Hacettepe Journal of Mathematics and Statistics
Volume 43 (4) (2014), 595 – 602

On the sum of best simultaneously proximinal subspaces

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Abstract

Let X be a Banach space and G a subspace of X. A point $g_0 \in G$ is said to be a best simultaneous approximation for a bounded set $A \subseteq X$ if $d(A,G) = \inf_{g\in G} \sup_{a\in A} \|a-g\| = \sup_{a\in A} \|a-g_0\|$. In this paper we prove that if F and G are two subspaces of a Banach space X such that G is reflexive and F is simultaneously proximinal, then $F+G$ is simultaneously proximinal provided that $F\cap G$ is finite dimensional and $F+G$ is closed. Some other related results are also presented.

Received 21/11/2010 : Accepted 30/05/2013

2000 AMS Classification: Primary 41A50, Secondary 41A28

Keywords: Approximation, simultaneously .

1. Introduction

Let X be a normed space and G a closed subspace of X. For $x \in X$, a point $g_0 \in G$ satisfying $\|x-g_0\| = d(x,G) = \inf_{g\in G} \|x-g\|$ is called a point of best approximation to x from G. The set $P_G(x) = \{g \in G : \|x-g\| = d(x,G)\}$ is called the set of all best approximations to x from G. The set G is called proximinal in X if for each $x \in X$, $P_G(x) \neq \phi$, see [12] . Note that for $x \in \overline{G}$, $d\left(x,\overline{G}\right) = 0$. Consequently if G is proximinal it follows that G is closed. If a bounded set A is given in X one might like to approximate all elements of A simultaneously by a single element of G. This type of problem arises when a function being approximated is not known precisely but it is known to belong to

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a set. For the definition regarding simultaneous approximation one may refer to Narang [12].

1.1. Definition. Let G be a closed subspace of a normed space X and A be a bounded subset of X. A point $g_0 \in G$ is called a point of simultaneous approximation to A from G if

$$d(A,G) = \inf_{g\in G}\sup_{a\in A}\|a-g\| = \sup_{a\in A}\|a-g_0\|.$$

A subspace $G \subset X$ is called simultaneously proximinal if every bounded set $A \subset X$ admits a point of best simultaneous approximation.

Taking the set A to be a singleton it follows that simultaneously proximinal subspaces are proximinal and hence closed.

The problem of best simultaneous approximation has been studied by many authors [1, 11, 12, 13, 14, 15]. Most of these works deal with characterization of best simultaneous approximation in spaces of continuous functions with values in a Banach space X. In [11], some existence and uniqueness of best simultaneous approximation of bounded functions with values in a uniformly convex Banach space by corresponding continuous function were obtained. In [13], some results were obtained in the spaces of $L^p(I,X)$, $1 \le p < \infty$. In [5], Cheney and Wulbert raised the question:

Problem: If F and G are proximinal subspaces of a Banach space X and $F+G$ is closed, does it follow that $F+G$ is proximinal in X ?

In [7], Feder gave a negative answer to this problem. Further Feder proved that if G is reflexive and F is proximinal such that $E\cap F$ is finite dimensional then $F+G$ is proximinal. In [10], P. K. Lin proved that a Banach space G is reflexive if and only if for every Banach space X with $G \subseteq X$, F is proximinal in X and $G+F$ is closed implies $G+F$ is proximinal in X. In [2], it has been proved that if F and G are reflexive proximinal subspaces of a Banach spaces X such that F and G are p-orthogonal, then $L^p(I,F)+L^p(I,G)$ is proximinal in $L^p(I,X)$.

In this paper we generalize Feder's result to simultaneous approximation of bounded sets. Some other related results are presented.

Throughout this paper we write X/Y for the quotient space and the coset in X/Y will be denoted by $\overline{x}$ for $x \in X$, where X is a normed space and X^* the space of all bounded linear functionals on X.

2. Best Simultaneous Approximation

In this section we generalize Feder's result to simultaneous approximation of bounded sets. Some other results concerning simultaneous approximations of the sum of two simultaneously proximinal subspaces will be proved.

2.1. Theorem. *Let F be a simultaneously proximinal subspace of a normed space X and E be a subspace of X. Assume $E+F$ is closed. If $E+F$ is simultaneously proximinal in X, then $(E+F)/F$ is simultaneously proximinal in X/F.*

Proof. Let B be a nonempty bounded subset of X/F. Then $B = H/F$ for some $H \subset X$. We will show that there exists a bounded subset A of X such that $B = A/F$, similar assertion as in Lemma 3.2,[8]. Let $C = \bigcup_{b\in B} b = \bigcup_{h\in H} h+F$. Claim $B = \{\overline{x} = x+F : x \in C\}$. Indeed if $b \in B$, then $b = h_b+F$ for some $h_b \in H$. But F is a subspace. Thus $h_b = h_b+0 \in h_b+F \subset C$. Hence $b = h_b+F \in \{\overline{x} = x+F : x \in C\}$ and $B \subseteq \{\overline{x} = x+F : x \in C\}$. Similarly if $x \in C$, then $x \in h_x+F$ for some $h_x \in H$. This implies that $x = h_x+f_x$ for some $f_x \in F$. Hence $x+F = h_x+f_x+F = h_x+F \in B$. Therefore $\{\overline{x} = x+F : x \in C\} \subseteq B$.

Now Clearly C is not bounded unless F is trivial. Note that B is bounded. So there exists $M > 0$ such that $\|b\| \leq M$ for all $b \in B$. Consider the set $A = \{x \in C : \|x\| \leq M + 1\} \subseteq C$. Now we claim that for all $x \in C$, $\overline{x} \cap A = (x + F) \cap A \neq \phi$. Given $x \in C$. Since

$$\|x + F\| = \inf_{f \in F} \|x + f\| \leq M,$$

there exists $f_x \in F$ such that $\|x + f_x\| < M + 1$. But $x + f_x \in x + F \subseteq C$. Hence $x + f_x \in (x + F) \cap A \neq \phi$. Claim $B = A/F$. Since $A \subseteq C$, we have $A/F \subseteq \{\overline{x} = x + F : x \in C\} = B$. To show the other inclusion, let $b \in B = \{\overline{x} = x + F : x \in C\}$. Then $b = x_b + F$ for some $x_b \in C$. But $(x_b + F) \cap A \neq \phi$. Thus there exists $a \in A$ such that $a = x_b + f_a \in x_b + F$. Therefore $b = x_b + F = (x_b + f_a) + F = a + F \in A/F$. Hence $B \subseteq A/F$. Consequently $A/F = B$.

Now, since $E + F$ is simultaneously proximinal, there exist $g_0 \in E + F$ such that

$$d(A, E + F) = \inf_{g \in E + F} \sup_{a \in A} \|a - g\| = \sup_{a \in A} \|a - g_0\| \leq \sup_{a \in A} \|a - g\|$$

for all $g \in E + F$. Now

$$\begin{aligned} \sup_{a \in A} \|\overline{a} - \overline{g_0}\| &= \sup_{a \in A} \|\overline{a - g_0}\| \\ &\leq \sup_{a \in A} \|a - g_0\| \\ &\leq \sup_{a \in A} \|a - g\| \end{aligned}$$

for all $g \in E + F$. Since $E + F$ is a subspace of X it follows that,

$$\sup_{a \in A} \|\overline{a} - \overline{g_0}\| \leq \sup_{a \in A} \|a - g - w\|,$$

for all $g \in E + F$, and $w \in F$, which implies that

$$\sup_{a \in A} \|\overline{a} - \overline{g_0}\| \leq \sup_{a \in A} \inf_{w \in F} \|a - g - w\| = \sup_{a \in A} \|\overline{a} - \overline{g}\|.$$

Consequently,

$$\sup_{a \in A} \|\overline{a} - \overline{g_0}\| \leq \sup_{a \in A} \|\overline{a} - \overline{g}\|$$

and $(E + F)/F$ is simultaneously proximinal in X/F. □

2.2. Theorem. *If G is a reflexive subspace of a Banach space X, then G is simultaneously proximinal in X.*

Proof. Let A be a bounded subset of X. Then there exists a sequence $g_n \in G$ such that:

$$\lim_{n \to \infty} \sup_{a \in A} \|a - g_n\| = d(A, G).$$

Thus $S_n = \sup_{a \in A} \|a - g_n\|$ is a convergent sequence of real numbers, and hence bounded. So there exists $M_1 > 0$ such that

$$\|S_n\| = \sup_{a \in A} \|a - g_n\| \leq M_1,$$

for every positive integer $n \in \mathbb{N}$. Now for $a \in A$ and $n \in \mathbb{N}$, the inequality

$$\begin{aligned} \|g_n\| &\leq \|g_n - a\| + \|a\| \\ &\leq \sup_{a \in A} \|g_n - a\| + \sup_{a \in A} \|a\| \\ &\leq M_1 + M_2 = M, \end{aligned}$$

which implies $\{g_n\}$ is a bounded sequence in G. Since G is reflexive, the ball

$$B(0,M)=\{g\in G:\|g\|\leq M\}\subseteq G \ =G^{**}.$$

Thus $B(0,M)$ is weakly- closed and norm bounded. Since G is reflexive by Corollary 8 [6, p 425] $B(0,M)$ is weakly compact and so there exists (g_{nk}) subsequence of (g_n) such that $g_{nk}\xrightarrow{w} g_0$ for some $g_0\in G$. Now for any bounded linear functional $f\in X^*$, $\|f\|=1$ and $a\in A$,

$$\lim_{n_k\to\infty}|f(g_{n_k}-a)|\leq \lim_{n_k\to\infty}\sup_{a\in A}|f(g_{n_k}-a)|.$$

Therefore

$$\sup_{a\in A}\lim_{n_k\to\infty}|f(g_{n_k}-a)|\leq \lim_{n_k\to\infty}\sup_{a\in A}|f(g_{n_k}-a)|.$$

But (g_{n_k}) converges weakly to g_0 so,

$$\lim_{n_k\to\infty}f(g_{n_k}-a)=\lim_{n_k\to\infty}f(g_{n_k})-f(a)=f(g_0)-f(a)=f(g_0-a).$$

Consequently

$$\begin{aligned}\sup_{a\in A}|f(g_0-a)|&=\sup_{a\in A}\lim_{n_k\to\infty}|f(g_{n_k}-a)|\\&\leq \lim_{n_k\to\infty}\sup_{a\in A}|f(g_{n_k}-a)|\\&\leq \lim_{n_k\to\infty}\sup_{a\in A}\sup_{\|f\|=1}|f(g_{n_k}-a)|.\end{aligned}$$

Since $\|(g_{n_k}-a)\|=\sup_{f\in X^*,\|f\|=1}|f(g_{n_k}-a)|$, (Corollary 15, [6, p 65]) we have,

$$\begin{aligned}\sup_{a\in A}|f(g_0-a)|&\leq \lim_{n_k\to\infty}\sup_{a\in A}\|(g_{n_k}-a)\|\\&=\lim_{n\to\infty}\sup_{a\in A}\|(g_n-a)\|=d(A,G).\end{aligned}$$

Consequently using (Corollary 15, [6, p 65]) again we get

$$\sup_{a\in A}\sup_{f\in X^*,\|f\|=1}|f(g_0-a)|=\sup_{a\in A}\|(g_0-a)\|\leq d(A,G).$$

But

$$d(A,G)\leq \sup_{a\in A}\|(g_0-a)\|.$$

Therefore,

$$d(A,G)=\sup_{a\in A}\|(g_0-a)\|,$$

and G is simultaneously proximinal in X. □

2.3. Theorem. *Let E and F be two subspaces of a Banach space X, E is reflexive, and F is simultaneously proximinal. Assume $E+F$ is closed in X. Then $(E+F)/F$ is simultaneously proximinal in X/F.*

Proof. Since E is reflexive, $E\cap F$ is reflexive. It is well known that $E/(E\cap F)$ is algebraically isomorphic to $(E+F)/F$ and the map $T:E/(E\cap F)\to(E+F)/F$ defined by $T(x+(E\cap F))=x+F$ has norm less than one. Since $E+F$ and X are Banach spaces, by the Open Mapping Theorem $(E+F)/F$ and $E/(E\cap F)$ are isomorphic. So since E is reflexive, It follows that $(E+F)/F$ is reflexive. Hence by Theorem 2, $(E+F)/F$ is simultaneously proximinal in X/F. □

2.4. Definition. A Banach space X is called a *p-sum* of two non-zero subspaces $X_1$and X_2 if $X = X_1 \oplus X_2$, $X_1 \cap X_2 = \{0\}$ and for any $x \in X$, $x = x_1 + x_2$ and $\|x\|^p = \|x_1\|^p + \|x_2\|^p$, $1 \le p < \infty$.

2.5. Proposition. *Let $X = X_1 \oplus X_2$, be a p-sum of X_1 and X_2 and A be a bounded subset of X such that $A = A_1 \oplus A_2$, where $A_1 \subseteq X_1$, $A_2 \subseteq X_2$. Then* $\sup\limits_{a\in A} \|a\|^p = \sup\limits_{a_1\in A_1} \|a_1\|^p + \sup\limits_{a_2\in A_2} \|a_2\|^p$.

Proof. Let $a \in A$, then $a = a_1 + a_2$, where $a_1 \in A_1$ and $a_2 \in A_2$. Suppose

$$\sup_{a\in A} \|a\|^p < \sup_{a_1\in A_1} \|a_1\|^p + \sup_{a_2\in A_2} \|a_2\|^p .$$

Then

$$\sup_{a_1\in A_1} \|a_1\|^p + \sup_{a_2\in A_2} \|a_2\|^p - \sup_{a\in A} \|a\|^p > 0.$$

Now given $\epsilon > 0$, there exists $x_1 \in A_1$ and $x_2 \in A_2$ such that

$$\sup_{a_1\in A_1} \|a_1\|^p - \epsilon < \|x_1\|^p \text{ and } \sup_{a_2\in A_2} \|a_2\|^p - \epsilon < \|x_2\|^p .$$

Consequently,

$$\begin{aligned}\|x_1\|^p + \|x_2\|^p &= \|x_1 + x_2\|^p \\ &> \sup_{a_1\in A_1} \|a_1\|^p + \sup_{a_2\in A_2} \|a_2\|^p - 2\epsilon.\end{aligned}$$

Now if we take

$$2\epsilon = \sup_{a_1\in A_1} \|a_1\|^p + \sup_{a_2\in A_2} \|a_2\|^p - \sup_{a\in A} \|a_1 + a_2\|^p > 0,$$

we have

$$\|x_1 + x_2\|^p > \sup_{a\in A} \|a_1 + a_2\|^p .$$

This is a contradiction. Hence

$$\sup_{a\in A} \|a\|^p = \sup_{a_1\in A_1} \|a_1\|^p + \sup_{a_2\in A_2} \|a_2\|^p .$$

□

2.6. Theorem. *Let $X = X_1 \oplus X_2$, be a p-sum of X_1 and X_2. If $E \subseteq X_1$ and $F \subseteq X_2$ are simultaneously proximinal in X_1 and X_2, respectively, then $E \oplus F$ is simultaneously proximinal in X.*

Proof. Let A be a bounded subset of X. Since X is a p-sum of X_1 and X_2, for each $a \in A$, there exists $a_1 \in X_1$ and $a_2 \in X_2$ such that $a = a_1 + a_2$. Therefore, if we take $A_1 = \{a_1 \in X_1 : a = a_1 + a_2 \in A\} \subseteq X_1$ and $A_2 = \{a_2 \in X_2 : a = a_1 + a_2 \in A\} \subseteq X_2$, then $A = A_1 \oplus A_2$. Since A is bounded, it can be easily shown that A_1 and A_2 are bounded sets by making use of the observation

$$\|a_1 + a_2\|^p = \|a_1\|^p + \|a_2\|^p \ge \max\left(\|a_1\|^p , \|a_2\|^p\right)$$

for any and $a_1 \in X_1$ and $a_2 \in X_2$.

Now Since E and F are simultaneously proximinal in X_1 and X_2, respectively, there exists $e_1 \in E$ and $f_1 \in F$ such that:

$$\sup_{a_1\in A_1} \|a_1 - e_1\|^p \le \sup_{a_1\in A_1} \|a_1 - e\|^p \tag{1}$$

$$\sup_{a_2\in A_2} \|a_2 - f_1\|^p \le \sup_{a_2\in A_2} \|a_2 - f\|^p \tag{2}$$

for every $e \in E$ and $f \in F$. Therefore,

$$\begin{aligned}\sup_{a\in A}\|a-(e_1+f_1)\|^p &= \sup_{a\in A}\|a_1+a_2-(e_1+f_1)\|^p \\ &= \sup_{a\in A}\left(\|a_1-e_1\|^p+\|a_2-f_1\|^p\right)\end{aligned}$$

Now using Proposition 6 and inequalities (1) and (2) we get:

$$\begin{aligned}\sup_{a\in A}\left(\|a_1-e_1\|^p+\|a_2-f_1\|^p\right) &= \sup_{a\in A}\|a_1-e_1\|^p+\sup_{a\in A}\|a_2-f_1\|^p \\ &\le \sup_{a_1\in A_1}\|a_1-e\|^p+\sup_{a_2\in A_2}\|a_2-f\|^p \\ &= \sup_{a\in A}\left(\|a_1-e\|^p+\|a_2-f\|^p\right) \\ &= \sup_{a\in A}\|a_1+a_2-(e+f)\|^p\,,\end{aligned}$$

for all $e \in E$ and $f \in F$. Hence $e_1 + f_1$ is a best simultaneous approximation of $A \subseteq X$ in $E \oplus F$. □

Now we prove the main result of this paper.

2.7. Theorem. *Let F and G be two subspaces of a Banach space X. Assume that F is simultaneously proximinal and G is reflexive such that $F \cap G$ is finite dimensional, and $F + G$ is closed. Then $F + G$ is simultaneously proximinal.*

Proof. First we prove the theorem in the case $F \cap G = \{0\}$. Let A be a bounded subset of X. Then there exists a sequence $\{h_n\}$in F and a sequence $\{g_n\}$ in G such that

$$\sup_{a\in A}\|a-(h_n+g_n)\| \longrightarrow d\left(A, F+G\right).$$

Note that $S_n = \sup\limits_{a\in A}\|a-(h_n+g_n)\|$ is a bounded sequence in $\mathbb{R}$, being convergent. Hence there exists $M_1 > 0$ such that $|S_n| \le M_1$. The inequality

$$\|h_n+g_n\| \le \|h_n+g_n-a\|+\|a\|\,,$$

implies that

$$\begin{aligned}\|h_n+g_n\| &\le \sup_{a\in A}\|h_n+g_n-a\|+\sup_{a\in A}\|a\| \\ &\le M_1+M_2 = M\end{aligned}$$

and so $(h_n + g_n)$ is a bounded sequence in $F + G$.

Since the projection $P : F + G \longrightarrow G$, $P(h + g) = g$ is bounded and $P(F) = 0$, by the closed graph theorem it follows that:

$$\begin{aligned}\|g_n\| &= \|P\left(h_n+g_n\right)\| \\ &\le \|h_n+g_n\| \le M,\end{aligned}$$

which implies that g_n is bounded and so is h_n. Since G is reflexive $\{g_n\}$ has a weakly convergent subsequence $g_{n_k} \xrightarrow{w} g_0$ for some $g_0 \in G$. Using Corollary 14 [6, p 422] there is a sequence of convex combinations $\hat{g}_n = \sum\limits_{i\in I_n}\lambda_i g_i$, where $I_n = \{i : p_n < i \le p_{n+1}\}$, p_n is an increasing sequence of integers $\lambda_i \ge 0$, $\sum\limits_{i\in I_n}\lambda_i = 1$ such that $\|\hat{g}_n - g_0\| \longrightarrow 0$. Now let $\hat{h}_n = \sum_{i \in I_n} \lambda_i h_i$. Then

$$\begin{aligned} \sup_{a\in A}\left\|a-\left(\hat{h}_n+\hat{g}_n\right)\right\| &= \sup_{a\in A}\left\|\sum_{i\in I_n}\lambda_i\left(a-h_i-g_i\right)\right\| \\ &\le \sup_{a\in A}\sum_{i\in I_n}\lambda_i\left\|a-h_i-g_i\right\| \\ &\le \sum_{i\in I_n}\lambda_i\sup_{a\in A}\left\|a-h_i-g_i\right\|. \end{aligned}$$

Hence

$$\sup_{a\in A}\left\|a-\hat{h}_n-\hat{g}_n\right\| \longrightarrow d\left(A,F+G\right).$$

Also

$$\sup_{a\in A}\left\|a-\left(\hat{h}_n+g_0\right)\right\| \le \sup_{a\in A}\left(\left\|a-\hat{h}_n-\hat{g}_n\right\|+\left\|\hat{g}_n-g_0\right\|\right),$$

implies that

$$\sup_{a\in A}\left\|a-\hat{h}_n-g_0\right\| \longrightarrow d\left(A,F+G\right).$$

Let h_0 be a point of simultaneous approximation to the set $A-g_0$ in F. Then

$$\sup_{a\in A}\left\|a-\left(h_0+g_0\right)\right\| = \sup_{a\in A}\left\|a-h_0-g_0\right\| \le \sup_{a\in A}\left\|a-\hat{h}_n-g_0\right\| \longrightarrow d\left(A,F+G\right).$$

This implies that

$$\sup_{a\in A}\left\|a-\left(h_0+g_0\right)\right\| = d\left(A,F+G\right).$$

Hence $F+G$ is simultaneously proximinal in X.

Finally if $F\cap G$ is finite dimensional using Corollary 34-10 [4, p 137] , we can find a closed subspace H of G such that $F+G=F+H$ with $F\cap G=\{0\}$. □

We note that for two closed subspaces G and H of a Banach space X, $G+H$ need not be closed but if G is of finite dimension then $G+H$ is closed, Indeed let us consider the quotient space X/H, equipped with the quotient norm $\|.\|_{X/H}$ and the quotient map $\pi : X \to X/H$. Since the linear subspace $V=\pi(G)\subset X/H$ is finite dimensional, it follows that V is closed in X/H. By continuity of the quotient map π, its preimage $\pi^{-1}(V)=G+H$ is closed, see $[3, p\ 160, 167], [9, p\ 82]$. As a corollary of Theorem 8, we have:

2.8. Corollary. *Let F and G be two subspaces of a Banach space X. Assume that F is simultaneously proximinal and G is of finite dimension. Then $F+G$ simultaneously proximinal.*

Acknowledgement: The authors would like to thank the referees for their valuable comments and suggestions that improved the presentation of the paper.

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