2. Corollary. *Let (X,G) be a complete G-metric space and let T_1,T_2,T_3 be selfmaps of X satisfying for all $x,y,z\in X$,*

$$G(T_1x,T_2y,T_3z)\le k(M(x,y,z)),$$

where

$$\begin{aligned} M(x,y,z) = \max\{&G(x,y,z),G(x,x,T_1x),G(y,y,T_2y),G(z,z,T_3z), \\ &G(x,x,T_2y),G(z,z,T_1x),G(y,y,T_1x),G(y,y,T_3z),G(z,z,T_2y)\}, \end{aligned}$$

where k is a constant satisfying $0\le k<1$. Then T_1,T_2 and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

2.3. Corollary. *Let (X,G) be a complete G-metric space and let T_1,T_2,T_3 be selfmaps of X satisfying for all $x,y,z\in X$,*

$$G(T_1^mx,T_2^my,T_3^mz)\le\phi(M(x,y,z)),$$

where

$$\begin{aligned} M(x,y,z) = \max\{&G(x,y,z),G(x,x,T_1^mx),G(y,y,T_2^my),G(z,z,T_3^mz), \\ &G(x,x,T_2^my),G(z,z,T_1^mx),G(y,y,T_1^mx),G(y,y,T_3^mz),G(z,z,T_2^my)\}, \end{aligned}$$

and $\phi\in\Phi$. Then T_1,T_2 and T_3 have a unique common fixed point (say u) and each T_i is G-continuous at u.

Proof. From Theorem 2.1, we conclude that the maps T_1^m,T_2^m and T_3^m have a unique common fixed point say u. For any $i=1,2,3, T_iu=T_i(T_i^mu)=T_i^{m+1}u=T_i^m(T_iu)$, meaning that T_iu is also a fixed point of T_i^m. By the uniqueness of u, we get $T_iu=u$. □

By taking $T_1=S$ and $T_2=T_3=T$ in Theorem 2.1, we obtain the following result:

2.4. Corollary. *Let (X, G) be a complete G-metric space and let S, T be selfmaps of X satisfying for all $x, y, z \in X$,*

$$G(Sx, Ty, Tz) \leq \phi(M(x, y, z)),$$

where

$$\begin{aligned} M(x, y, z) = \max\{&G(x, y, z), G(x, x, Sx), G(y, y, Ty), G(z, z, Tz),\\ &G(x, x, Ty), G(z, z, Sx), G(y, y, Sx), G(y, y, Tz), G(z, z, Ty)\}, \end{aligned}$$

and $\phi \in \Phi$. Then S, T have a unique common fixed point (say u). Also S and T are G-continuous at u.

By taking $T = T_1 = T_2 = T_3$ in Theorem 2.1 and in the above Corollaries, we obtain the following results:

2.5. Corollary. *Let (X, G) be a complete G-metric space and let T be selfmap of X satisfying for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq \phi(M(x, y, z)),$$

where

$$\begin{aligned} M(x, y, z) = \max\{&G(x, y, z), G(x, x, Tx), G(y, y, Ty), G(z, z, Tz),\\ &G(x, x, Ty), G(z, z, Tx), G(y, y, Tx), G(y, y, Tz), G(z, z, Ty)\}, \end{aligned}$$

and $\phi \in \Phi$. Then T has a unique fixed point (say u). Also T is G-continuous at u.

2.6. Corollary. *Let (X, G) be a complete G-metric space and let T be selfmap of X satisfying for all $x, y, z \in X$,*

$$G(Tx, Ty, Tz) \leq k(M(x, y, z)),$$

where

$$\begin{aligned} M(x, y, z) = \max\{&G(x, y, z), G(x, x, Tx), G(y, y, Ty), G(z, z, Tz),\\ &G(x, x, Ty), G(z, z, Tx), G(y, y, Tx), G(y, y, Tz), G(z, z, Ty)\}, \end{aligned}$$

where k is a constant satisfying $0 \leq k < 1$. Then T has a unique fixed point (say u). Also T is G-continuous at u.

2.7. Remark. Special cases of Corollary 2.6 are Theorems 2.1, 2.2 of [2], Theorems 2.1, 2.3, 2.5, 2.8, 2.9 of [5],Theorem 2.3 of [7] and Theorems 2.1, 2.4, 2.6, 2.8 of [9].

2.8. Example. Let $X = [0, 1]$ with G-metric defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$. Define $\phi : R \to R$ as $\phi(t) = \frac{2t}{3}$ and $T_1, T_2, T_3 : X \to X$ by

$$T_1 x = \begin{cases} \frac{x}{15} & \text{if } x \in [0, \frac{1}{3}) \\ \frac{x}{8} & \text{if } x \in [\frac{1}{3}, 1] \end{cases},$$

$$T_2 x = \begin{cases} \frac{x}{10} & \text{if } x \in [0, \frac{1}{3}) \\ \frac{x}{4} & \text{if } x \in [\frac{1}{3}, 1] \end{cases},$$

$$T_3 x = \begin{cases} \frac{x}{5} & \text{if } x \in [0, \frac{1}{3}) \\ \frac{x}{2} & \text{if } x \in [\frac{1}{3}, 1] \end{cases}.$$

Now, we prove that the mappings T_1, T_2 and T_3 satisfy condition (2.1). We discuss the following eight cases:

Case-I: If $x, y, z \in [0, \frac{1}{3})$, then

$$M(x,y,z) = \max\Big\{ \max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{9y}{10}, \frac{4z}{5}, \left|x - \frac{y}{10}\right|, \left|z - \frac{x}{15}\right|, \left|y - \frac{x}{15}\right|, \left|y - \frac{z}{5}\right|, \left|z - \frac{y}{10}\right| \Big\}.$$

So,

$$\begin{aligned} G(T_1x, T_2y, T_3z) &= \max\left\{ \left|\frac{x}{15} - \frac{y}{10}\right|, \left|\frac{y}{10} - \frac{z}{5}\right|, \left|\frac{z}{5} - \frac{x}{15}\right| \right\} \\ &= \frac{1}{5}\max\left\{ \left|\frac{x}{3} - \frac{y}{2}\right|, \left|\frac{y}{2} - z\right|, \left|z - \frac{x}{3}\right| \right\} \\ &\le \frac{2}{3}\max\Big\{ \max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{9y}{10}, \frac{4z}{5}, \\ &\quad \left|x - \frac{y}{10}\right|, \left|z - \frac{x}{15}\right|, \left|y - \frac{x}{15}\right|, \left|y - \frac{z}{5}\right|, \left|z - \frac{y}{10}\right| \Big\} \\ &= \frac{2}{3}M(x,y,z) \\ &= \phi(M(x,y,z)). \end{aligned}$$

Case-II: If $x, y, z \in [\frac{1}{3}, 1]$, then

$$M(x,y,z) = \max\Big\{ \max\{|x-y|, |y-z|, |z-x|\}, \frac{7x}{8}, \frac{3y}{4}, \frac{z}{2}, \left|x - \frac{y}{4}\right|, \left|z - \frac{x}{8}\right|, \left|y - \frac{x}{8}\right|, \left|y - \frac{z}{2}\right|, \left|z - \frac{y}{4}\right| \Big\}.$$

So,

$$\begin{aligned} G(T_1x, T_2y, T_3z) &= \max\left\{ \left|\frac{x}{8} - \frac{y}{4}\right|, \left|\frac{y}{4} - \frac{z}{2}\right|, \left|\frac{z}{2} - \frac{x}{8}\right| \right\} \\ &= \frac{1}{2}\max\left\{ \left|\frac{x}{4} - \frac{y}{2}\right|, \left|\frac{y}{2} - z\right|, \left|z - \frac{x}{4}\right| \right\} \\ &\le \frac{2}{3}\max\Big\{ \max\{|x-y|, |y-z|, |z-x|\}, \frac{7x}{8}, \frac{3y}{4}, \frac{z}{2}, \\ &\quad \left|x - \frac{y}{4}\right|, \left|z - \frac{x}{8}\right|, \left|y - \frac{x}{8}\right|, \left|y - \frac{z}{2}\right|, \left|z - \frac{y}{4}\right| \Big\} \\ &= \frac{2}{3}M(x,y,z) \\ &= \phi(M(x,y,z)). \end{aligned}$$

Case-III: If $x \in [0, \frac{1}{3})$ and $y, z \in [\frac{1}{3}, 1]$, then

$$M(x,y,z) = \max\Big\{ \max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{3y}{4}, \frac{z}{2}, \left|x - \frac{y}{4}\right|, \left|z - \frac{x}{15}\right|, \left|y - \frac{x}{15}\right|, \left|y - \frac{z}{2}\right|, \left|z - \frac{y}{4}\right| \Big\}.$$

So,

$$\begin{aligned}
G(T_1x,T_2y,T_3z) &= \max\left\{\left|\frac{x}{15}-\frac{y}{4}\right|,\left|\frac{y}{4}-\frac{z}{2}\right|,\left|\frac{z}{2}-\frac{x}{15}\right|\right\}\\
&= \frac{1}{2}\max\left\{\left|\frac{2x}{15}-\frac{y}{2}\right|,\left|\frac{y}{2}-z\right|,\left|z-\frac{2x}{15}\right|\right\}\\
&\leq \frac{2}{3}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{14x}{15},\frac{3y}{4},\frac{z}{2},\right.\\
&\left.\left|x-\frac{y}{4}\right|,\left|z-\frac{x}{15}\right|,\left|y-\frac{x}{15}\right|,\left|y-\frac{z}{2}\right|,\left|z-\frac{y}{4}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Case-IV: If $y\in[0,\frac{1}{3})$ and $x,z\in[\frac{1}{3},1]$, then

$$\begin{aligned}
M(x,y,z) = \max&\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{7x}{8},\frac{9y}{10},\frac{z}{2},\right.\\
&\left.\left|x-\frac{y}{10}\right|,\left|z-\frac{x}{8}\right|,\left|y-\frac{x}{8}\right|,\left|y-\frac{z}{2}\right|,\left|z-\frac{y}{10}\right|\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x,T_2y,T_3z) &= \max\left\{\left|\frac{x}{8}-\frac{y}{10}\right|,\left|\frac{y}{10}-\frac{z}{2}\right|,\left|\frac{z}{2}-\frac{x}{8}\right|\right\}\\
&= \frac{1}{2}\max\left\{\left|\frac{x}{4}-\frac{y}{5}\right|,\left|\frac{y}{5}-z\right|,\left|z-\frac{x}{4}\right|\right\}\\
&\leq \frac{2}{3}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{7x}{8},\frac{9y}{10},\frac{z}{2},\right.\\
&\left.\left|x-\frac{y}{10}\right|,\left|z-\frac{x}{8}\right|,\left|y-\frac{x}{8}\right|,\left|y-\frac{z}{2}\right|,\left|z-\frac{y}{10}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Case-V: If $z\in[0,\frac{1}{3})$ and $x,y\in[\frac{1}{3},1]$, then

$$\begin{aligned}
M(x,y,z) = \max&\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{7x}{8},\frac{3y}{4},\frac{4z}{5},\right.\\
&\left.\left|x-\frac{y}{4}\right|,\left|z-\frac{x}{8}\right|,\left|y-\frac{x}{8}\right|,\left|y-\frac{z}{5}\right|,\left|z-\frac{y}{4}\right|\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x, T_2y, T_3z) &= \max\left\{\left|\frac{x}{8}-\frac{y}{4}\right|, \left|\frac{y}{4}-\frac{z}{5}\right|, \left|\frac{z}{5}-\frac{x}{8}\right|\right\}\\
&= \frac{1}{4}\max\left\{\left|\frac{x}{2}-y\right|, \left|y-\frac{4z}{5}\right|, \left|\frac{4z}{5}-\frac{x}{2}\right|\right\}\\
&\leq \frac{2}{3}\max\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{7x}{8}, \frac{3y}{4}, \frac{4z}{5},\right.\\
&\qquad \left.\left|x-\frac{y}{4}\right|, \left|z-\frac{x}{8}\right|, \left|y-\frac{x}{8}\right|, \left|y-\frac{z}{5}\right|, \left|z-\frac{y}{4}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Case-VI: If $x, y \in [0, \frac{1}{3})$ and $z \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
M(x,y,z) = \max&\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{9y}{10}, \frac{z}{2},\right.\\
&\left.\left|x-\frac{y}{10}\right|, \left|z-\frac{x}{15}\right|, \left|y-\frac{x}{15}\right|, \left|y-\frac{z}{2}\right|, \left|z-\frac{y}{10}\right|\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x, T_2y, T_3z) &= \max\left\{\left|\frac{x}{15}-\frac{y}{10}\right|, \left|\frac{y}{10}-\frac{z}{2}\right|, \left|\frac{z}{2}-\frac{x}{15}\right|\right\}\\
&= \frac{1}{2}\max\left\{\left|\frac{2x}{15}-\frac{y}{5}\right|, \left|\frac{y}{5}-z\right|, \left|z-\frac{2x}{15}\right|\right\}\\
&\leq \frac{2}{3}\max\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{9y}{10}, \frac{z}{2},\right.\\
&\qquad \left.\left|x-\frac{y}{10}\right|, \left|z-\frac{x}{15}\right|, \left|y-\frac{x}{15}\right|, \left|y-\frac{z}{2}\right|, \left|z-\frac{y}{10}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Case-VII: If $x, z \in [0, \frac{1}{3})$ and $y \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
M(x,y,z) = \max&\left\{\max\{|x-y|, |y-z|, |z-x|\}, \frac{14x}{15}, \frac{3y}{4}, \frac{4z}{5},\right.\\
&\left.\left|x-\frac{y}{4}\right|, \left|z-\frac{x}{15}\right|, \left|y-\frac{x}{15}\right|, \left|y-\frac{z}{5}\right|, \left|z-\frac{y}{4}\right|\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x,T_2y,T_3z) &= \max\left\{\left|\frac{x}{15}-\frac{y}{4}\right|,\left|\frac{y}{4}-\frac{z}{5}\right|,\left|\frac{z}{5}-\frac{x}{15}\right|\right\}\\
&= \frac{1}{4}\max\left\{\left|\frac{4x}{15}-y\right|,\left|y-\frac{4z}{5}\right|,\left|\frac{4z}{5}-\frac{4x}{15}\right|\right\}\\
&\le \frac{2}{3}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{14x}{15},\frac{3y}{4},\frac{4z}{5},\right.\\
&\qquad \left.\left|x-\frac{y}{4}\right|,\left|z-\frac{x}{15}\right|,\left|y-\frac{x}{15}\right|,\left|y-\frac{z}{5}\right|,\left|z-\frac{y}{4}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Case-VIII: If $y,z\in[0,\frac{1}{3})$ and $x\in[\frac{1}{3},1]$, then

$$\begin{aligned}
M(x,y,z) = \max&\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{7x}{8},\frac{9y}{10},\frac{4z}{5},\right.\\
&\left.\left|x-\frac{y}{10}\right|,\left|z-\frac{x}{8}\right|,\left|y-\frac{x}{8}\right|,\left|y-\frac{z}{5}\right|,\left|z-\frac{y}{10}\right|\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
G(T_1x,T_2y,T_3z) &= \max\left\{\left|\frac{x}{8}-\frac{y}{10}\right|,\left|\frac{y}{10}-\frac{z}{5}\right|,\left|\frac{z}{5}-\frac{x}{8}\right|\right\}\\
&= \frac{1}{5}\max\left\{\left|\frac{5x}{8}-\frac{y}{2}\right|,\left|\frac{y}{2}-z\right|,\left|z-\frac{5x}{8}\right|\right\}\\
&\le \frac{2}{3}\max\left\{\max\{|x-y|,|y-z|,|z-x|\},\frac{7x}{8},\frac{9y}{10},\frac{4z}{5},\right.\\
&\qquad \left.\left|x-\frac{y}{10}\right|,\left|z-\frac{x}{8}\right|,\left|y-\frac{x}{8}\right|,\left|y-\frac{z}{5}\right|,\left|z-\frac{y}{10}\right|\right\}\\
&= \frac{2}{3}M(x,y,z)\\
&= \phi(M(x,y,z)).
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied for all $x,y,z\in X$ and 0 is the unique common fixed point of T_1,T_2 and T_3.

3. Properties Q and R

In this section, we shall show that maps satisfying the conditions of theorem (2.1) and Corollary (2.3) possess properties R and Q respectively.

3.1. Theorem. *Under the conditions of Theorem 2.1, T_1,T_2 and T_3 have Property R.*

Proof. From Theorem 2.1, T_1,T_2 and T_3 have a fixed point. Therefore, $(F(T_1^n)\cap F(T_2^n)\cap F(T_3^n))$ is non empty for each $n\in N$. Let $n>1$ and suppose that $p\in F(T_1^n)\cap F(T_2^n)\cap F(T_3^n)$. We claim that $p\in F(T_1)\cap F(T_2)\cap F(T_3)$. To prove this, it is sufficient to show that p is a fixed point of T_1. Suppose that $T_1p \neq p$. Then using (2.1)

$$\begin{aligned} G(T_1p,p,p) &= G(T_1^{n+1}p, T_2^n p, T_3^n p) \\ &\leq \phi(\max\{G(T_1^n p, T_2^{n-1}p, T_3^{n-1}p), G(T_1^n p, T_1^n p, T_1^{n+1}p), \\ &\quad G(T_2^{n-1}p, T_2^{n-1}p, T_2^n p), G(T_3^{n-1}p, T_3^{n-1}p, T_3^n p), \\ &\quad G(T_1^n p, T_1^n p, T_2^n p), G(T_3^{n-1}p, T_3^{n-1}p, T_1^{n+1}p), \\ &\quad G(T_2^{n-1}p, T_2^{n-1}p, T_1^{n+1}p), G(T_2^{n-1}p, T_2^{n-1}p, T_3^n p), \\ &\quad G(T_3^{n-1}p, T_3^{n-1}p, T_2^n p)\}) \\ &= \phi(\max\{G(p, T_2^{n-1}p, T_3^{n-1}p), G(p,p,T_1p), G(T_2^{n-1}p, T_2^{n-1}p, p), \\ &\quad G(T_3^{n-1}p, T_3^{n-1}p, p), G(p,p,p), G(T_3^{n-1}p, T_3^{n-1}p, T_1p), \\ &\quad G(T_2^{n-1}p, T_2^{n-1}p, T_1p), G(T_2^{n-1}p, T_2^{n-1}p, p), G(T_3^{n-1}p, T_3^{n-1}p, p)\}). \end{aligned}$$

If $p = T_2^{n-1}p$, then $G(T_2^{n-1}p, T_2^{n-1}p, p) = 0$. If $p \neq T_2^{n-1}p$, then from (G3) and (G4),

$$G(T_2^{n-1}p, T_2^{n-1}p, p) \leq G(T_2^{n-1}p, p, T_3^{n-1}p).$$

Similarly, either

$$G(T_3^{n-1}p, T_3^{n-1}p, p) = 0 \text{ or}$$

$$G(T_3^{n-1}p, T_3^{n-1}p, p) \leq G(p, T_2^{n-1}p, T_3^{n-1}p),$$

$$G(T_2^{n-1}p, T_2^{n-1}p, T_1p) = 0 \text{ or}$$

$$G(T_2^{n-1}p, T_2^{n-1}p, T_1p) \leq G(T_1p, T_2^{n-1}p, T_3^{n-1}p),$$

and

$$G(T_3^{n-1}p, T_3^{n-1}p, T_1p) = 0 \text{ or}$$

$$G(T_3^{n-1}p, T_3^{n-1}p, T_1p) \leq G(T_1p, T_2^{n-1}p, T_3^{n-1}p).$$

Therefore

$$G(T_1p,p,p) \leq \phi(\max\{G(T_1p,p,p), G(p, T_2^{n-1}p, T_3^{n-1}p), G(T_1p, T_2^{n-1}p, T_3^{n-1}p)\}).$$

Then either

$$G(T_1p,p,p) \leq \phi(G(T_1p,p,p)),$$

$$\text{or } G(T_1p,p,p) \leq \phi(G(p, T_2^{n-1}p, T_3^{n-1}p)),$$

$$\text{or } G(T_1p,p,p) \leq \phi(G(T_1p, T_2^{n-1}p, T_3^{n-1}p)).$$

The condition

$$G(T_1p,p,p) \leq \phi(G(p, T_2^{n-1}p, T_3^{n-1}p)),$$

implies that

$$\begin{aligned} G(T_1p,p,p) &= G(T_1^{n+1}p, T_2^n p, T_3^n p) \\ &\leq \phi(G(T_1^n p, T_2^{n-1}p, T_3^{n-1}p)) \leq .... \leq \phi(G(T_1p,p,p)), \end{aligned}$$

and if

$$G(T_1p,p,p) \leq \phi(G(T_1p, T_2^{n-1}p, T_3^{n-1}p)),$$

implies that

$$G(T_1p,p,p) = G(T_1p,T_2^np,T_3^np)$$
$$\leq \phi(G(T_1p,T_2^{n-1}p,T_3^{n-1}p)) \leq .... \leq \phi(G(T_1p,p,p)).$$

Thus in either case, we obtain

$$G(T_1p,p,p) \leq \phi(G(T_1p,p,p)),$$

which is a contradiction. Hence $T_1p = p$. Then, by Theorem 2.1, $T_2p = T_3p = p$. Therefore, $p \in F(T_1) \cap F(T_2) \cap F(T_3)$, and T_1, T_2 and T_3 satisfy property R.

□

3.2. Corollary. *Under the conditions of Corollary 2.3, S and T have Property Q.*

3.3. Remark. For $T_1 = T_2 = T_3 = T$ in Theorem 3.1, we obtain Theorems 3.1, 3.2 of [2] as special cases of our result.

References

[1] Abbas, M. and Rhoades, B. E.: Common fixed point results for noncommuting mappings without continuity in generalized metric spaces, Applied Mathematics and Computation,**215**, 262-269, 2009.

[2] Chugh, R., Kadian, T., Rani, A., Rhoades, B. E.: Property P in G-metric spaces, Fixed Point Theory and App., **2010**, Article ID 401684, 12 pages, 2010.

[3] Gugnani, M., Aggarwal, M. and Chugh, R.: Common Fixed Point Results in G-Metric Spaces and Applications, International Journal of Computer Applications, **43**, No. 11, 38-42, 2012.

[4] Jeong, G. S. and Rhoades, B. E.: More maps for which $F(T) = F(T^n)$, Demonstratio Mathematica, **XL**, No. 3, 671-680, 2007.

[5] Mustafa, Z.: A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, 2005.

[6] Mustafa, Z., Obiedat, H., Awawdeh, F.: Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and Applications, **2008**, Article ID 189870, 12 pages,2008.

[7] Mustafa, Z., Shatanawi, W., Bataineh, M.: Existence of fixed point results in G-metric spaces, International Journal of Mathematics and Mathematical Sciences, **2009**, Article ID 283028, 10 pages, 2009.

[8] Mustafa, Z., Sims, B.: A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, **7**,No. 2, 289-297, 2006.

[9] Mustafa, Z., Sims, B.: Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, **2009**, Article ID 917175, 10 pages, 2009.

[10] Rhoades, B. E. and Abbas, M.: Maps satisfying generalized contractive condition of integral type for which $F(T) = F(T^n)$, International Journal of Pure and Applied Mathematics,**45**, No. 2, 225-231, 2008.